1 Complements of Linear Algebra

1.1. Determine the eigenvalues of each of the following matrices and, if they are real, determine the corresponding eigenvectors together with the algebraic and geometric multiplicities.

a) $\begin{bmatrix} 2 & -7 \\ 3 & -8 \end{bmatrix}$ b) $\begin{bmatrix} 2 & 4 \\ -2 & 6 \end{bmatrix}$ c) $\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$

d) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ e) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ f) $\begin{bmatrix} 1 & -1 & -2 \\ 0 & 3 & 0 \\ -2 & 5 & 1 \end{bmatrix}$

g) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ h) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ i) $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Solution: a) $\lambda_1 = -5,\ a.m. = 1$; eigenvectors $u = (c, c)$, with $c \neq 0,\ g.m. = 1$;
$\lambda_2 = -1,\ a.m. = 1$; eigenvectors $u = (7/3c, c)$, with $c \neq 0,\ g.m. = 1$;
b) The characteristic polynomial does not have real eigenvalues ($\lambda_1 = 4 + 2i,\ \lambda_2 = 4 - 2i$);
c) $\lambda_1 = 1 + \sqrt{2};\ a.m. = 1$; eigenvectors $u = ((1 + \sqrt{2})c, c)$, with $c \neq 0,\ g.m. = 1$;
$\lambda_2 = 1 - \sqrt{2};\ a.m. = 1$; eigenvectors $u = ((1 - \sqrt{2})c, c)$, with $c \neq 0,\ g.m. = 1$;
d) $\lambda = 0,\ a.m. = 2$; eigenvectors $u = (c, 0)$, with $c \neq 0,\ g.m. = 1$;
e) $\lambda_1 = 2,\ a.m. = 1$; eigenvectors $u = (c, 0, 0)$, with $c \neq 0,\ g.m. = 1$;
$\lambda_2 = 3,\ a.m. = 1$; eigenvectors $u = (0, c, 0)$, with $c \neq 0,\ g.m. = 1$;
$\lambda_3 = 4,\ a.m. = 1$; eigenvectors $u = (0, 0, c)$, with $c \neq 0,\ g.m. = 1$;
f) $\lambda_1 = 3,\ a.m. = 2$; eigenvectors $u = (-c, 0, c)$, with $c \neq 0,\ g.m. = 1$;
$\lambda_2 = -1,\ a.m. = 1$; eigenvectors $u = (c, 0, c)$, with $c \neq 0,\ g.m. = 1$;
g) $\lambda_1 = 0,\ a.m. = 2$; eigenvectors $u = (-c_1 - c_2, c_1, c_2)$, with $c_1^2 + c_2^2 \neq 0$, $g.m. = 2$;
$\lambda_2 = 3,\ a.m. = 1$; eigenvectors $u = (c, c, c)$, with $c \neq 0,\ g.m. = 1$;
h) $\lambda = 1,\ a.m. = 3$; eigenvectors $u = (c, 0, 0)$, with $c \neq 0,\ g.m. = 1$;
i) $\lambda_1 = 2,\ a.m. = 2$; eigenvectors $u = (c_1, c_1, c_2)$, with $c_1^2 + c_2^2 \neq 0,\ g.m. = 2$;
$\lambda_2 = 0,\ a.m. = 1$; eigenvectors $u = (-c, c, 0)$, with $c \neq 0,\ g.m. = 1$.

1.2. Show that

a) Every eigenvalue of A is also an eigenvalue of A^T.

b) If λ is an eigenvalue of A and $|A| \neq 0$ then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

c) If λ is an eigenvalue of A then λ^k is an eigenvalue of A^k, $k \in \mathbb{N}$.

d) If u and v are eigenvectors associated to an eigenvalue λ of A, then

i) tu is an eigenvector of A associated to λ provided that $t \in \mathbb{R} \setminus \{0\}$,

ii) $u + v$ is an eigenvector of A associated to λ provided that $u + v \neq 0$.

1.3. Let $A = \begin{bmatrix} 1 & 4 \\ 6 & -1 \end{bmatrix}$

a) Determine the eigenvalues and eigenvectors of A.

b) Determine the eigenvalues and eigenvectors of A^{100}.

c) Is A diagonalizable?

d) Compute the trace of A^{100}.

Solution: a) $\lambda_1 = -5$, eigenvectors $u = (-2/3c, c)$, with $c \neq 0$, ;
$\lambda_2 = 5$, eigenvectors $u = (c, c)$, with $c \neq 0$, ;
b) $\lambda = 5^{100}$, eigenvectors $u = (-2/3c_1, c_1) + (c_2, c_2)$, with c_1, c_2 not simultaneously zero;
c) Yes, A has distinct eigenvalues;
d) 2×5^{100}.

1.4. Determine which of the following matrices are diagonalizable.

a) $\begin{bmatrix} 2 & 2 & 2 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ b) $\begin{bmatrix} 6 & 0 \\ 1 & 6 \end{bmatrix}$ c) $\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$

d) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ e) $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ f) $\begin{bmatrix} 1 & -1 & -2 \\ 0 & 3 & 0 \\ -2 & 5 & 1 \end{bmatrix}$

g) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ h) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ i) $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Solution: a) Yes; b) No; c) Yes; d) No; e) Yes; f) No; g) Yes; h) No; i) Yes.

1.5. Two square n-by-n matrices A and B are called *similar* if there is an invertible n-by-n matrix P such that $A = PBP^{-1}$. Suppose that A and B are similar. Show that

a) $|A| = |B|$.

b) A and B have the same characteristic polynomial.

c) (λ, v) is an eigenpair of B iff (λ, Pv) is an eigenpair of A.

1.6. Consider a matrix A and a vector $\boldsymbol{x}$ given by

$$A = \begin{bmatrix} a & a & 0 \\ a & a & 0 \\ 0 & 0 & b \end{bmatrix} \qquad \text{e} \qquad \boldsymbol{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad (a, b \in \mathbb{R})$$

a) Write down the characteristic polynomial $p(\lambda)$.

b) Determine the eigenvalues and eigenvectors of matrix A.

c) Compute $\boldsymbol{x}^T A \boldsymbol{x}$.

d) Consider that $a, b \geq 0$. Without performing any calculations show that there exists a a vector $\boldsymbol{x} \neq \mathbf{0}$ such that $\boldsymbol{x}^T A \boldsymbol{x} = 0$

e) Classify the quadratic form $\boldsymbol{x}^T A \boldsymbol{x}$ for all possible values of a, b.

Solution: a) $P(\lambda) = (b - \lambda)(-\lambda)(2a - \lambda)$; b) $\lambda_1 = b, \lambda_2 = 0, \lambda_3 = 2a$; c) $\mathbf{x}^T A \mathbf{x} = ax^2 + 2axy + ay^2 + bz^2$;
e) PSD if ($a \geq 0$ *e* $b \geq 0$) ; NSD if ($a \leq 0$ *e* $b \leq 0$);
Ind. in the remaining cases, i.e. ($a > 0$ *e* $b < 0$) ou ($a < 0$ *e* $b > 0$).

1.7. Classify the following quadratic forms:

a) $q(x, y) = x^2 + 2xy + y^2$

b) $q(x, y) = x^2 - 2xy + y^2$

c) $q(x, y) = x^2 - y^2$

d) $q(x, y, z) = x^2 + 4xy - 2xz + 7y^2 - 3z^2$

e) $q(x, y, z) = x^2 - 4xy + 4xz - z^2$

f) $q(x, y) = 6x^2 + 4xy + 3y^2$

g) $q(x, y) = x^2 + 4xy + y^2$

h) $q(x, y) = 2x^2 + 6xy + 4y^2$

i) $q(x, y, z) = 3y^2 + 4xz$

j) $q(x, y) = x^2 + 4xy + ay^2, \quad a \in \mathbb{R}$

Solution: a)PSD b)PSD c)Ind. d)Ind. e)Ind. f)PD g)Ind. h)Ind. i)Ind. j)Ind. if $a < 4$, PSD if $a = 4$ and PD if $a > 4$.

1.8.

