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Macroeconomics II

Lecture 07 (March 2020)

Solow model
Diagram and functioning of the model

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Theoretical Lecture 7

The Solow model of economic growth

- **the hypotheses of the Solow model;**
- **the Cobb-Douglas production function (to remind);**
- **the production function *per capita*;**
- **the Solow model;**
- **equilibrium in the Solow model: the *steady state.***

Reading:

Jones & Vollrath (2013), *Introduction to Economic Growth*, Norton, chap 2, pp. 20 – 53

Complementary reading

Solow, R. (1956), A Contribution to the Theory of Economic Growth. The Quarterly Journal of Economics. Vol. 70, No 1 (Feb. 1956), pp. 65-94

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Robert M. Solow
(1924-)

American economist, born in Brooklyn, New York, educated at Harvard (B.A. 1947, M.A. 1949, Ph.D. 1951). He is best known for his path-breaking work on capital and growth. Since 1950, he has taught at MIT; he has never had or wanted any other job. He was president of the American Economic Association in 1979 and was awarded the Nobel Prize for his work on economic growth in 1987. Recently, he wrote: "Maybe the main function of economics in general is not, as we usually think, the systematic building of theories and models, or their empirical estimation. Maybe we are intellectual sanitation workers. The world is full of nonsense ... Maybe the higher function of economics is to hold out against nonsense, ... All those theories and models we invent and teach are just nature's way of making people who know nonsense when they see it."

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Robert Solow, Nobel Prize Lecture, December 1987

"Growth theory did not begin with my articles of 1956 and 1957, and it certainly did not end there. Maybe it began with *The Wealth of Nations*; and probably even **Adam Smith** had predecessors. More to the point, in the 1950s I was following a trail that had been marked out by Roy **Harrod** and by Evsey **Domar**, and also by Arthur **Lewis** in a slightly different context. Actually I was trying to track down and relieve **a certain discomfort that I felt with their work**. (…)"

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Neoclassical economists and the models of economic growth

Robert **Solow** (1956) and Trevor **Swan** (1956)

Criticisms to Harrod-Domar model by **Robert Solow**

•capital-output ratio as a constant is **not** a realistic hypothesis

•relevance of labour: it is implicit (complementary factors), but not explicit in the model

The response by Robert **Solow**: **new hypotheses**

•GDP depends on physical capital and labour, with decreasing returns

•Production function with substitutable factors

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(…) That was the spirit in which I began tinkering with the theory of economic growth, trying to improve on the Harrod-Domar model. I can not tell you why **I thought first about replacing the constant capital-output (and labour-output) ratio by a richer and more realistic representation of the technology"** **(Solow)**

production function

Harrod - Domar: Y = F(K,L) = min (AK, BL), complementary factors

Neoclassical models:. substitutable factors

Solow basic model: Y(t) = F(K(t), L(t))

with technology: Y(t) = F (K(t), A(t).L(t))

A and L enter multiplicatively. AL is the effective labour, and technology (exogenous) enters as *labour-augmenting* or *Harrod-neutral* technological progress;

equilibrium paths (*steady-state*): not mechanical equilibrium, but instead *equilibrium paths*, and, behind it, there are economic dynamics that support such *steady-state* equilibrium)

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The hypotheses of the Solow model

- the economy is represented by a production function with two substitutable production factors (capital, labour);
- the production function exhibits constant returns;
- the production factors have positive decreasing marginal productivities;
- domestic savings is a fixed proportion of GDP;
- the macroeconomic equilibrium in the closed economy is: I = S;
- there is no unemployment;
- the growth rate of active population is equal to the growth rate of total population;
- the economy is closed.

the basic Solow model

The model is explained by two equations:

•**production function**

•**capital accumulation equation**

A. The production function (basic: with no technical progress, yet …)

Cobb-Douglas (remind the properties: lecture 06)

$Y = F(K, L) = K^{\alpha} L^{1-\alpha}$, $0 < \alpha < 1$

α partial elasticity of Y relative to K (math. interpretation)

α economic interpretation?

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per capita production function

per capita means ***per worker***

$Y/L = F(K/L)$

since $y = Y/L$ and $k = K/L$, the production function Cobb-Douglas *per capita* is:

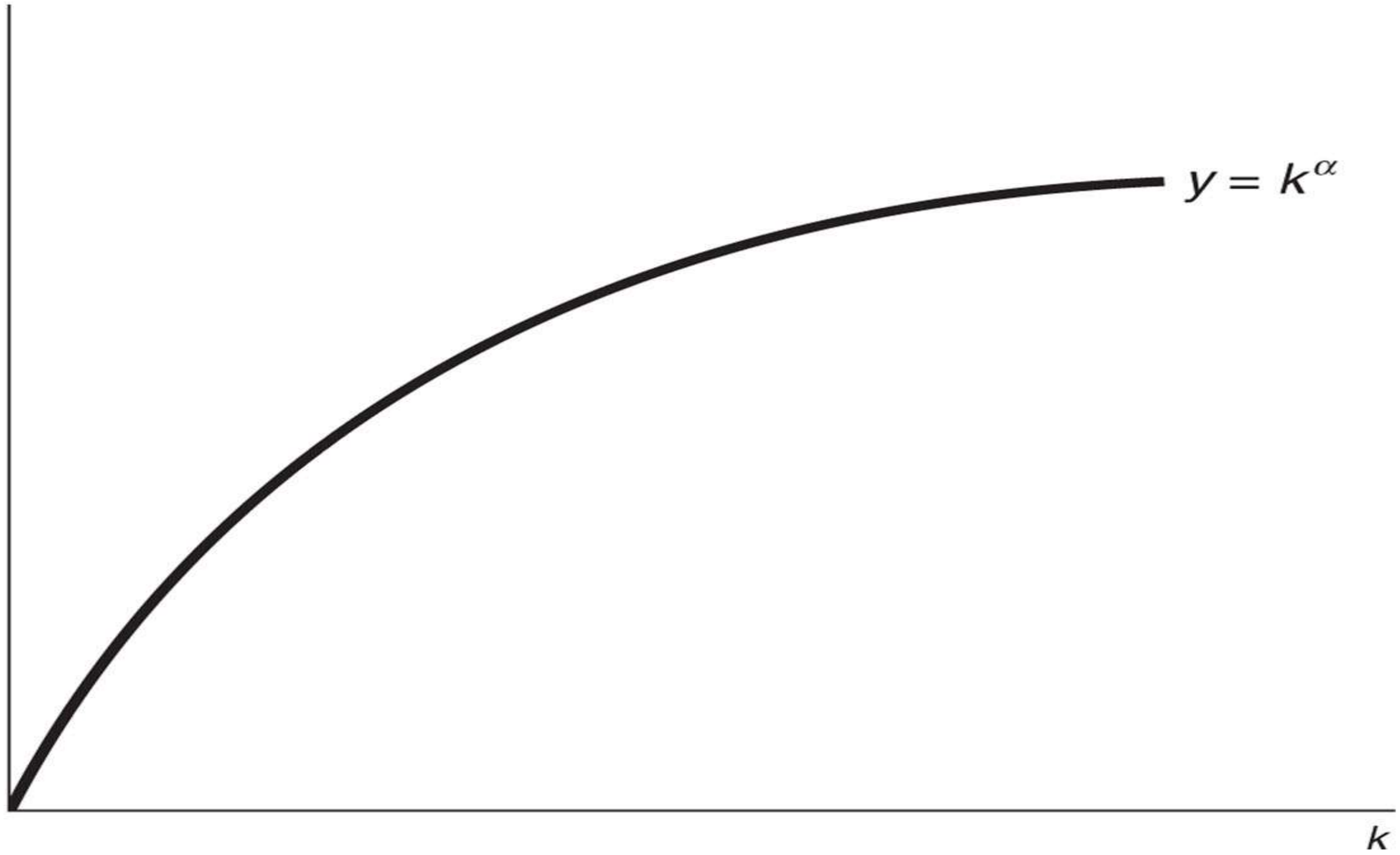
$y = k^{\alpha}$, $0 < \alpha < 1$

interpretation:

