ISEG - Lisbon School of Economics and Management

Statistics I

2nd Semester of 2019/2020

Exam 1

25 October 2019 Duration: 70 minutes

Name: Solution Number: ________

Justify your answers carefully and present all the calculations you consider necessary.

Q	1	2	3a	3b	3c	3d	4a	4b	5a	5b
Val	2	2	2	2	2	2	2	2	2	2

1. To guarantee the quality of a certain product, a factory has three different testing lines, line A, line B and line C, where the product is sequentially tested. In each line, the product is classified as "Damaged", "Undefined", and "Flawless". The product is ready to be sold if (i) it is never classified as "Damaged" and (ii) it is classified as "Flawless" (at least) in line C. For example, if an object receives the classification A - undefined, B - undefined, and C - flawless it is ready to be sold, but if it receives A - flawless, B - flawless, and C - undefined it fails the test. It is known that

   - in line A, 2% of the products are classified as damaged;
   - from those that are not classified as damaged in line A, 1% of these products are classified as damaged in line B;
   - from those that are not classified as damaged in line B, 2% these products are classified as damaged in line C and 1% as undefined.

   What is the probability that a randomly chosen product is ready to be sold?

Consider the events

D_i: "the product is considered as damaged in line i" with

U_i: "the product is classified as Undefined in line i" $i=A,B,C$

F_i: "the product is classified as Flawless in line i"

$P(D_A) = 0.02$ $P(D_B | \bar{D}_A) = 0.01$ $P(D_C | \bar{D}_B) = 0.02$ $P(U_C | \bar{D}_B) = 0.01$

$P(\bar{D}_A) = 0.98$ $P(\bar{D}_B | \bar{D}_A) = 0.99$ $P(F_C | \bar{D}_B) = 0.97$

$$P(\bar{D}_A \cap \bar{D}_B \cap F_C) = P(\bar{D}_A) \times P(\bar{D}_B | \bar{D}_A) \times \overbrace{P(F_C | \bar{D}_A \cap \bar{D}_B)}^{P(F_C | \bar{D}_B)}$$

$$= 0.98 \times 0.99 \times 0.97 = 0.941$$

2. Let A, and B be two events of a sample space S. Show that if

$$P(B|A) = P(B|\overline{A})$$

then A and B are independent.

$$P(B|A) = P(B|\bar{A}) \Leftrightarrow \frac{P(A\cap B)}{P(A)} = \frac{P(\bar{A}\cap B)}{P(\bar{A})}$$

$$\Leftrightarrow P(A\cap B)\,P(\bar{A}) = P(\bar{A}\cap B)\,P(A)$$

$$\Leftrightarrow P(A\cap B)\,(1-P(A)) = (P(B) - P(A\cap B))\,P(A)$$

$$\Leftrightarrow P(A\cap B) - \cancel{P(A\cap B)\,P(A)} = P(A)P(B) - \cancel{P(A)P(A\cap B)}$$

$$\Leftrightarrow P(A\cap B) = P(A)P(B)$$

Therefore, A and B are two independent events.

3. A firm might invest in project 1, with a probability of 1/3, or in project 2, with a probability of 2/3. Let X_1 be a discrete random variable that represents the return of project 1, and X_2 a continuous random variable that represents the return of project 2, both in millions of €. The cumulative distribution function of X_1 and the cumulative distribution function of X_2 are given by

$$F_{X_1}(x) = \begin{cases} 0, & x < 0 \\ 1/3, & 0 \le x < 1 \\ 1, & x \ge 1 \end{cases}, \quad F_{X_2}(x) = \begin{cases} 0, & x < 0 \\ x/3, & 0 \le x < 1 \\ \frac{13+2x^2}{45}, & 1 \le x < 4 \\ 1, & x \ge 4 \end{cases}.$$

(a) Compute the probability function of X_1 and the probability density function of X_2.

$$f_{X_1}(x) = F_{X_1}(x) - F_{X_1}(x^-) = \begin{cases} 1/3, & x = 0 \\ 2/3, & x = 1 \\ 0, & \text{c.c.} \end{cases}$$

$$f_{X_2}(x) = F'_{X_2}(x) = \begin{cases} 1/3, & 0 < x < 1 \\ \frac{4}{45}x, & 1 < x < 4 \\ 0, & \text{c.c.} \end{cases}$$

(b) Compute the expected return of project 2.