Classify the following symmetric matrices (with respect to being PD, PSD, ND, NSD, Ind.)

a) $\begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix}$ b) $\begin{bmatrix} -5 & 1 \\ 1 & 5 \end{bmatrix}$ c) $\begin{bmatrix} -5 & 1 \\ 1 & -5 \end{bmatrix}$ d) $\begin{bmatrix} 1 & 2 \\ 2 & a \end{bmatrix}, a \in \mathbb{R}$ e) $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 7 & 0 \\ -1 & 0 & -3 \end{bmatrix}$

f) $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 7 & -5 \\ -1 & -5 & 4 \end{bmatrix}$ g) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -4 \end{bmatrix}$ h) $\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & a & 0 \\ 0 & a & 2 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}, a \in \mathbb{R}$ i) $\begin{bmatrix} -1 & 1 & 1 \\ 1 & -3 & 0 \\ 1 & 0 & -2 \end{bmatrix}$

j) $\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

Solution: a) PD b) Ind. c) ND d)Ind. if $a < 4$, PSD if $a = 4$ and ND if $a > 4$ e) Ind. f) PSD
g) Ind. h)Ind. if $a < -\sqrt{2} \vee a > \sqrt{2}$, PD if $-\sqrt{2} < a < \sqrt{2}$ and PSD if $a = \pm\sqrt{2}$ i) ND j) Ind.

2 Functions of several variables: Topology, limits and continuity

2.1. Determine the domain of the following functions and represent it graphically.

$a) f(x,y) = \dfrac{\sqrt{9 - x^2 - y^2}}{1 - \ln x}$ $b) f(x,y) = \dfrac{\sqrt{e - e^x}}{\ln(4 - x^2 - y^2)}$ $c) f(x,y) = \ln(x - y)\sqrt{(y - x)(x^2 + y^2 - 1)}$

$d) f(x,y) = \dfrac{\sqrt[3]{x + y}}{\ln x^2 - \ln(3 - x)^2}$ $e) f(x,y) = \ln(x - y)^2$

Solution: a) $\{(x,y) \in \mathbb{R}^2 : x > 0 \wedge x \neq e \wedge x^2+y^2 \leq 9\}$ b) $\{(x,y) \in \mathbb{R}^2 : x \leq 1 \wedge x^2+y^2 < 4 \wedge x^2+y^2 \neq 3\}$ c) $\{(x,y) \in \mathbb{R}^2 : x-y > 0 \wedge x^2+y^2-1 \leq 0\}$ d) $\{(x,y) \in \mathbb{R}^2 : x \neq 0 \wedge x \neq 3 \wedge x \neq 3/2\}$ e) $\{(x,y) \in \mathbb{R}^2 : x \neq y\}$

2.2. Determine the interior, exterior and boundary of the following subsets of $\mathbb{R}^2$. Classify them with respect to being open, closed and bounded.

$A = \{(x,y) \in \mathbb{R}^2 : (x-1)^2+(y+2)^2 < 4\}$ $B = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \geq 9\}$

$C = \{(x,y) \in \mathbb{R}^2 : \frac{x^2}{9}+\frac{y^2}{16} \leq 1\}$ $D = \{(x,y) \in \mathbb{R}^2 : \frac{(x-2)^2}{4}+\frac{(y+1)^2}{9} = 1\}$

Solution: $\text{Int}(A) = A$, $\text{Bdy}(A) = \{(x,y) \in \mathbb{R}^2 : (x-1)^2+(y+2)^2 = 4\}$, $ext(A) = \{(x,y) \in \mathbb{R}^2 : (x-1)^2+(y+2)^2 > 4\}$, A is open and bounded. $\text{Int}(B) = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 > 9\}$, $\text{Bdy}(B) = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = 9\}$, $\text{Ext}(B) = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 9\}$, B is closed and not bounded. $\text{Int}(C) = \{(x,y) \in \mathbb{R}^2 : \frac{x^2}{9}+\frac{y^2}{16} < 1\}$, $\text{Bdy}(C) = \{(x,y) \in \mathbb{R}^2 : \frac{x^2}{9}+\frac{y^2}{16} = 1\}$, $\text{Ext}(C) = \{(x,y) \in \mathbb{R}^2 : \frac{x^2}{9}+\frac{y^2}{16} > 1\}$, C is closed and bounded. $\text{Int}(D) = \emptyset$, $fr(D) = D$, $\text{Ext}(D) = \{(x,y) \in \mathbb{R}^2 : \frac{(x-2)^2}{4}+\frac{(y+1)^2}{9} \neq 1\}$, D is closed and bounded.

2.3.

Represent graphically and analitically the the domain D_f, as well as $\text{Int}(D_f)$, $\text{Bdy}(D_f)$ and D_f'. State in each case if D_f open, closed, bounded and compact.

$a) f(x,y) = \dfrac{|x|-4}{\ln(4-x^2-y^2)}$ $b) f(x,y) = \sqrt{x(1-x)}+\dfrac{\ln(x^2-y)}{\sqrt{y+x}}$

$c) f(x,y) = \sqrt{y+x-1}.\ln(4-(x+1)^2-(y-1)^2)$ $d) f(x,y) = \sqrt{x-|y|}+\sqrt{2-y^2-x}$

$e) f(x,y) = x\sqrt{y^2-4}+\sqrt[4]{16-x^2-y^2}$

Solution: a) $D_f = \{(x,y) \in \mathbb{R}^2 : 4-x^2-y^2 > 0 \wedge \ln(4-x^2-y^2) \neq 0\} = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 4 \wedge x^2+y^2) \neq 3\}$, $\text{Int}(D_f) = D_f$, $\text{Bdy}(D_f) = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = 4 \vee x^2+y^2 = 3\}$, $D_f' = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 4\}$. D_f is open, not closed, bounded, not compact.

b) $D_f = \{(x,y) \in \mathbb{R}^2 : x(1-x) \geq 0 \wedge x^2-y > 0 \wedge y+x > 0\} = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1 \wedge -x < y < x^2\}$,

$\text{Int}(D_f) = \{(x,y) \in \mathbb{R}^2 : 0 < x < 1 \wedge -x < y < x^2\}$, $\text{Bdy}(D_f) = \{(x,y) \in \mathbb{R}^2 : (y = x^2 \ \wedge \ 0 \le x \le 1) \vee (y = -x \ \wedge \ 0 \le x \le 1) \vee (x = 1 \ \wedge \ -x \le y \le x^2)\}$, $D'_f = \{(x,y) \in \mathbb{R}^2 : (0 \le x \le 1) \wedge (-x \le y \le x^2)\}$. D_f is not open nor closed, bounded, not conmpact.

c) $D_f = \{(x,y) \in \mathbb{R}^2 : y + x - 1 \ge 0 \wedge 4 - (x+1)^2 - (y-1)^2 > 0\} = \{(x,y) \in \mathbb{R}^2 : y \ge 1 - x \wedge (x+1)^2 + (y-1)^2 < 4\}$, $\text{Int}(D_f) = \{(x,y) \in \mathbb{R}^2 : y > 1 - x \wedge (x+1)^2 + (y-1)^2 < 4\}$, $\text{Bdy}(D_f) = \{(x,y) \in \mathbb{R}^2 : y = 1 - x \wedge (x+1)^2 + (y-1)^2 \le 4\} \cup \{(x,y) \in \mathbb{R}^2 : y \ge 1 - x \wedge (x+1)^2 + (y-1)^2 = 4\}$, $D'_f = \{(x,y) \in \mathbb{R}^2 : y \ge 1 - x \wedge (x+1)^2 + (y-1)^2 \le 4\}$. D_f is not open nor closed, bounded, not compact.

d)$D_f = \{(x,y) \in \mathbb{R}^2 : x - |y| \ge 0 \wedge 2 - y^2 - x \ge 0\}$, $\text{Int}(D_f) = \{(x,y) \in \mathbb{R}^2 : x - |y| > 0 \wedge 2 - y^2 - x > 0\}$, $\text{Bdy}(D_f) = \{(x,y) \in \mathbb{R}^2 : x - |y| = 0 \wedge 2 - y^2 - x \ge 0\} \cup \{(x,y) \in \mathbb{R}^2 : x - |y| \ge 0 \wedge 2 - y^2 - x = 0\}$, $D'_f = D_f$. D_f is not open, is closed, bounded, compact.

e)$D_f = \{(x,y) \in \mathbb{R}^2 : y^2 - 4 \ge 0 \wedge 16 - x^2 - y^2 \ge 0\} = \{(x,y) \in \mathbb{R}^2 : (y \ge 2 \vee y \le -2) \wedge x^2 + y^2 \le 16\}$, $\text{Int}(D_f) = \{(x,y) \in \mathbb{R}^2 : (y > 2 \vee y < -2) \wedge x^2 + y^2 < 16\}$, $\text{Bdy}(D_f) = \{(x,y) \in \mathbb{R}^2 : (y = 2 \vee y = -2) \wedge x^2 + y^2 \le 16\} \cup \{(x,y) \in \mathbb{R}^2 : (y \ge 2 \vee y \le -2) \wedge x^2 + y^2 = 16\}$, $D'_f = D_f$. D_f is not open, is closed, bounded, compact.