•if K/L is higher, the firms produce higher Y/L;

•for each additional unit of capital *per worker*, the increase of output *per worker* becomes lower;

FIGURE 2.1 A COBB-DOUGLAS PRODUCTION FUNCTION



B. the accumulation of capital equation

$K_{t+1} - K_t$ discrete

= $s.Y - \delta.K$

dK/dt continuous

$(dK/dt)/K = s.Y/K - \delta$ growth rate of the stock of capital

the accumulation of capital equation *per worker*

$k = K/L$

$\ln k = \ln K - \ln L$

$(dk/dt)/k = (dK/dt)/K - (dL/dt)/L$ growth rate of the stock of capital *per capita*

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growth of the active population

$L(t) = L_0 e^{nt}$

active population grows at the same rate as population

$(dL/dt)/L = n$

$(dk/dt)/k = (dK/dt)/K - (dL/dt)/L$

$= s.Y/K - n - \delta$

$= s.\ y/k - n - \delta$

$dk/dt = s.y - (n + \delta).k$

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Solow model

$y = k^{\alpha}$, $0 < \alpha < 1$ production function *per capita*

$dk/dt = s.y - (n + \delta).k$ stock of capital variation *per capita*

$$dk/dt = s.\ k^{\alpha} - (n + \delta).k$$

the model addresses the following question:

If an economy has, in some year, the stock of capital per worker $\mathbf{k_0}$, the growth rate of population is **n** and the stock of capital depreciates at the rate **δ**, ***what is the trend of output per worker in this economy***? How does this economy compares to another one with a different rate of investment?

$s.\,k^{\alpha}$ *actual* investment per capita (per worker)

$(n + \delta).k$ investment per capita (per worker) that is *required* to keep the stock of capital per capita (per worker) constant

when $s.k^{\alpha} > (n + \delta).k$, the economy increases the stock of capital per capita (worker); ***capital deepening*** of the economy;

when $s.k^{\alpha} = (n + \delta).k$, when k = k*, the stock of capital K is rising at the growth rate n>0 (the same as that of active population and that of population);
the stock of capital *per capita* (worker) is not growing (rate of growth = 0)
capital widening of the economy.

The notion of a steady state

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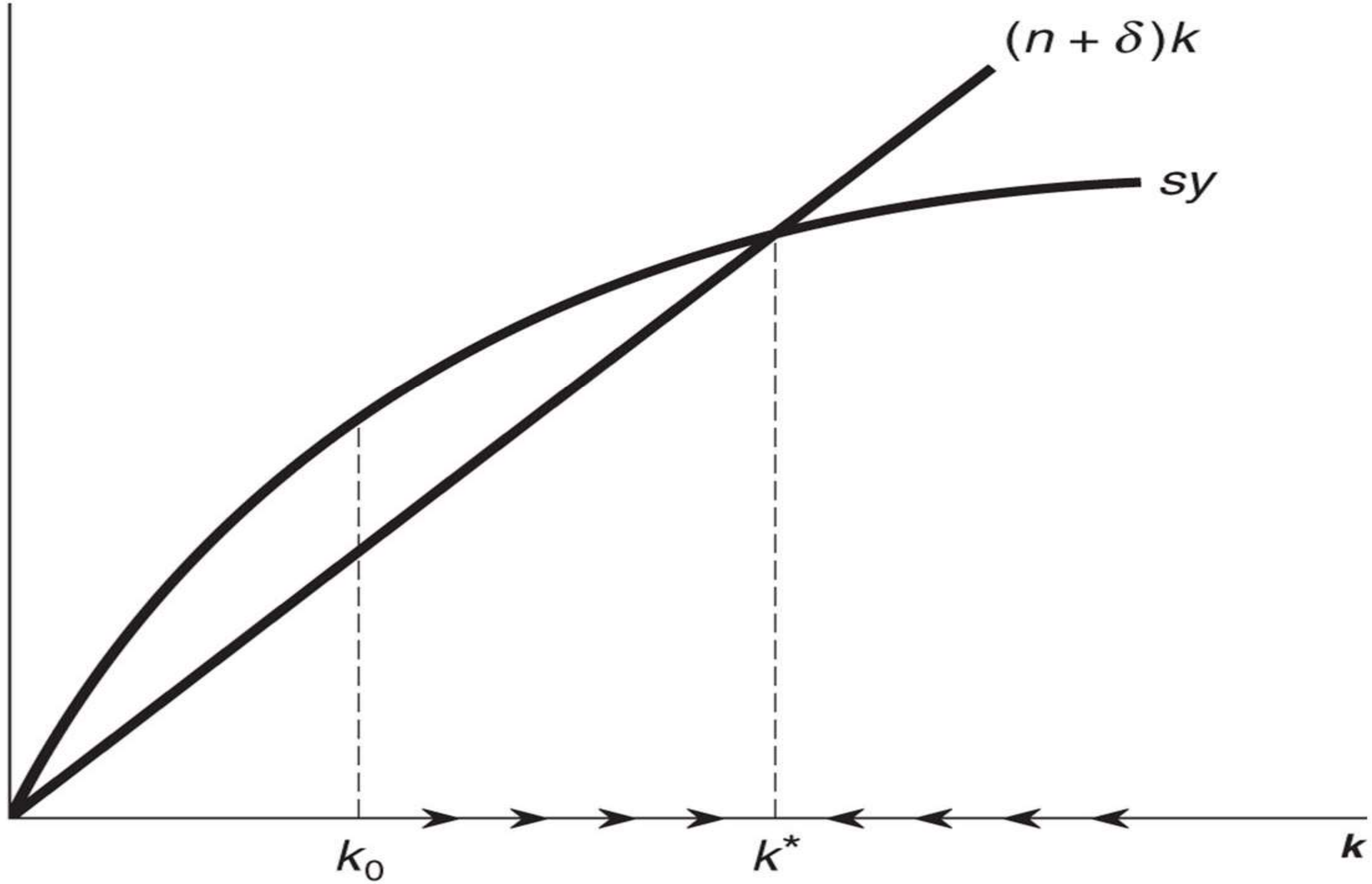
FIGURE 2.2 THE BASIC SOLOW DIAGRAM