$$E(X_2) = \int_{-\infty}^{+\infty} x \cdot f_{X_2}(x)\,dx = \int_0^1 x/3\,dx + \int_1^4 \frac{4}{45} x^2\,dx$$

$$= \frac{x^2}{6}\Big]_0^1 + \frac{4x^3}{3\times 45}\Big|_1^4 = 1/6 + \frac{4^4-4}{3\times 45} = 61/30$$

(c) Compute $P(X_2 > 1 | X_2 < 3)$.

$$P(X_2 > 1 \mid X_2 < 3) = \frac{P(1 < X_2 < 3)}{P(X_2 < 3)} = \frac{F_{X_2}(3) - F_{X_2}(1)}{F_{X_2}(3)}$$

$$= 16/31$$

(d) Let X be the amount, in millions of €, received by the company after its investment. Find the cumulative distribution function of X. Classify the random variable X.

A_i : "the firm invests in project i" $i=1,2$

$$F_X(x) = P(X \leqslant x) = P(X \leqslant x \mid A_1) \times P(A_1) + P(X \leqslant x \mid A_2) \times P(A_2)$$

$$= P(X_1 \leqslant x) \times P(A_1) + P(X_2 \leqslant x) \times P(A_2)$$

$$= 1/3\, F_{X_1}(x) + 2/3\, F_{X_2}(x) = \begin{cases} 0, & x < 0 \\ \frac{1+2x}{9}, & 0 \leqslant x < 1 \\ 1/3 + \frac{26+4x^2}{45\times 3}, & 1 \leqslant x < 4 \\ 1, & x \geqslant 4 \end{cases}$$

By construction, X is a mixed Random Variable.

4. Let X and Y be two random variables such that their joint probability distribution is represented by the following table

$y \setminus x$	0	1	$P(Y=y)$
0	a	b	0.2
1	c	d	0.8
$P(X=x)$	0.6	0.4	

Find the values of a, b, c and d such that:

a) the random variables X and Y are independent.

X and Y are two independent R.V. Therefore $P(X=x, Y=y) = P(X=x)\cdot P(Y=y)$

$a = P(X=0, Y=0) = 0.2 \times 0.6 = 0.12$

$b = P(X=1, Y=0) = 0.4 \times 0.2 = 0.08$

$c = P(X=0, Y=1) = 0.6 \times 0.8 = 0.48$

$d = P(X=1, Y=1) = 0.4 \times 0.8 = 0.32$

b) $f_{X|Y=0}(0) = \frac{1}{4}$.

$f_{X|Y=0}(0) = 1/4 \iff \dfrac{P(X=0, Y=0)}{P(Y=0)} = 1/4 \iff \dfrac{a}{0.2} = 1/4 \iff a = 1/20$

$P(Y=0) = \sum_{i=0}^{1} P(X=i, Y=0) = 0.2 \iff b = 3/20$

$P(X=0) = \sum_{i=0}^{1} P(X=0, Y=i) = 0.6 \iff c = 11/20$

$P(X=1) = \sum_{i=0}^{1} P(X=1, Y=i) = 0.4 \iff d = 5/20$

5. Let X be a continuous random variable such that its probability density function is f_X and its cumulative distribution function is F_X. Let Y be a random variable given by

$$Y = \begin{cases} -1, & X \leq a \\ 1, & X > a \end{cases},$$

for any $a \in \mathbb{R}$.

(a) Prove that $E(Y) = 1 - 2F_X(a)$.

If $Y = g(X)$ then $E(Y) = \int_{-\infty}^{+\infty} g(x)\cdot f_X(x)\,dx$

$= \int_{-\infty}^{a} (-1)\cdot f_X(x)\,dx + \int_{a}^{+\infty} 1\cdot f_X(x)\,dx = -1\cdot\int_{-\infty}^{a} f_X(x)\,dx + \int_{a}^{+\infty} f_X(x)\,dx$

$= -F_X(a) + 1 - F_X(a) = 1 - 2F_X(a)$

(b) Taking into account the previous question, compute $E(Y)$, when

$$a = 1 \quad \text{and} \quad f_X(x) = \begin{cases} \frac{1}{2}, & 0 < x < 2 \\ 0, & \text{otherwise} \end{cases}.$$

$$E(Y) = 1 - 2\,F_X(1)$$

$$= 1 - 2\int_{-\infty}^{1} f_X(x)\,dx$$

$$= 1 - 2\int_0^1 \tfrac{1}{2}\,dx = 0$$

————— This page can be used to complete your answers. —————