2.4. Determine the domain D_f of the function defined by

$$f(x,y) = \frac{\ln(x^2 + y^2 - 4)}{|x| - 4}.$$

Represent D_f graphically and show that it is open and unbounded. Check if $\text{Ext}(D_f)$ is also open and unbounded.

Solution: $D_f = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 - 4 > 0 \wedge |x| - 4 \neq 0\}$. $Ext(D_f) = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 4\}$.

2.5.

Determine the interior, the boundary and the limit points of the following subsets of $\mathbb{R}^2$.

$a) A = \{(0,0), (0,1), (1,0), (1,1)\}$ $\qquad$ $b) B = \{(x,y) \in \mathbb{R}^2 : (x,y) = (\frac{1}{n}, \frac{1}{n}), n \in \mathbb{N}\}$

$c) C = \{(x,y) \in \mathbb{R}^2 : x \ge 0, \ y = (-1)^n \frac{1}{n}, n \in \mathbb{N}\}$.

Solution: a) $\text{Int}(A) = \emptyset, \text{Bdy}(A) = A, A' = \emptyset$. b) $\text{Int}(B) = \emptyset$, $\text{Bdy}(B) = B \cup \{(0,0)\}$, $B' = \{(0,0)\}$. c) $\text{Int}(C) = \emptyset$, $\text{Bdy}(C) = C \cup \{(x,y) \in \mathbb{R}^2 : x \geq 0,\ y = 0\}$, $C' = fr(C)$.

2.6.

Compute the limits of the following sequences, or show that they no not exist.

a) $\lim \left(\left(\dfrac{2n^2+3}{1+2n^2} \right)^{n^2}, \ln \left(\dfrac{2n}{2n+1} \right)^{n+\frac{1}{2}} \right)$ b) $\lim \left(n^3 + n - n^2 - 1), \sqrt{n}. \dfrac{\sqrt{n}+3}{(\sqrt{n}+1)^2} \right)$

c) $\lim \left(n \left(e^{\frac{1}{n}} - 1 \right), \sin \dfrac{n\pi}{2} \right)$

Solution: a) $(e, -\frac{1}{2})$; b) $(+\infty, 1)$; c) does not exist.

2.7. Compute the following limits:

a) $\lim \overline{x}_n$, com $\overline{x}_n = \left[\dfrac{n}{2n+1}, \left(1 + \dfrac{2}{n}\right)^n, \ln \left(1 + \dfrac{1}{n}\right)^n \right]$

b) $\lim \overline{x}_n$ com $\overline{x}_n = \left[\sqrt{n} - \sqrt{n-1},\ \left(\sqrt[n]{e} - 1 \right).n,\ n. \ln \dfrac{n+2}{n}, \left(1 - \dfrac{n^2}{n^2+1}\right)^{\frac{1}{3}} . \left(\dfrac{n^2+1}{2n^2} \right)^{\frac{1}{3}} \right]$

Solution: a) $(1/2, e^2, 1)$; b) $(0, 1, 2, 0)$.

2.8.

For each of the following functions, investigate the existence of limit at the point $(0,0)$.

a) $f(x,y) = \dfrac{x^2 - y^2}{x(x+y)}$ b) $f(x,y) = \dfrac{x^3 - y^3}{x^2 + y^2}$ c) $f(x,y) = \dfrac{x^2 + y}{\sqrt{x^2 + y^2}}$

d) $f(x,y) = xy \dfrac{x^2 - y^2}{x^2 + y^2}$ e) $f(x,y) = \dfrac{xy}{x^2 + y^2}$ f) $f(x,y) = \dfrac{x^2 y}{x^4 + y^2}$

g) $f(x,y) = \dfrac{x^2(x+y)}{x^2 + y^2}$ h) $f(x,y) = \dfrac{x^2 - y^2 + 2x^3}{x^2 + y^2}$, i) $f(x,y) = \begin{cases} \dfrac{y}{x}\sqrt{x^2+y^2}, & \text{se } x > 0,\ y > 0 \\ 0, & \text{se } x < 0 \text{ ou } y < 0. \end{cases}$

Solution: a) does not exist b) 0 c)does not exist d) 0 e) does not exist f) does not exist g) 0 h) does not exist i) does not exist.

2.9.

Determine, if possible, a continuous extention of each of the functions in the previous exercise to the point $(0,0)$.

Solution:

When the limit does not exist it is not possible to define such extension. In the other cases simply define $f(0,0) = \lim_{(x,y)\to(0,0)} f(x,y)$.

2.10. Investigate the existence of the following limits

$a) \lim_{(x,y)\to(2,1)} \dfrac{(x-2)(y-1)}{(x-2)^2+(y-1)^2}$ $b) \lim_{(x,y)\to(1,-3)} \dfrac{\sqrt{(x-1)(y+3)}+\sin(x-1)(y+3)}{\sqrt{(x-1)(y+3)}}$

$c) \lim_{(x,y)\to(2,1)} \dfrac{3(x-2)^2(y-1)}{(x-2)^2+(y-1)^2}$ $d) \lim_{(x,y)\to(0,0)} \dfrac{x^2+y^2}{x}$

$e) \lim_{(x,y)\to(1,0)} \dfrac{x^2-y^2-1}{x-1}$ $f) \lim_{(x,y)\to(0,0)} \dfrac{x^2y+x^2+y^2}{x^2+y^2}$

$g) \lim_{(x,y)\to(0,1)} \dfrac{x^2\sqrt{|y-1|}}{x^2+(y-1)^2}$ $h) \lim_{(x,y,z)\to(0,0,0)} \dfrac{xyz}{x^2+y^2+z^2}$

Solution: a) does not exist b) 1 c) 0 d) does not exist e) does not exist f) 1 g) 0 h) 0.

2.11. Consider the function $f:\mathbb{R}^2\to\mathbb{R}$ defined by $f(x,y)=\dfrac{x^3-y^3}{x-y}$.

a) Determine the domain D_f..

b) Show that $x^3-y^3=(x-y)p(x,y)$, where $p(x,y)$ is a polynomial.

c) Can f be extended by continuity to the line $y=x$?

Solution: a) $D_f = \mathbb{R}^2 \setminus \{(x,x) : x \in \mathbb{R}\}$. c) yes, just define $f(x,x) = 3x^2$.

2.12. Consider the function $f(x,y) = \frac{x^2 y}{x^2 - y^2}$.

a) Compute $\lim\limits_{\substack{(x,y)\to(0,0),\\ y=x+x^2}} f(x,y)$. What can you conclude about $\lim\limits_{(x,y)\to(0,0)} f(x,y)$?

b) Compute $\lim\limits_{\substack{(x,y)\to(0,0),\\ y=mx}} f(x,y)$, $|m| \neq 1$. What can you conclude about $\lim\limits_{(x,y)\to(0,0)} f(x,y)$?

Solution:

a) $-\frac{1}{2}$. If it exists $\lim\limits_{(x,y)\to(0,0)} f(x,y)$, is $-\frac{1}{2}$. b) 0. It does not exist ($0 \neq -\frac{1}{2}$).

3 Differential Calculus in $\mathbb{R}^n$

3.1.

Determine the first order partial derivatives of the following functions, defining them in the largest possible domain.

$$\text{a)} f(x,y,z) = 3xy + x^2 - zy + z^2; \qquad \text{b)}\ f(x,y) = \begin{cases} x^2 - yx, & y \neq x \\ x, & y = x. \end{cases}$$

Solution: a) $f'_x = 3y + 2x$; $f'_y = 3x - z$; $f'_z = -y + 2z$; $\mathcal{D}f'_x = \mathcal{D}f'_y = \mathcal{D}f'_z = \mathbb{R}^3$
b) $f'_x = 2x - y$ if $x \neq y$ and $f'_x = 0$ if $x = y = 0$; $f'_y = -x$ if $x \neq y$ and $f'_y = 0$ if $x = y = 0$
$\mathcal{D}f'_x = \mathcal{D}f'_y = \mathbb{R}^2 \setminus \{(a,a) : a \neq 0\}$

3.2. Show that $f(x,y) = \dfrac{x - y + 1}{x + y}$ is a solution of the equation

$$\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} = \frac{2}{x+y}$$

in any of the sets defined by $x + y > 0$ or $x + y < 0$.