Introduction to Economic Growth, 3rd Edition

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the dynamic equilibrium in the Solow model (*steady-state*)

$dk/dt = s.y - (n + \delta).k = 0$

in the steady state, k = K/L remains constant

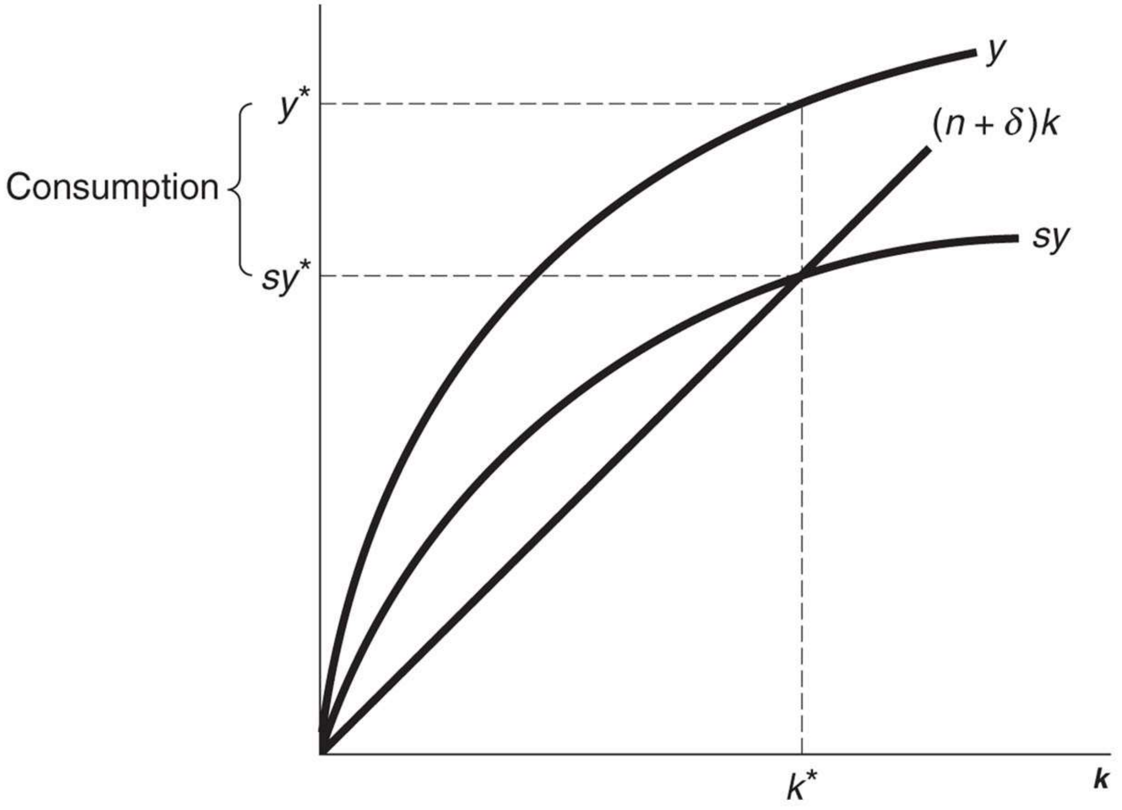
$s.y = (n + \delta).k$

K grows at the same rate as L

K grows at the rate n

capital widening

FIGURE 2.3 **THE SOLOW DIAGRAM AND THE PRODUCTION FUNCTION**



comparative statics

FIGURE 2.4 AN INCREASE IN THE INVESTMENT RATE

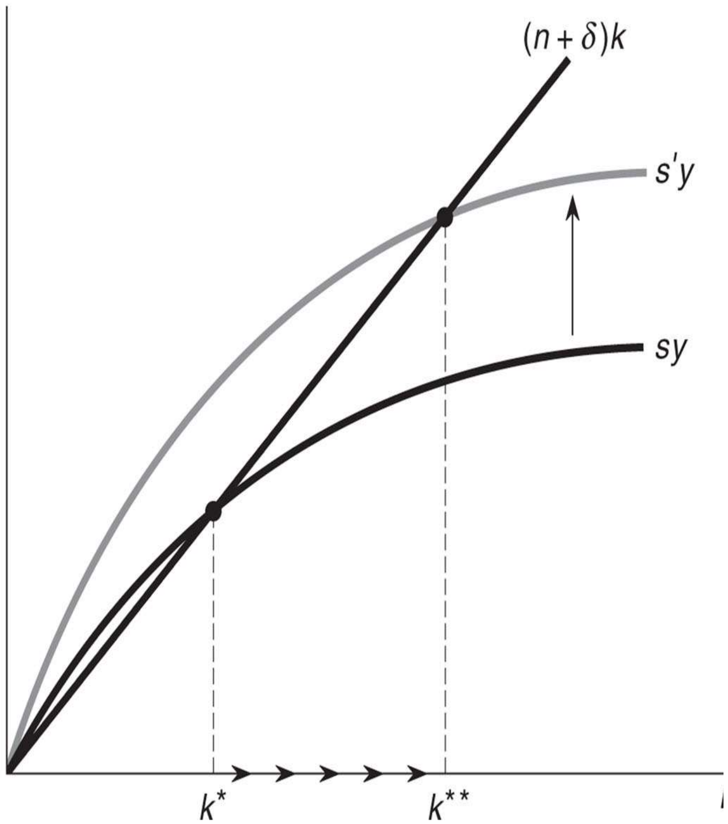
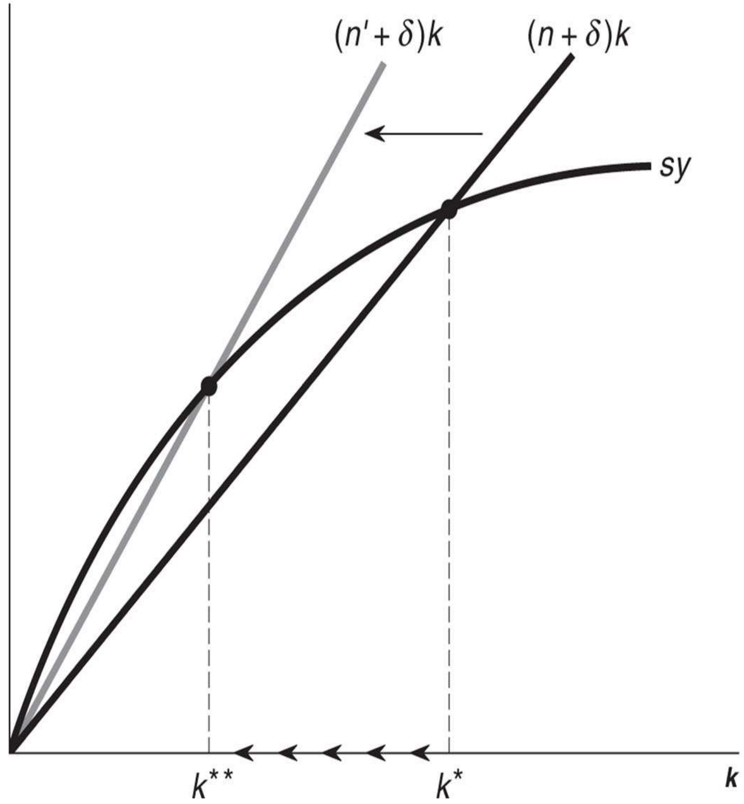