3.3. Consider the function

$$f(x,y) = \begin{cases} \dfrac{e^{x-y} - (x - y + 1)}{x - y}, & x \neq y \\ 0, & x = y \end{cases}$$

a) Discuss the continuity of $f(x,y)$ at $(1,1)$.

b) Check that $f'_x(a,a) + f'_y(a,a) = 0,\ \forall a \in \mathbb{R}$.

Solution:

a) f is continuous at $(1,1)$.

3.4. Given the function

$$f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2} & , x^2+y^2 \neq 0 \\ 0 & , x = y = 0 \end{cases},$$

compute the directional derivatives at $(0,0)$, whenever they exist.

Solution: $\partial_{\overrightarrow{v}} f(0,0)$ exists for $\overrightarrow{v} = (\alpha,\alpha)$ and $\overrightarrow{v} = (\alpha,-\alpha)$, $\alpha \in \mathbb{R}\setminus\{0\}$. In that case, $\partial_{\overrightarrow{v}} f(0,0) = 0$.

3.5. Consider the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4} & , x \neq 0 \\ 0 & , x = 0 \end{cases},$$

a) Show that f admits a directional derivative at $(0,0)$ along any direction and compute it.

b) Show that f is not continuous at $(0,0)$.

c) Without performing any calculations, state the value of $\dfrac{\partial f}{\partial x}(0,0)$ and of $\dfrac{\partial f}{\partial y}(0,0)$.

Solution: a) $\partial_{(\alpha,\beta)} f(0,0) = \begin{cases} \dfrac{\beta^2}{\alpha} & \text{se } \alpha \neq 0 \\ 0 & \text{se } \alpha = 0 \end{cases}$, $(\alpha,\beta) \in \mathbb{R}^2\setminus\{(0,0)\}$. c) $\dfrac{\partial f}{\partial x}(0,0) = \dfrac{\partial f}{\partial y}(0,0) = 0$.

3.6.

Study the differentiability of the following functions at the proposed points and obtain the expression of the first order differentials (in case they are differentiable).

a) $f(x,y) = x^2 + y^2$, at point (0,0);

b) $f(x,y) = \begin{cases} x+y, & x \neq y \\ \\ x+1, & x = y \end{cases}$, at (1,1);

c) $f(x,y) = \begin{cases} xy - 2y + 3x, & x \neq y \\ \\ x^2y^2 + 3x - 2y, & x = y \end{cases}$, at (0,0);

d) $y = (x^2 + 1, x)$, at $x = 1$;

Solution:
a) f is differentiable at $(0,0)$; $Df(0,0)(\mathbf{h}) = 0$;
b) f is not differentiable at em $(1,1)$;
c) f is differentiable at $(0,0)$; $Df(0,0)(\mathbf{h}) = 3h_1 - 2h_2$;
d) y is differentiable at $x = 1$; $Df(1)(\mathbf{h}) = (2h, h)$.

3.7. Write down the expressions of the first order differentials of each given function, at the proposed points:

a) $f(x,y) = y^x$, at a generic point (a,b), with $b > 0$;

b) $f(x_1,x_2,x_3) = \dfrac{x_1 - x_2 + x_3}{\sqrt{x_3 - 1}}$, at (1,-3,2).

Note: Admit that the functions are differentiable.

Solution: a) $Df(a,b)(\mathbf{h}) = b^a \log b.h_1 + ab^{a-1}.h_2$ b) $Df(1,-3,2)(\mathbf{h}) = h_1 - h_2 - 2h_3$.

3.8. Show that the following functions are continuous but not differentiable at the given points:

a) $f(x,y) = \begin{cases} \dfrac{-3x(y-2)^2 + x^3}{x^2 + (y-2)^2}, & \text{if } (x,y) \neq (0,2) \\ \\ 0, & \text{if } (x,y) = (0,2), \end{cases}$, at $(0,2)$.

b) $g(x,y) = \begin{cases} \dfrac{2xy}{\sqrt{x^2+y^2}}, & \text{if } x^2 + y^2 \neq 0 \\ \\ 0, & \text{if } x = y = 0, \end{cases}$ at $(0,0)$

c) $h(x,y) = \sqrt{|x|}\cos y$, at $(0,0)$.

3.9. Use the chain rule to compute

a) $\dfrac{df}{dt}$, where $f = x^2y^3$, knowing that $x = te^t$ e $y = t^2 + 1$;

b) $\dfrac{df}{dt}$, where $f = u^2 + v^3$, knowing that $u = \dfrac{x}{y}$, $v = (x + 2y)^3$ e $x = \dfrac{1}{t}$, $y = tg\ t$;

c) $\dfrac{dz}{dt}$, knowing that $z = \dfrac{2xy}{x^2 + y^2}$ e $x = \cos t$, $y = \sin t$.

d) $\nabla f(1, 1)$, where $f(x, y) = \sin(2u - v^3 + w)$, knowing that $u = e^{x^2-y}$, $v = xy^2$ e $w = x^3y^2$;

e) $\dfrac{\partial f}{\partial y}(0, 1, 1)$, where $f(x, y, z) = (u^2 - 3v)^5$, knowing that $u = e^{\frac{xy}{z}}$ e $v = \ln(y^2z^3)$;

f) $\nabla f(1, 2, 3)$, where $f(x, y, z) = g(u, v, w)$, with $u = 5x + 3z$, $v = 8x + 2y$, $w = -y + z$ and knowing that $\nabla g(14, 12, 1) = (4, 5, 6)$.

Solution:

a) $\frac{df}{dt} = 2te^{2t}(t + 1)(t^2 + 1)^3 + 6t^3e^{2t}(t^2 + 1)^2$;
b) $\frac{df}{dt} = -2\frac{1}{t^3}\frac{1}{tg^2\ t} - 2\frac{1}{t^2}\frac{\sec^2 t}{tg^3\ t} + 9(-\frac{1}{t^2} + 2\sec^2 t)(\frac{1}{t} + 2tg\ t)^8$; c) $\frac{dz}{dt} = 2 - 4\sin^2 t$; d) $\nabla f(1, 1) = (4\cos 2, -6\cos 2)$; e) $\frac{\partial f}{\partial y}(0, 1, 1) = -30$; f) $\nabla f(1, 2, 3) = (60, 4, 18)$.

3.10. If a function $f(u, v, w)$ is differentiable at $u = x - y$, $v = y - z$ and $w = z - x$, show that setting $F(x, y, z) = f(x - y, y - z, z - x)$ we have

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} = 0.$$

3.11. Consider the function

$$g(x, y) = \begin{cases} \dfrac{(x-1)^2y^2}{(x-1)^2 + y^2}, & (x, y) \neq (1, 0) \\ \\ 0, & (x, y) = (1, 0) \end{cases}$$

a) Determine the partial derivatives $g'_x(x, y)$ and $g'_y(x, y)$, as well as their domain of definition.

b) Show that $g'_x(x, y)$ and $g'_y(x, y)$ are continuous over $\mathbb{R}^2$.

c) Study the differentiability of f at $(1, 0)$.

d) Discuss the continuity of f at $(1, 0)$.

Solution:
a) $g'_x(x,y) = \begin{cases} \dfrac{2(x-1)y^4}{((x-1)^2+y^2)^2}, & (x,y) \neq (1,0) \\ 0, & (x,y) = (1,0) \end{cases}$ $g'_y(x,y) = \begin{cases} \dfrac{2(x-1)^4 y}{((x-1)^2+y^2)^2}, & (x,y) \neq (1,0) \\ 0, & (x,y) = (1,0) \end{cases}$
Therefore, $D_{g'_x} = D_{g'_y} = \mathbb{R}^2$. c) g is differentiable at $(1,0)$. d) g continuous at (1,0).