FIGURE 2.5 AN INCREASE IN POPULATION GROWTH



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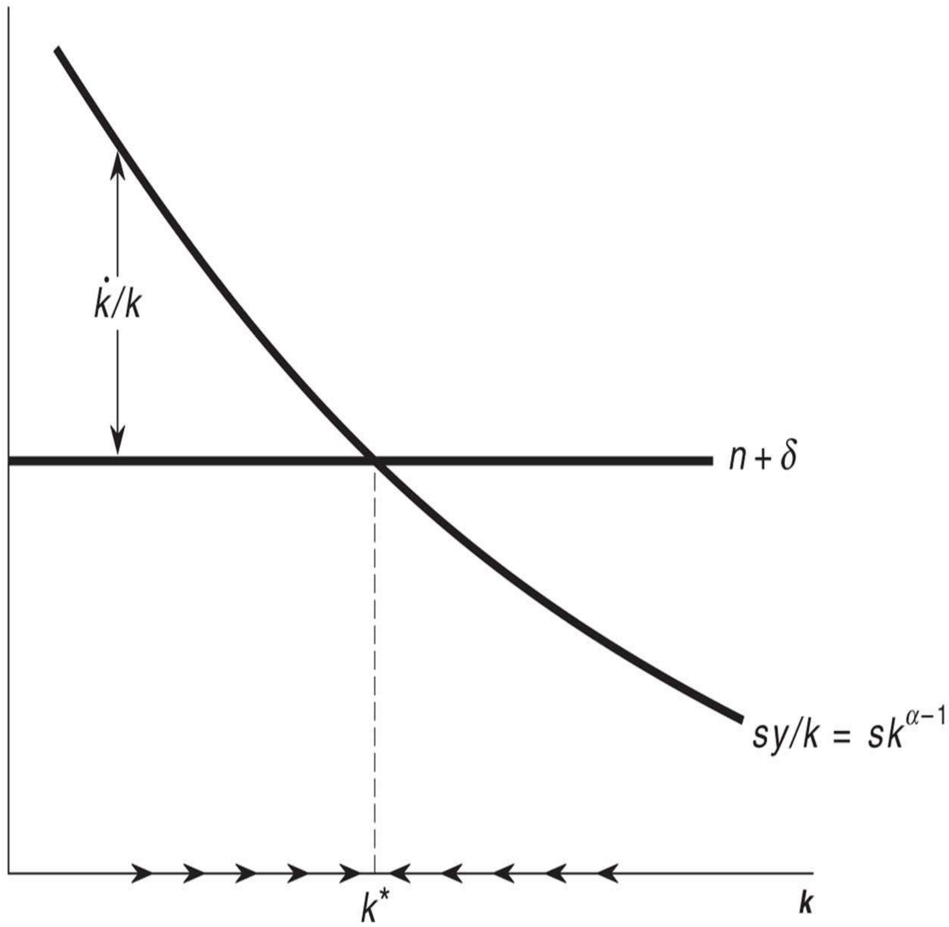
Solow model: the transition dynamics

FIGURE 2.8 TRANSITION DYNAMICS



$dk/dt = s.k^{\alpha} – (n + \delta).k$

↓

$(dk/dt)/k = s.k^{\alpha-1} – (n + \delta)$

decreasing growth rate
in the transition to *steady-state*

diminishing returns to
capital accumulation

$y = k^{\alpha}$

$0 < \alpha < 1$

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the steady-state: interpretation and empirical evidence

$s.k^{\alpha} = (n + \delta).k$

$k^* = (s/(n + \delta))^{1/(1-\alpha)}$

$y^* = (s/(n + \delta))^{\alpha/(1-\alpha)}$

interpretation:

•countries with higher <u>savings rate</u> are richer

•countries with higher <u>population growth rates</u> are poorer

Does the model describe reality?

FIGURE 2.6 GDP PER WORKER VERSUS THE INVESTMENT RATE

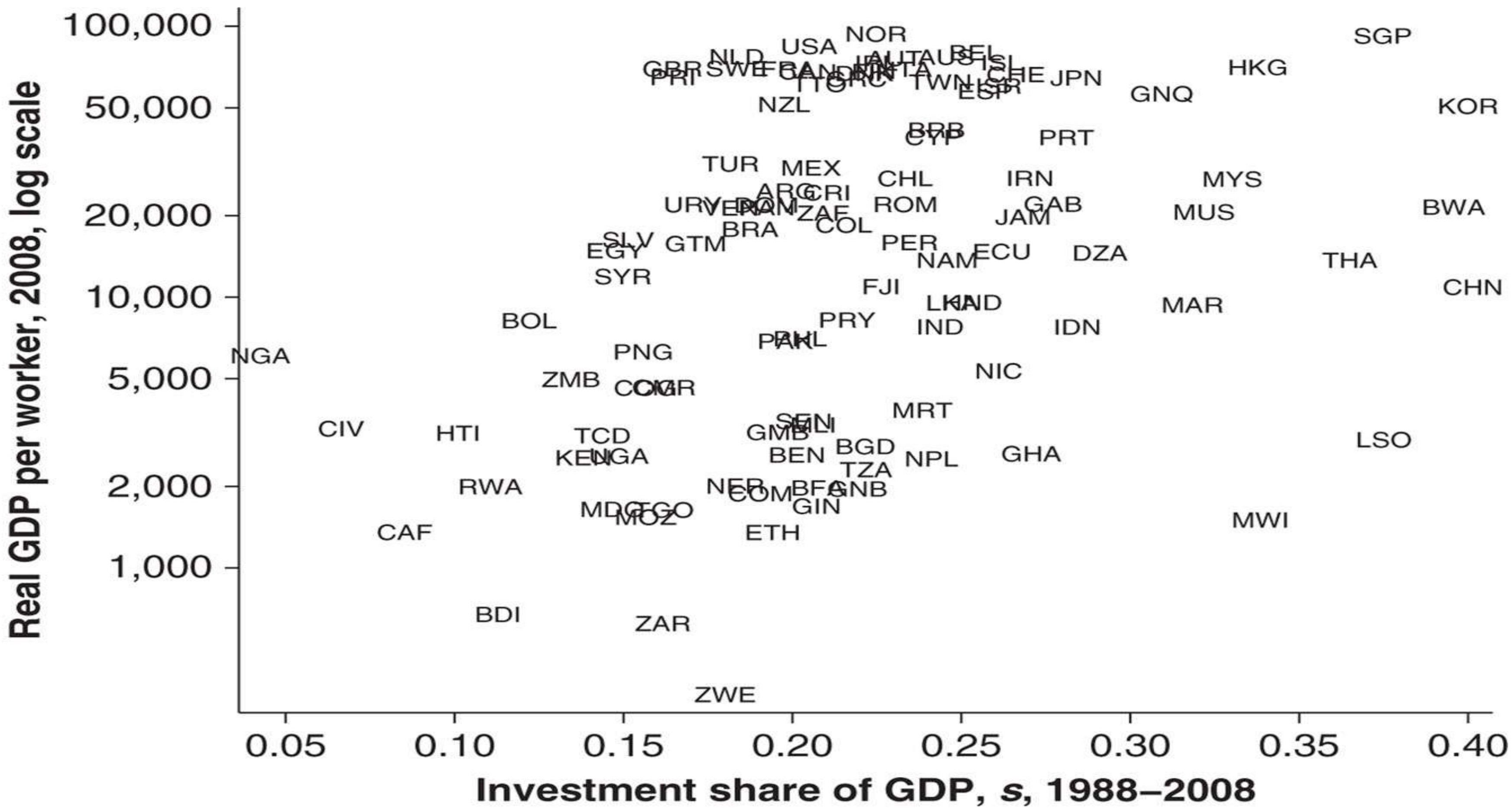

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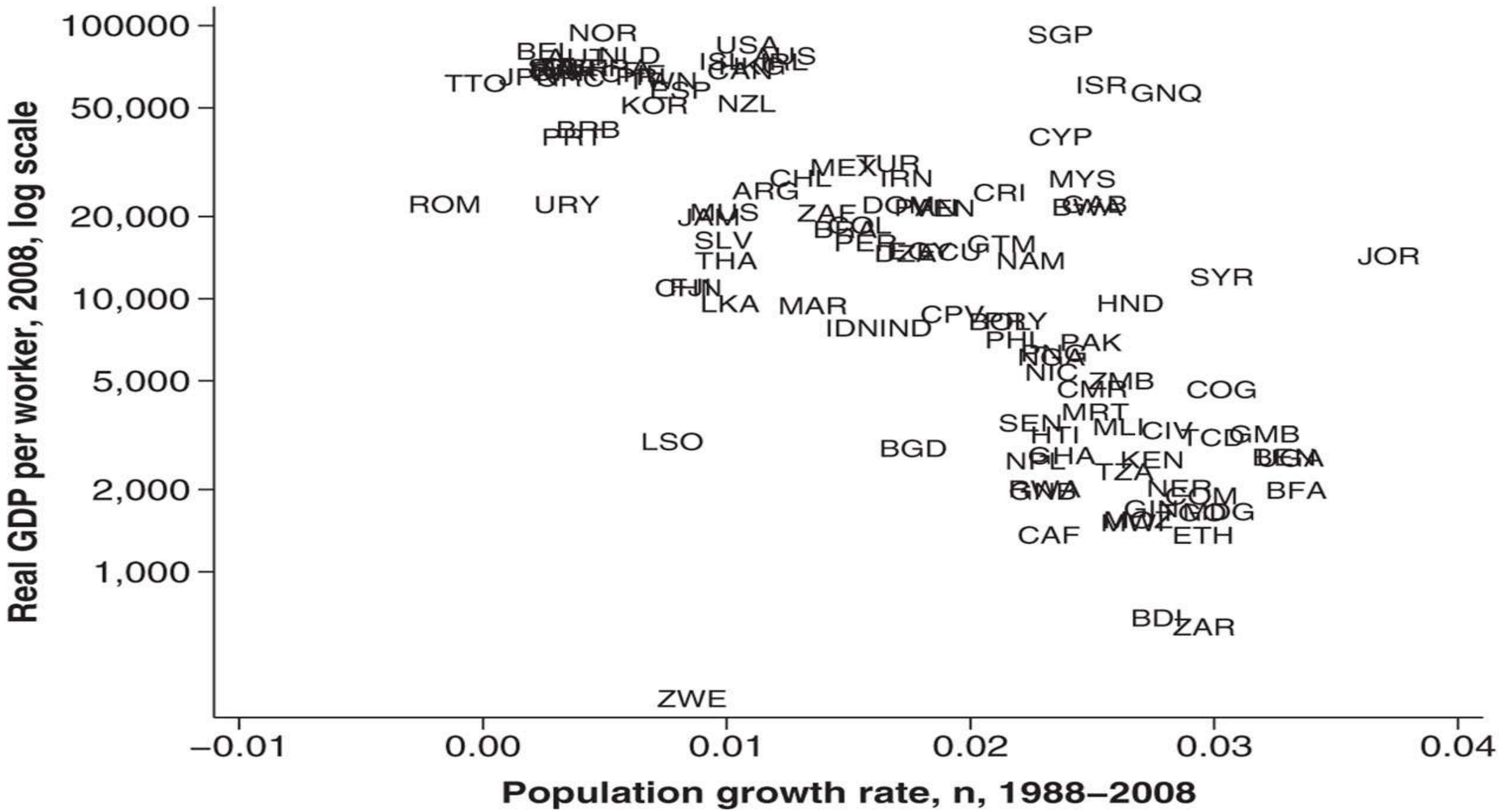
FIGURE 2.7 GDP PER WORKER VERSUS POPULATION GROWTH RATES



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Solow model with technological progress

$Y = F(K, AL)$

<u>labour augmenting</u> (or <u>Harrod-neutral</u>) technological progress

$A(t) = A_0 e^{gt}$; $r(A) = g$

g is the growth rate (<u>constant</u>) of the technology (like "<u>manna from heaven</u>")

$Y = K^{\alpha} (AL)^{1-\alpha}$ **=>** $\mathbf{y = Y/L = k^{\alpha} A^{1-\alpha}}$, $0 < \alpha < 1$

and, then (taking logs and derivatives):

$$\mathbf{r(y) = \alpha . r(k) + (1 - \alpha) . g}$$

capital accumulation

$dK/dt = s.Y - \delta.K$

$(dK/dt)/K = s.(Y/K) - \delta$

Q: what relation among the growth rates of K, Y, L and A in *steady-state*?

growth rate of K is constant ⇔ Y/K constant ⇔ y/k constant ⇔ r(y) = r(k)

r(y) = r(k) = g

in the **balanced growth path**, (*steady state*), y and k grow at the same rate as the technology

technological progress is the factor that explains the sustained growth of GDP per capita.

technological progress in the Solow model

$k = K/L$ → $k^{\#} = \mathbf{K/AL} = k/A$ capital-technology ratio

$y = Y/L$ → $y^{\#} = \mathbf{Y/AL} = y/A$ output-technology ratio

$$y^{\#} = k^{\#\alpha}$$

$$(dk^{\#}/dt)/k^{\#} = (dK/dt)/K - (dA/dt)/A - (dL/dt)/L = (s.Y - \delta K)/K - g - n$$

$$dk^{\#}/dt = ((s.Y - \delta.K)/K - g - n).k^{\#} = (s.Y/K - \delta - g - n)\ .\ k^{\#} =$$

$$= (s.(Y/AL)/(K/AL) - \delta - g - n).k^{\#} = (s.y^{\#}/k^{\#} - (\delta + g + n)).\ k^{\#}$$

$$dk^{\#}/dt = s.y^{\#} - (\delta + g + n).\ k^{\#}$$

Steady state

$dk^{\#}/dt = 0$

$k^{\#*} = (s/(n+g+\delta))^{1/(1-\alpha)}$

$k^{\#} < k^{\#*}$

investment is **higher** that the required to keep the capital-technology ratio, $k^{\#}$ constant