3.12. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined by

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\dfrac{1}{\sqrt{x^2+y^2}}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

a) Compute $\dfrac{\partial f}{\partial x}(0,0)$ and $\dfrac{\partial f}{\partial y}(0,0)$.

b) Determine $\dfrac{\partial f}{\partial y}(x,y)$ and show that it is discontinuous at $(0,0)$.

c) Check that$f$ is differentiable at $(0,0)$.

d) Compute $\partial_{(\frac{3}{5},\frac{4}{5})} f(0,0)$.

e)Discuss the continuity of f at $(0,0)$.

Solution:
a) $f'_x(0,0) = f'_y(0,0) = 0$;
$b) f'_y(x,y) = \begin{cases} 2y\sin\frac{1}{\sqrt{x^2+y^2}} - y\frac{1}{\sqrt{x^2+y^2}}\cos\frac{1}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$; d) 0; e) f is continuous at $(0,0)$.

3.13. Use the function

$$g(x,y) = \begin{cases} \dfrac{\sin x}{y}, & y \neq 0 \\ 0, & y = 0 \end{cases}$$

and the point $(0,0)$ to show that a function with finite partial derivatives at a given point is not necessarily continuous at that point. Is the given function differentiable at $(0,0)$? Why?

Solution:

The function is not continuous at $(0,0)$, and so it is also not differentiable.

3.14. Considerer the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{xy}{|x|+|y|}, & \text{if } (x,y) \neq (0,0) \\ \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

a) Show that f is continuous at $(0,0)$.

b) Determine $\dfrac{\partial f}{\partial x}(0,0)$ and $\dfrac{\partial f}{\partial y}(0,0)$.

c) Show that f is not differentiable at (0,0). Without performing any calculations, what can you conclude about the continuity of $\dfrac{\partial f}{\partial x}$ e $\dfrac{\partial f}{\partial y}$ at (0,0)?

Solution:

b) $f'_x(0,0) = f'_y(0,0) = 0$ c) Since f is not differentiable at $(0,0)$, at least one of the functions f'_x or f'_y is not continuous at $(0,0)$.

3.15. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x,y) = \begin{cases} \dfrac{x(x-y)}{x+y} & \text{if } x+y \neq 0 \\ \\ 0 & \text{if } x+y = 0 \end{cases}$$

a) Study the continuity of f at $(0,0)$

b) Compute the partial derivative $\dfrac{\partial f}{\partial x}$ and discuss its continuity at $(0,0)$.

c) Study the differentiability of f at $(0,0)$.

d) Show that $\left(\dfrac{\partial f}{\partial x}(0,0), \dfrac{\partial f}{\partial y}(0,0)\right) . \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \neq \delta_{\vec{(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})}} f(0,0)$. Comment on the result.

Solution:

a) f is not continuous at (0,0).
b) $f'_x(x,y) = \frac{x^2+2xy-y^2}{(x+y)^2}$ if $x+y \neq 0$ and $f'_x(x,y) = 1$ if $(x,y) = (0,0)$ (it does not exist $f'_x(a,-a), a \neq 0$); $f'_x(x,y)$ is not continuous at $(0,0)$.
c) f is not differentiable at $(0,0)$.
d) The two values onlym had to be equal if f was differentiable at $(0,0)$.

3.16. Compute $\dfrac{\partial^2 f}{\partial x^2}$ and $\dfrac{\partial^4 f}{\partial x^2 \partial z \partial y}$, for $f(x,y,z) = z^2x^2y + xye^z$.

Solution: $\dfrac{\partial^2 f}{\partial x^2} = 2yz^2$, $\dfrac{\partial^4 f}{\partial x^2 \partial z \partial y} = 4z$.

3.17. Compute f''_{x^2}, f''_{xy} and f'''_{xyx} for each of the following functions, indicating the corresponding domain of definition:

$$a)\ f(x,y) = x\sin(x+y); \qquad b)\ f(x,y) = \begin{cases} y\sin x, & y \neq 0 \\ 2, & y = 0 \end{cases}$$

Solution:

a) $f''_{x^2} = 2\cos(x+y) - x\sin(x+y)$, $f''_{xy} = \cos(x+y) - x\sin(x+y)$ and $f'''_{xyx} = -2\sin(x+y) - x\cos(x+y)$.
b) $f''_{x^2} = -y\sin x$, $f''_{xy} = \cos x$ and $f'''_{xyx} = -\sin x$;

3.18. Compute the differential of order 2, 3 and 4 of the function $f(x,y) = \sqrt{xy}$ at $(1,1)$.

Solution:
$D^2f(1,1)(\mathbf{h}^2) = -\frac{1}{4}h_1^2 + \frac{1}{2}h_1h_2 - \frac{1}{4}h_2^2$,
$D^3f(1,1)(\mathbf{h}^3) = \frac{3}{8}h_1^3 - \frac{3}{8}h_1^2h_2 - \frac{3}{8}h_1h_2^2 + \frac{3}{8}h_2^3$,
$D^4f(1,1)(\mathbf{h}^4) = -\frac{15}{16}h_1^4 + 4\frac{3}{16}h_1^3h_2 + 6\frac{1}{16}h_1^2h_2^2 + 4\frac{3}{16}h_1h_2^3 - \frac{15}{16}h_2^4$.

3.19. Determine the differential of order n of the function $f(x,y) = \sin(x+y)$, at the point (0,0).

Solution: $D^nf(0,0)(\mathbf{h}) = \sin(n\pi/2)\displaystyle\sum_{i=0}^{n} \binom{n}{i} h_1^i h_2^{n-i}$

3.20. Show that $f(x,y) = log(e^x + e^y)$ satisfies the (differential) equation

$$\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 0$$

everywhere in $\mathbb{R}^2$.

3.21. Let $f \in C^2(\mathbb{R}^2)$ be a real function such that $\dfrac{\partial f}{\partial u}(0,0) = \dfrac{\partial f}{\partial v}(0,0) = 1$. Also, let $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by

$$g(x,y) = f(y\text{sen } x, y^2).$$

Show that the Hessian matrix of g at $(0,0)$ is given by $\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$.

3.22. Consider $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by

$$f(x,y) = xy^2 + g(u,v,w), \text{ with } u = \text{sen } y^2,\ v = \ln x \text{ and } w = ye^x.$$

Assuming that g is of class $C^2(\mathbb{R}^3)$, compute $\dfrac{\partial^2 f}{\partial y \partial x}(1,0)$.

Solution: $\frac{\partial^2 f}{\partial y \partial x}(1,0) = e\left(\frac{\partial^2 g}{\partial w \partial v}(0,0,0) + \frac{\partial g}{\partial w}(0,0,0)\right)$.

3.23. Show that the following functions are homogeneous or positively homogeneous. Determine in each case the degree of homogeneity and verify Euler's identity.

a) $f(x,y) = \log \dfrac{(x+y)^2}{xy}$ b) $f(x,y,z) = \dfrac{\sqrt{x^2+y^2}}{z^2}$ c) $f(x,y) = \begin{cases} (x+y)\sin\left(\dfrac{xy}{x^2+y^2}\right), & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

Solution:

a) f is homogeneous with degree 0;
b) f is positively homogeneous with degree -1;
c) f is homogeneous with degree 1.

3.24. Study the function $g(x,y,z) = x^2 + x^\alpha y^{\beta-3} - z^{3\alpha} y^\beta$ e de $h(x,y) = \dfrac{x^3 y^\alpha + x^{\beta-1}}{y^{3-\beta}}$, with respect to its homogeneity in terms of the parameters $\alpha, \beta \in \mathbb{R}$,

a) Using the definition.

b) Using Euler's identity.

Solution:

g is homogeneous with degree 2 for $\alpha = -\frac{3}{2}$ and $\beta = \frac{13}{2}$;
h is homogeneous with degree $\alpha + \beta$ for $\beta = \alpha + 4, \alpha \in \mathbb{R}$.

3.25. Assuming that $g(u, v)$ is differentiable $\left(\frac{x}{y}, \frac{z}{x}\right)$, with $x, y \neq 0$, show that

$$f(x, y, z) = x^2.g\left(\frac{x}{y}, \frac{z}{x}\right),$$

satisfies the identity $x\, f'_x + y\, f'_y + z\, f'_z = 2.f$. Interpret this results in terms of homogeneity.

Solution: f is positively homogeneous with degree 2.

3.26. Let $f(\mathbf{x}) : \mathbb{R}^n \setminus \{\mathbf{0}\} \to \mathbb{R}$ be an homogeneous, non constant function with degree 0. Show that $\lim_{\mathbf{x}\to\mathbf{0}} f(\mathbf{x})$ does not exist.