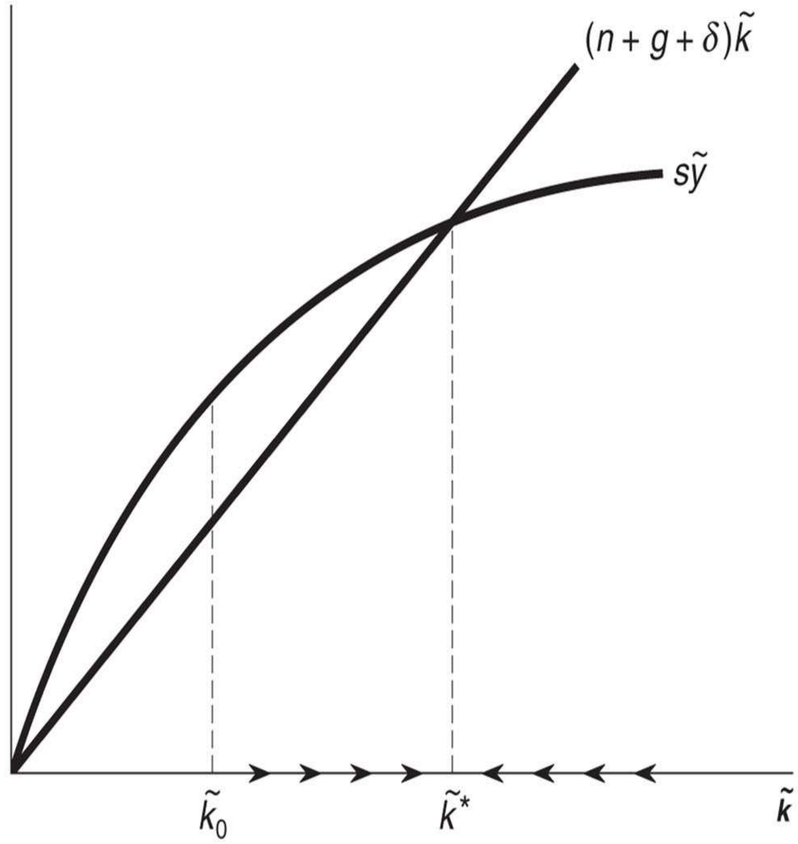
in *steady state* $r(k^{\#}) = 0$

Since $k^{\#} = k/A$, then:

$\mathbf{r(k) = r(y) = r(A) = g}$

(*balanced growth path*)

FIGURE 2.9 THE SOLOW DIAGRAM WITH TECHNOLOGICAL PROGRESS



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Conclusive remarks

In the ***steady state***:

$k^{\#*} = (s/(n+g+\delta))^{1/(1-\alpha)}$

$y^{\#*} = (s/(n+g+\delta))^{\alpha/(1-\alpha)}$

$y^* = A(t) . (s/(n+g+\delta))^{\alpha/(1-\alpha}$ (since $y^{\#} = y/A$) (remember: $y=Y/L$; $y^{\#} = Y/AL$)

the GDP per worker, y, in ***steady state*** (that is, in the *balanced growth path*) is determined by technology (**A(t**) and its rate of growth (**g**), by the investment rate (**s**) and by the population growth rate (**n**).

In ***steady state*** the **growth rate** of the GDP **per worker**, **y**, is equal to the growth rate of technology (**g**); the growth rate of GDP (Y) is equal to the growth rate of technology (g) plus the population growth rate (n).

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main conclusions/contributions of Solow model

each economy has characteristics that determine its ***steady state*** (when the economy grows at a rate such that the amount of capital per worker will not change over time)

the main characteristics of the economy that determine its *steady state* are:
investment (= saving) rate, rate of population growth, rate of technological progress

Solow model is a theory of income differences among countries
•countries may have different *steady states*;
•countries (even with the same steady state) may be not at their *steady state*: in this case, these economies may have different income levels;
•if an economy has an income level different from its *steady state* income level, then it moves into its *steady state* income level;

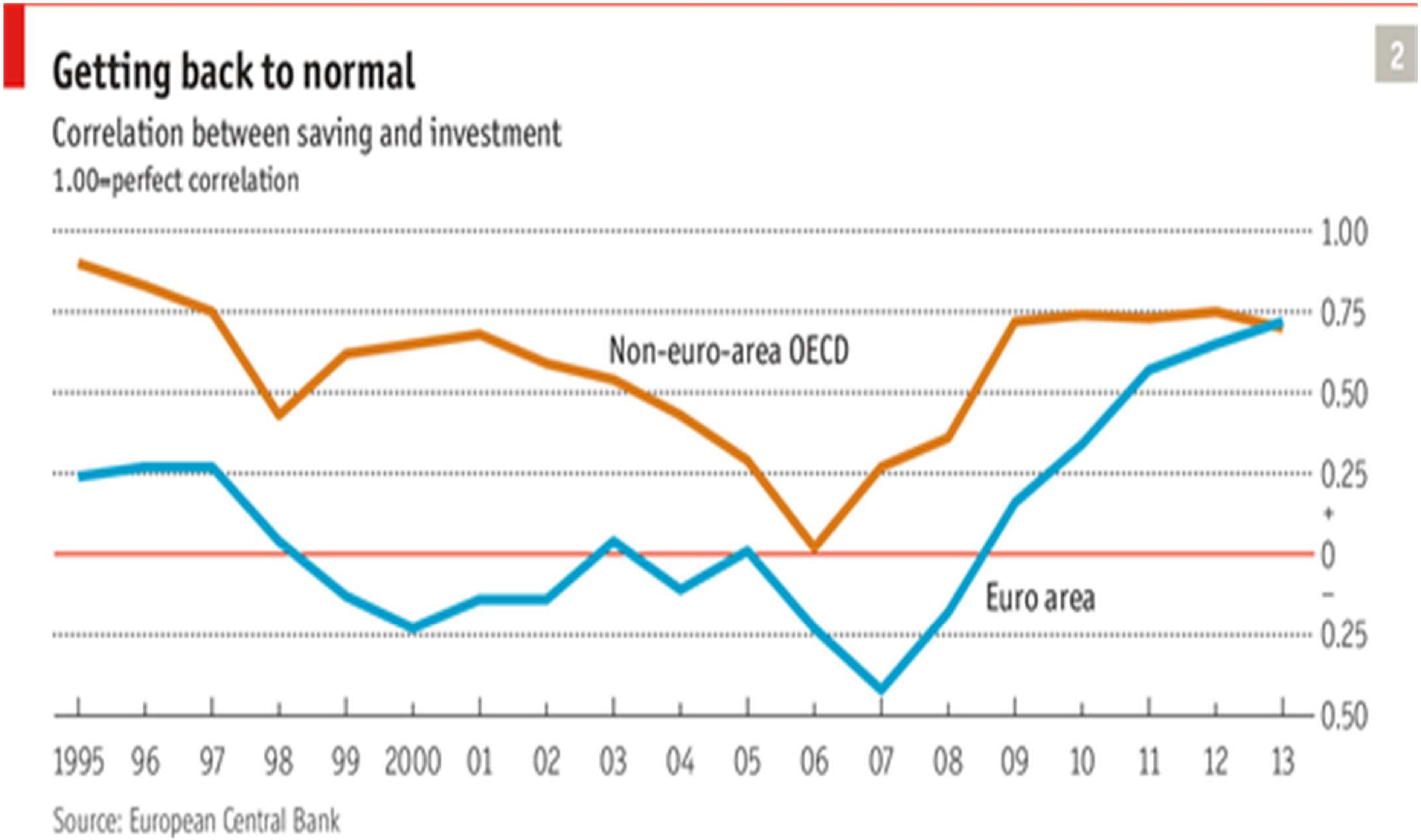
Solow model is a theory of relative growth rates
two "similar" countries with the same rate of investment have the same *steady state* income level, but may have different income levels (if they are not at their *steady state*) ; in this case, the poorer income level country (that more distant to its *steady state*) grows faster into its *steady state* income level: conditional convergence

Comparison of H-D with the Solow model

Harrod-Domar: output growth **adjusts** to the capital-output ratio (which is **exogenous**)

Solow: capital-output ratio is **endogenous**, and **adjusts** to the output growth, which is **exogenous** (determined by **exogenous** factors: population growth and technological progress)

Is is true that S=I?
in a real **open** economy it is not



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Complementary notes (annex to the lecture)

Question:

What is the effect of a permanent rise of the investment rate on:

- the *level* of GDP *per capita,* and on
- the *growth rate* of GDP *per capita*?

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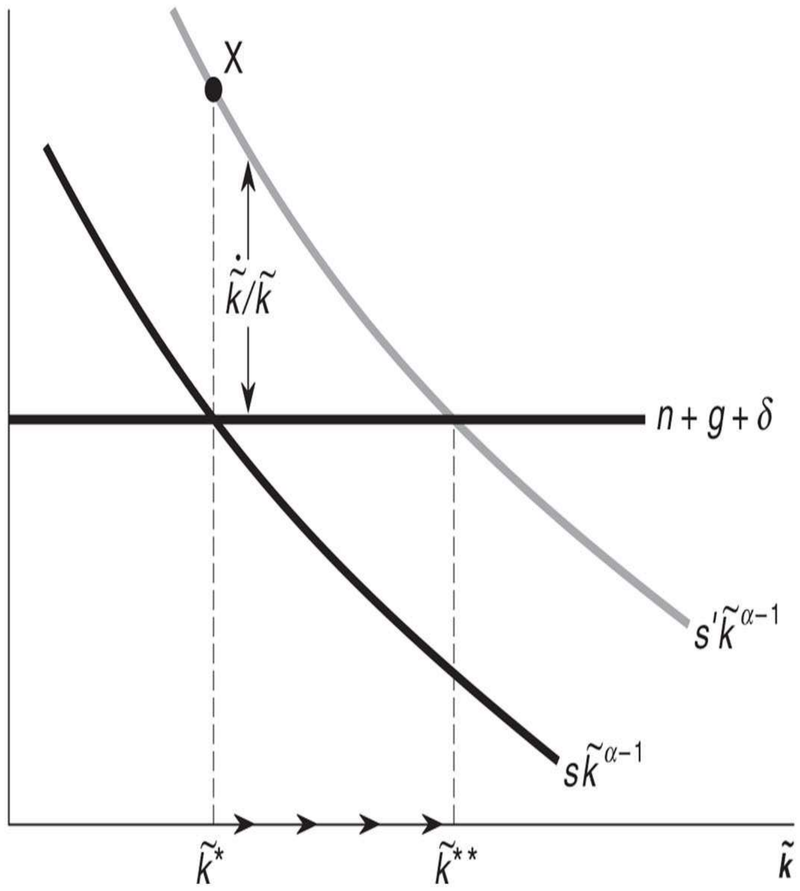
FIGURE 2.11 AN INCREASE IN THE INVESTMENT RATE: TRANSITION DYNAMICS



$(dk^{\#}/dt)/k^{\#} = s.k^{\#\alpha-1} - (n + g + \delta)$

decreasing growth rate
in the transition to *steady-state*

diminishing returns to
capital accumulation

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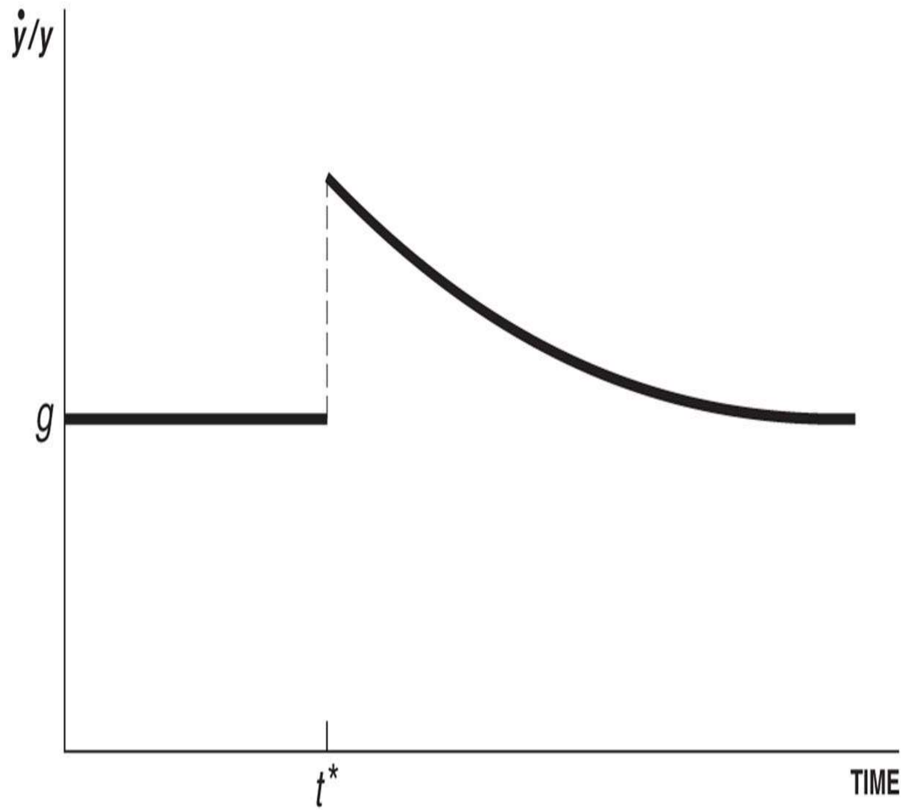
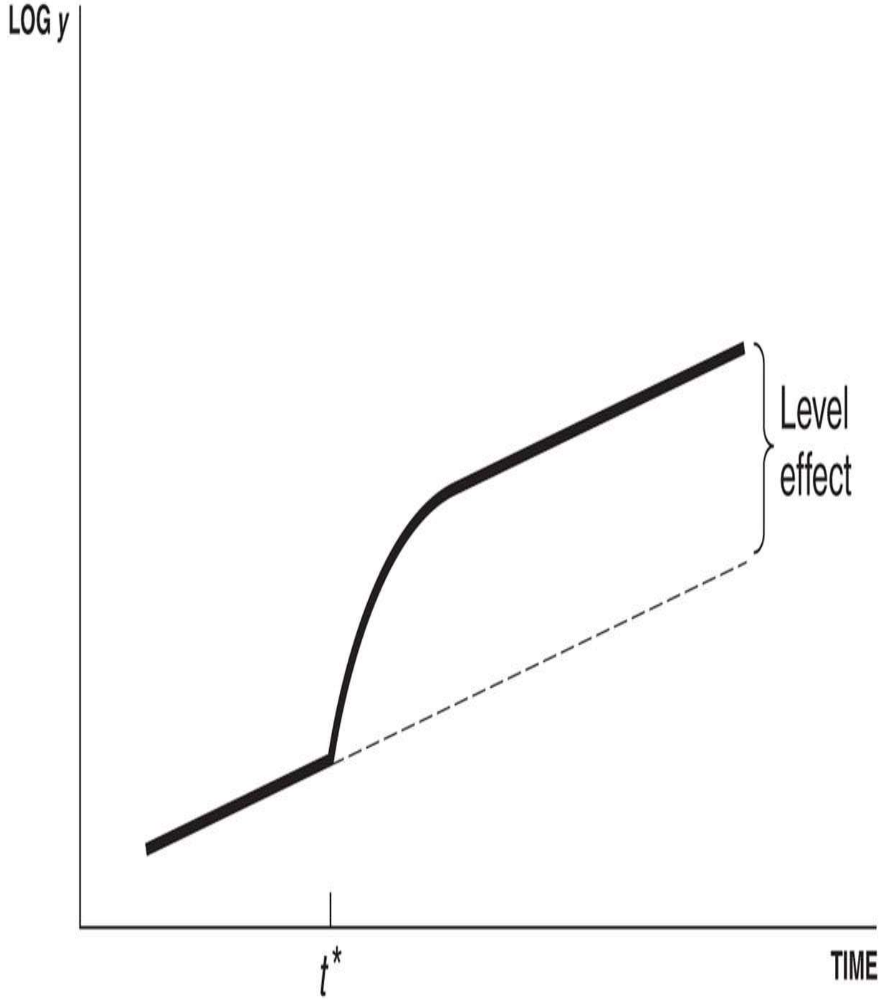
FIGURE 2.12 THE EFFECT OF AN INCREASE IN INVESTMENT ON GROWTH