3.27. Consider the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by

$$f(x, y) = \ln\left(\frac{xy}{x + y}\right).$$

Write down Taylor's formula with degree 2, around (1,1).

Solution: $\ln \frac{(1+h)(1+k)}{2+h+k} = -\ln 2 + \frac{1}{2}h + \frac{1}{2}k + \frac{1}{2}\left(-\frac{3}{4}h^2 + \frac{1}{2}hk - \frac{3}{4}k^2\right) + r_3(h, k).$

4 Optimization Problems

4.1. Determine and classify the critical points of the following functions from $\mathbb{R}^2$ to $\mathbb{R}$.

$a) x^2 + y^2$	$b) x^2 - y^2$	$c) x^3 + y^3$	$d) x^3 - y^3$
$e) x^4 + y^4$	$f) x^4 - y^4$	$g) 3xy - x^3 - y^3$	$h) x \ln x + y \ln y$
$i) x^3 + ye^y$	$j) 2x^3 + xy^2 + 5x^2 + y^2$	$k) x^4 + y^4 - 4xy + 1$	$l) x^2y^2$

Solution: a) $(0,0)$ is a minimum point; b) c) d) $(0,0)$ is a saddle point ; e) $(0,0)$is a minimum point; f) $(0,0)$ is a saddle point; g) $(0,0)$ is a saddle point and $(1,1)$ is a maximum point; h) $(1/e, 1/e)$ is a minimum point; i) $(0,-1)$ is a saddle point; j) $(0,0)$ is a minimum point, $(-5/3, 0)$ éis a maximum point, $(-1,2)$ e $(-1,-2)$ are saddle points; k) $(0,0)$ is a saddle point, $(1,1)$ e $(-1,-1)$ are minimum points; l) $(0,b)$ e $(a,0)$ $\forall a, b \in \mathbb{R}$, are minimum points;

4.2. Determine and classify the critical points of the following functions, in terms of the parameter $a \in \mathbb{R}\setminus\{0\}$

$$a) f(x,y) = e^{x^2-ay^2} \qquad b) f(x,y) = ax^2 - y^2$$

$$c) f(x,y) = x^3 - ax^2 - 3y^2 \qquad d) f(x,y) = \tfrac{16}{5}x^5 + ay^2 - x$$

Solution: a) Critical point: $(0,0)$. if $a < 0$, minimum point; if $a > 0$, saddle point. b) Critical point: $(0,0)$. If $a > 0$, $(0,0)$ is a saddle point; if $a < 0$, $(0,0)$ is a maximum point. c) Critical points: $(0,0)$ and $(\frac{2a}{3}, 0)$. If $a > 0$, $(0,0)$ is a maximum point and $(\frac{2a}{3}, 0)$ is a saddle point; if $a < 0$, $(0,0)$ is a saddle point and $(\frac{2a}{3}, 0)$ is a maximum point. d) Critical points: $(-\frac{1}{2}, 0)$ and $(\frac{1}{2}, 0)$. if $a < 0$, $(-\frac{1}{2}, 0)$ is a maximum point and $(\frac{1}{2}, 0)$ is a saddle point; if $a > 0$, $(-\frac{1}{2}, 0)$ is a saddle point and $(\frac{1}{2}, 0)$ is a minimum point.

4.3. Consider the function $f(x,y) = (y-\alpha)\, xe^x$.

a) Knowing that $(0,1)$ is a critical point, determine α and classify this critical point.

b) Show that f is unbounded.

Solution: a) $\alpha = 1$. The critical point is a saddle point.

4.4. Let function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by $f(x,y) = 4\alpha(y-2)^2 + (\beta^2-1)(2x-2)^2$, where $\alpha \neq 0$, $\beta \neq 1$, $\beta \neq -1$. Show thate $(1,2)$ is the only critical point and classify it in terms of all possible values of α and β.

Solution: If $|\beta| < 1$ and $\alpha < 0$ then $(1,2)$ is a local maximum; if $|\beta| > 1$ and $\alpha > 0$ then $(1,2)$ is a local minimum; in all other cases it is a saddle point.

4.5. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by $f(x,y) = x^2 e^{y^3-3y}$.

a) Determine all critical points of function f.

b) Show that f attains its global minimum at points of the form $(0, b)$.

c) Justify that

(i) f is unbounded over $\mathbb{R}^2$;

(ii) f has a maximum and minimum over $B = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 9\}$.

Solution: a) Critical points: $(0, b)$ with $b \in \mathbb{R}$.

4.6. Determine the global extrema of f over the set M, where

$$a) f(x, y, z) = x - 2y + 2z, \qquad M = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

$$b) f(x, y) = 4x^2 + y^2, \qquad M = \{(x, y) \in \mathbb{R}^2 : 2x^2 + y^2 = 1\}$$

$$c) f(x, y) = xy, \qquad M = \left\{(x, y) \in \mathbb{R}^2 : \tfrac{x^2}{8} + \tfrac{y^2}{2} = 1\right\}$$

$$d) f(x, y, z) = x^2 + 2y - 2z, \qquad M = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 8\}$$

$$e) f(x, y) = x^2 + 2xy + y^2, \qquad M = \{(x, y) \in \mathbb{R}^2 : (x - 3)^2 + y^2 = 2\}$$

$$f) f(x, y, z) = 2x + 2y^2 + z^2, \qquad M = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 2\}$$

$$g) f(x, y, z) = e^{-x^2-y^2}, \qquad M = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

$$h) f(x, y) = 4xy - 2x^2 - 2y^2, \qquad M = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$

$$i) f(x, y) = x^2 + 2xy + y^2, \qquad M = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 8\}$$

Solution: a) max. = 3, min.=-3; b) max.= 2, min. = 1; c) max. = 2, min. = -2; d) max. =10, min. = -8; e) max. = 25, min. = 1; f) max. = $9/2$, min. = $-2\sqrt{2}$. g) max. = 1, min. = $1/e$; h) max. = 0, min. = -4; i) max. = 16, min.= 0

4.7. Determine the global extrema of f over the set A, where

$a) f(x,y,z) = x - 2y + 2z, \quad A = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$
(note: compare with a) from previous exercise)

$b) f(x,y) = 4x^2 + y^2, \quad A = \{(x,y) \in \mathbb{R}^2 : 2x^2 + y^2 \leq 1\}$
(note: compare with b) from previous exercise)

$c) f(x,y) = x^2 + 2xy + y^2, \quad A = \{(x,y) \in \mathbb{R}^2 : (x-3)^2 + y^2 \leq 2\}$
(note: compare with e) from previous exercise)

Solution: a) max. = 3, min. = -3; b) max. = 2, min. = é 0; c) max. = 25, min. = 1.

4.8. Determine the maximum and minimum distance to the origin of the points in the ellipse $5x^2 + 6xy + 5y^2 = 8$.

Solution: The maximum distance is 2 and the minimum distance is 1.

4.9. Solve the optimization problem $\min(x + 4y + 3z)$ subject to the condition $x^2 + 2y^2 + \frac{1}{3}z^2 = b$, $(b > 0)$.

Solution: The minimum value is $-6\sqrt{b}$, attained at $(-\frac{\sqrt{b}}{6}, -\frac{\sqrt{b}}{3}, -\frac{3\sqrt{b}}{2})$.

4.10. Determine the point in the ellipse $x^2 + 2xy + 2y^2 = 2$ with smallest x coordinate.

Solution: $(-2, 1)$.

4.11. determine teh global extrema of $f(x,y) = e^{x^2+y^2+z^2}$ over the set $\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = 1,\ z = 2 - y\}$.

Solution: The maximum value is e^{10}, attained at $(0,-1,3)$; the minimum is e^2, attained at $(0,1,1)$.