the rise of the investment rate **has no effect** on the long-run **growth rate** of GDP *per capita*

FIGURE 2.13 THE EFFECT OF AN INCREASE IN INVESTMENT ON y



the rise of the investment rate

<u>has positive effect</u> on the long-run

level of GDP *per capita*

Comparing both effects

FIGURE 2.12 THE EFFECT OF AN INCREASE IN INVESTMENT ON GROWTH

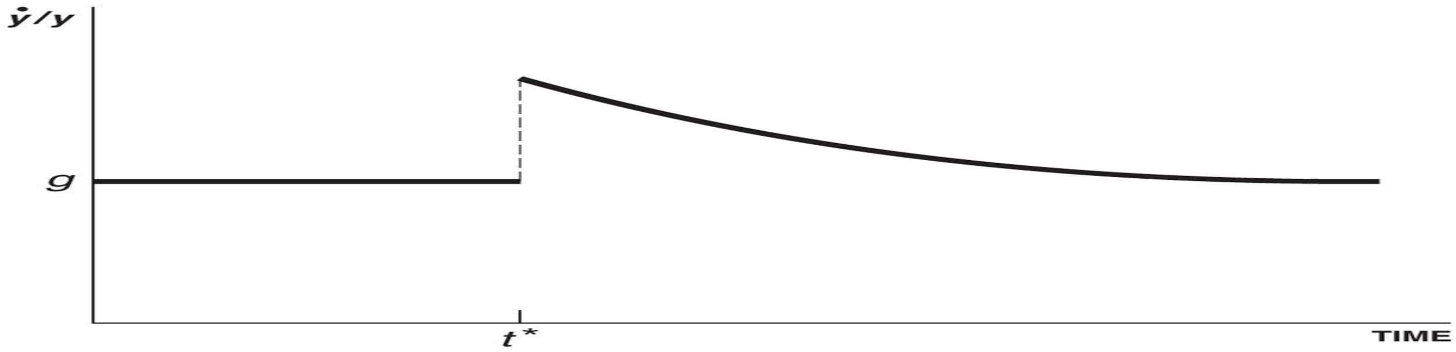
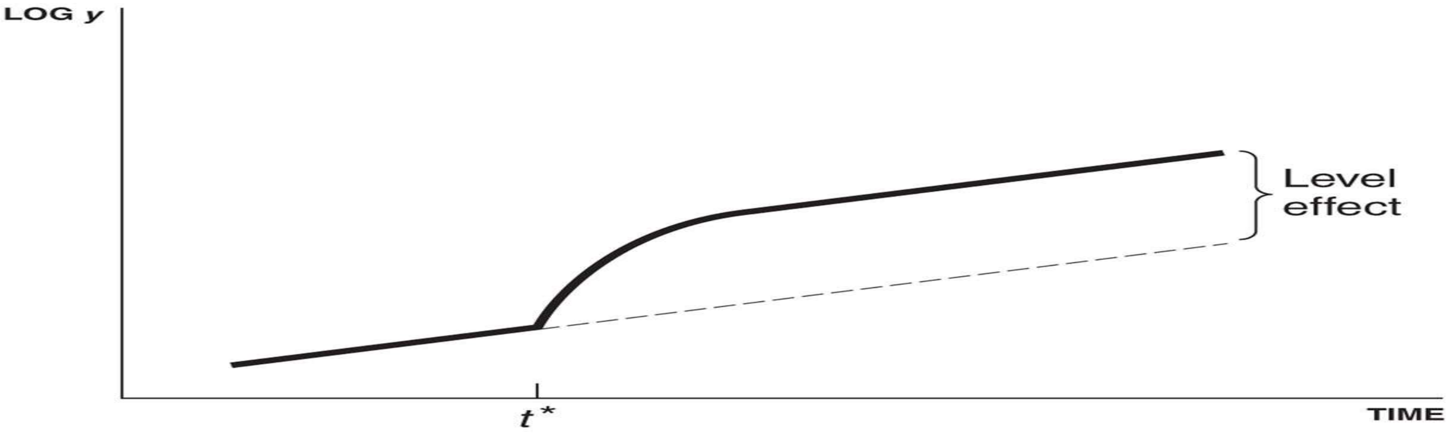


FIGURE 2.13 THE EFFECT OF AN INCREASE IN INVESTMENT ON y



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Conclusion: what is crucial in the Solow model

assumption of diminishing returns to capital: economies that have less capital per worker (relative to their long-run capital per worker) tend to have higher rates of return and, then, higher growth rates;

in the Solow model, economic policy may change the **growth rate** of GDP per capita but such change is **temporary** during the process to a new steady state; they **do not have long-run effect on economic growth**;

in the Solow model, economic policy may change the **long-run level of GDP *per capita***, **permanently**.