5 Multiple Integrals

5.1. Compute the following integrals.

a) $\int_0^1 \int_{-1}^1 \int_1^2 x\, dxdydz$ b) $\int_{-1}^1 \int_0^1 ye^{xy} dxdy$ c) $\int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} e^{-x-y-z} dxdydz$

d) $\int_0^5 \int_0^{+\infty} (x^2 e^{-2yx} + 3ye^{-y^2}) dydx$ e) $\int_1^2 \int_1^2 (1 + x + \frac{y}{2}) dxdy$ f) $\int_0^1 \int_0^1 \int_0^{\frac{\pi}{2}} y \cos x\, dxdydz.$

Solution: a) 3 b) $e - \frac{1}{e} - 2$ c) 1 d) $\frac{55}{4}$ e) $\frac{13}{4}$ f) $\frac{1}{2}$

5.2. Compute $\iint_A f(x,y)dxdy$, where

$a) f(x,y) = \frac{2x}{y^6}$, $A = \{(x,y) \in \mathbb{R}^2 : 1 \leq y \leq 3,\ 0 \leq x \leq y^4\}$

$b) f(x,y) = e^{\frac{y}{x}}$, $A = \{(x,y) \in \mathbb{R}^2 : 1 \leq x \leq 2,\ x \leq y \leq x^3\}$

$c) f(x,y) = x^2 y^5$, $A = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1,\ x^2 \leq y \leq 1\}$

$d) f(x,y) = ye^x + x^2 y$, $A = \{(x,y) \in \mathbb{R}^2 : 0 \leq y \leq 1,\ 0 \leq x \leq y^2\}$

$e) f(x,y) = xe^{-xy}$, $A = \{(x,y) \in \mathbb{R}^2 : 1 \leq x \leq 2,\ \frac{1}{x} \leq y < +\infty\}$

$f) f(x,y) = x^3 + 4y$, A is the region bounded by the lines $y = x^2$ and $y = 2x$.

Solution: a) $\frac{26}{3}$ b) $\frac{e^4}{2} - 2e$ c) $\frac{2}{45}$ d) $\frac{e}{2} - \frac{23}{24}$ e) $\frac{1}{e}$ f) $\frac{32}{3}$

5.3. Let $A = \{(x,y) \in \mathbb{R}^2 : y \leq 1 - x,\ y \leq 1 + x,\ y \geq 0\}$, and compute

a) $\iint_A (x-1)y\, dxdy$ b) $\iint_A (y - 2y^2)e^{xy}\, dxdy$ c) $\iint_A (x+y)\, dxdy.$

Solution: a) $-\frac{1}{3}$ b) 0 c) $\frac{1}{3}$

5.4. Compute $\int_{-1}^1 \int_{-1}^1 f(x,y)dxdy$, with $f(x,y) = \begin{cases} xy & x \geq 0 \text{ e } y \geq 0 \\ 1 - x - y, & \text{other } (x,y) \end{cases}$.

Solution: $\frac{17}{4}$

5.5. Using a double integral compute the area of set A, where

a) $A = \{(x,y) \in \mathbb{R}^2 : -1 \leq x \leq 1 \ \wedge \ x^2 \leq y \leq 1\}$;

b) $A = \{(x,y) \in \mathbb{R}^2 : 1 - x \leq y \leq 1 \ \wedge \ x \leq 1\}$;

c) $A = \{(x,y) \in \mathbb{R}^2 : x^2 \leq y \leq 2 - x^2\}$;

d) $A = \{(x,y) \in \mathbb{R}^2 : x + y + 2 \geq 0 \ \wedge \ x + y^2 \leq 0\}$;

e) $A = \{(x,y) \in \mathbb{R}^2 : y \leq x^2 \ \wedge \ y - x \geq 0 \ \wedge \ 2y - x \leq 3 \ \wedge \ x \geq 0\}$;

f) $A = \{(x,y) \in \mathbb{R}^2 : x \geq 0 \ \wedge \ x^2 + y^2 \leq 1\}$;

g) $A = \{(x,y) \in \mathbb{R}^2 : y^2 + x \leq 2 \ \wedge \ x - y \geq 0\}$;

h) $A = \left\{(x,y) \in \mathbb{R}^2 : y^2 \leq x \ \wedge \ x \leq \frac{y^2}{4} + 3\right\}$;

Solution: a) $\frac{4}{3}$ b) $\frac{1}{2}$ c) $\frac{8}{3}$ d) $\frac{9}{2}$ e) $\frac{35}{48}$ f) $\frac{\pi}{2}$ g) $\frac{9}{2}$ h) 8

5.6. Compute $\int_0^4 \int_{2x}^8 \sin(y^2) dy dx$ (suggestion: Draw the integration region and invert the integration order)

Solution: $\frac{1-\cos 64}{4}$

5.7. Compute $\int \int_A g(x,y) dx dy$, where $A = [0,2] \times [0,4]$ e $g(x,y) = \begin{cases} x - y, & x \leq y \leq 2x \\ 0, & \text{otherwise} \end{cases}$.

Solution: $-\frac{4}{3}$

6 Differential equations

6.1. Determine the general solution of the following differential equations with separable variables.

$a) y' = xy - x$ $\quad b)\ dx\, e^y = dy(x+1) - dx$ $\quad c) \frac{dy}{dx} + \frac{1+y^3}{xy^2(1+x^2)} = 0$

$d) \sqrt{1-x^2}dy - \sqrt{1-y^2}dx = 0$ $\quad e) e^{x^4} yy' = x^3\left(9+y^4\right)$ $\quad f) e^y(4+x^2)y' = x\left(2+e^y\right)$

$g) e^{3x}dy + \left(4+y^2\right)dx = 0$ $\quad h) 4xe^y dx + \left(x^4+4\right)dy = 0$

Solution: a) $y(x) = 1 + e^{\frac{1}{2}x^2}C$ b) $\ln\left(e^{y(x)}+1\right) - y(x) + \ln(x+1) = C$ c) $\frac{1}{3}\ln|1+y^3| + \ln|x| - \frac{1}{2}\ln(1+x^2) = C$ d) $\arcsin(y(x)) - \arcsin x = C$ e) $\frac{1}{6}\arctan\left(\frac{1}{3}y^2(x)\right) + \frac{1}{4}e^{-x^4} = C$ f) $\ln\left(2+e^{y(x)}\right) - \frac{1}{2}\ln\left(4+x^2\right) = C$ g) $\frac{1}{2}\arctan\left(\frac{1}{2}y(x)\right) - \frac{1}{3}e^{-3x} = C$ h) $-e^{-y(x)} + \arctan\frac{1}{2}x^2 = C$

6.2. Solve the following initial value problems.

a) $y' + 4y = 0, \quad y(0) = 6$ $\quad$ b) $\dfrac{dy}{dt} + y\sin t = 0, \quad y(\pi/3) = 3/2$

c) $(1+x^2)y' + y = 0, \quad y(1) = 1$ $\quad$ d) $2y' + 4xy = 4x, \quad y(0) = -2$

e) $y' + y\sin x = \sin x\cos x, \quad y(\frac{\pi}{2}) = 0$

Solution: a) $y(x) = 6e^{-4x}$ b) $y(t) = \frac{3}{2}e^{\cos t - \frac{1}{2}}$ c) $y(x) = e^{\frac{\pi}{4} - \arctan(x)}$ d) $y(x) = 1 - 3e^{-x^2}$ e) $y(x) = \cos x + 1 - e^{\cos x}$.

6.3. show that any homogeneous first order linear differential equation can be written as a

6.4. Determine de general solution of the following differential equations.

$a) y'' - 7y' + 12y = 0$ $\quad b) y'' + 4y = 0$ $\quad c) y'' - 4y' + 4y = 0$

$d) y'' + 2y' + 10y = 0$ $\quad e) y'' + y' - 6y = 8$ $\quad f) y'' + 3y' + 2y = e^{5x}$

$g) y'' - y = \sin x$ $\quad h) y'' - y = e^{-x}$ $\quad i) y'' - 6y = 36(x-1)$

$j) y'' - 9y = 9x^2$ $\quad k) y'' + 3y' + 2y = \sin x$ $\quad l) y'' + 3y' + 2y = e^{-x}$

$m) y'' - 4y' + 4y = 6e^{2x}$

Solution: a) $y(x) = C_1e^{3x} + C_2e^{4x}$ b) $y(x) = C_1\cos 2x + C_2\sin 2x$ c) $y(x) = (C_1 + C_2x)e^{2x}$ d) $y(x) = (C_1\cos 3x + C_2\sin 3x)e^{-x}$ e) $y(x) = C_1e^{-3x} + C_2e^{2x} - \frac{4}{3}$ f) $y(x) = C_1e^{-2x} + C_2e^{-x} + \frac{1}{42}e^{5x}$ g) $y(x) = C_1e^{-x} + C_2e^{x} - \frac{1}{2}\sin x$ h) $y(x) = C_1e^{x} + C_2e^{-x} - \frac{1}{2}xe^{-x}$ i) $y(x) = C_1e^{\sqrt{6}x} + C_2e^{-\sqrt{6}x} - 6x + 6$ j) $y(x) = C_1e^{3x} + C_2e^{-3x} - x^2 - \frac{2}{9}$ k) $y(x) = C_1e^{-2x} + C_2e^{-x} - \frac{3}{10}\cos x + \frac{1}{10}\sin x$ l) $y(x) = C_1e^{-2x} + C_2e^{-x} + xe^{-x}$ m) $y(x) = C_1e^{2x} + C_2xe^{2x} + 3x^2e^{2x}$.

6.5. Solve the following initial value problems.

$a)\begin{cases} y'' + y' - 2y = 0 \\ y(0) = -1, y'(0) = 1 \end{cases}$ $b)\begin{cases} y'' + 2y' + 5y = 0 \\ y(0) = 0, y'(0) = 1 \end{cases}$ $c)\begin{cases} y'' + 2y' + y = x^2 \\ y(0) = 0,\ y'(0) = 1 \end{cases}$

$d)\begin{cases} y'' + 4y = 4x + 1 \\ y(\frac{\pi}{2}) = 0,\ y'(\frac{\pi}{2}) = 0 \end{cases}$ $e)\begin{cases} 9y'' + y = 0 \\ y\left(\frac{3}{2}\pi\right) = 2,\ y'\left(\frac{3}{2}\pi\right) = 0 \end{cases}$ $f)\begin{cases} 2y'' - 4y' + 2y = 0 \\ y(0) = -1, y'(0) = 1 \end{cases}$

$g)\begin{cases} y'' - 2y' + 10y = 10x^2 \\ y(0) = 0, y'(0) = 3 \end{cases}$

Solution: a) $y(x) = -\frac{2}{3}e^{-2x} - \frac{1}{3}e^{x}$ b) $y(x) = \frac{1}{2}\sin(2x)e^{-x}$ c) $y(x) = -(6 + x)e^{-x} + x^2 - 4x + 6$ d) $y(x) = \frac{2\pi+1}{4}\cos 2x + \frac{1}{2}\sin 2x + x + \frac{1}{4}$ e) $y(x) = 2\sin(\frac{x}{3})$ f) $y(x) = (-1 + 2x)e^{x}$ g) $y(x) = e^{x}\left(\frac{3}{25}\cos 3x + \frac{62}{75}\sin 3x\right) + x^2 + \frac{2}{5}x - \frac{3}{25}$.

6.6. Solve the following boundary value problems.

$a)\begin{cases} y'' - 6y' + 9y = e^{3x} \\ y(0) = 0,\ y(1) = 0 \end{cases}$ $b)\begin{cases} y'' + 4y = 0 \\ y(0) = 0,\ y(\pi) = 0 \end{cases}$ $c)\begin{cases} y'' - 10y' + 25y = 50 \\ y(0) = 0,\ y(2) = 2. \end{cases}$

$d)\begin{cases} y'' - 8y' + 16y = 0 \\ y(0) = 1,\ y(1) = e^4. \end{cases}$

Solution: a) $y(x) = (-\frac{1}{2}x + \frac{1}{2}x^2)e^{3x}$ b) $y(x) = C\sin 2x$ c) $y(x) = (-2 + x)e^{5x} + 2$ d) $y(x) = e^{4x}$.

6.7. Knowing that $y = e^{2x}$ is a solution of the differential equation

$$y'' - \alpha y' + 10y = 0, \quad \alpha \in \mathbb{R},$$

determine α and the general equation of this equation.

Solution: $\alpha = 7$; $y(x) = C_1 e^{2x} + C_2 e^{5x}$.

6.8. Knowing that $y(x) = xe^{2x}$ is a solution of the differential equation $2y'' - \alpha y' + 8y = 0$, with $\alpha \in \mathbb{R}$, solve the boundary value problem $\begin{cases} 2y'' - \alpha y' + 8y = 16 \\ y(0) = 1;\ y(1) = 2 \end{cases}$.

Solution: $y(x) = -e^{2x} + xe^{2x} + 2$.

6.9. solve the following problem with periodic conditions.

$$\begin{cases} y'' + 4y = 4, \\ y(0) = y\left(\frac{\pi}{2}\right), y'(0) = y'\left(\frac{\pi}{2}\right) \end{cases}.$$

Solution: $y(x) = 1$.

6.10. Determine the general solution of the following differential equations.

$a) y' + y^2 \sin x = 0$ $\quad b) yy' + x = 0$ $\quad c) y'' - 2y' = 0$

$d) y'y - x(2y^2 + 1)e^{x^2} = 0$ $\quad e) \frac{dy}{dx} \cos y = -x \frac{\sin y}{1+x^2}$ $\quad f) y' + 6yx^5 - x^5 = 0$

Solution: a) $y^{-1}(x) = -\cos x + C$ b) $y^2(x) = -x^2 + C$ c) $y(x) = C_1 + C_2 e^{2x}$ d) $\frac{1}{4}\ln(2y^2+1) - \frac{1}{2}e^{x^2} = C$ e) $\ln|\sin y| + \frac{1}{2}\ln(1+x^2) = C$ f) $y(x) = \frac{1}{6} + Ce^{-x^6}$.

6.11. Determine the values of a and b for which e^{2x} and e^{-2x} are solutions to the differential equation $y'' + ay' + by = 0$. For those vales of a and b, compute the general solution to the differential equation.

Solution: $a = 0$ e $b = -4$; $y_h(x) = Ae^{2x} + Be^{-2x}, A, B \in \mathbb{R}$.

6.12. [Malthus populational growth]

(a) Compute the time evolution of a population level $y(t)$ knowing that: i) at each time t, the growth rate of the population , $\frac{dy/y}{dt}$, is equal to r (with $r > 0$); ii) at time $t = 0$ there are y_0 (millions of individuals).

(b) Estimate the value of the Portuguese population in the year 2020, knowing that in the year 2013 (by hypothesis $t = 0$) there is record of $y = 10.457$ (million individuals) and that $r = -0.00476$.

(c) Comment on the Malthusian hypothesis, by studying $\lim_{t\to\infty} y(t)$.

Solution:

a) $y(t) = y_0 e^{rt}$; b) $y(8) \approx 10.0663$; c) $+\infty$ if $r > 0$, y_0 if $r = 0$ and 0 if $r < 0$.

6.13. [Populational growth according to Verhulst]

(a) Compute the time evolution of a population level $y(t)$ knowing that: i) at each time t, the rate of growth of the population, $\frac{dy/y}{dt}$, is equal to r minus ay (r: natural growth rate; a: migratory or death rate; with $r > 0$ e $a > 0$); ii) at time $t = 0$ there is a record of y_0 million individuals.

(b) b)Estimate the value of the Portuguese population in the year 2020, knowing that in the year 2013 (by hypothesis $t = 0$) there is a record of $y = 10.457$ million individuals and that $r = -0.00229$, $a = 0.00033$.

c) Study $\lim_{t\to\infty} y(t)$.

6.14. [Domar's growth model]

Consider an economical model where i) the aggregated demand y_d, varies on time according to the equation $\frac{dy_d}{dt} = \frac{dI}{dt}\frac{1}{s}$, where $I = I(t)$ is the investment and s the marginal propensity to saving ($1/s$ is Keynesian multiplier); ii) the productive capacity ($y_c = \rho K$ - note: the productive capacity depends only on the stock of capital) follows the differential equation $\frac{dy_c}{dt} = \rho\frac{dK}{dt}$, where $K = K(t)$ is the *stock* of capital of the economy (naturally, $\frac{dK}{dt} = I(t)$). Obtain the time trajectory of the investment, $I(t)$, satisfying the equilibrium of Domar's model - the variation of the aggregated demand = variation of the productive capacity.

6.15. Consider the demand and supply functions of a given commodity, $Q_d = a - bP$; $Q_s = -c + dP$.

(a) Determine the time evolution of the price level $P(t)$ knowing that a each time t, the variation rate of $P(t)$ is proportional to the excess demand, i.e. $\dfrac{dP}{dt} = \alpha(Q_d - Q_s)$, and that $P(0) = P_0 \neq P_e = (a+c)/(b+d)$ (equilibrium price).

(b) Verify under which conditions we have $\lim_{t\to\infty} P(t) = P_e$.