ISEG - Lisbon School of Economics and Management

Statistics I

1st Semester of 2019/2020

Exam 2

10 January 2020 Duration: 70 minutes

Name: ____ Number: __________

Please tick the boxes below to assure you understand the following:

- ☐ Remaining in the exam room during the second block of 70 minutes means that students desire to replace the grade of the midterm Exam by the grade of Exam 1.
- ☐ Those who want to keep the grade of the midterm Exam must deliver Exam 2 during the first block of 70 minutes.
- ☐ Justify your answers carefully and present all the calculations you consider necessary.

Q	1a	1b	2a	2b	2c	3a(i)	3a(ii)	3b	4a	4b
Val	2	2	2	2	2	2	2	2	2	2

1. Let X be a discrete random variable such that its probability function is given by

$$f_x(x) = \begin{cases} 0.1, & x = -1 \\ 0.3, & x = 2 \\ 0.15, & x = 3 \\ 0.05, & x = 5 \\ 0.4, & x = 7 \\ 0, & otherwise \end{cases}$$

a) Identify the mode of X.

$$mo(X) = \underset{x \in \mathbb{R}}{\text{Argmax}} \; f_X(x) = 7$$

b) Compute the median of X.

$$me(X) = q_{0.5} = \min\{x \in \mathbb{R} : F_X(x) \geq 0.5\}$$

$$F_X(x) = P(X \leq x) = \begin{cases} 0, & x < -1 \\ 0.1, & -1 \leq x < 2 \\ 0.4, & 2 \leq x < 3 \\ 0.55, & 3 \leq x < 5 \\ 0.6, & 5 \leq x < 7 \\ 1, & x \geq 7 \end{cases}$$

$$me(X) = \min\{x \in \mathbb{R} : x \geq 3\} = 3$$

2. Let (X, Y) be a two-dimensional discrete random variable such that its joint probability function is represented by

X/Y	-1	1	3
0	0.1	0.2	0.05
1	0.3	0.2	0.15

a) Compute $E(X)$ and $E(Y)$.

$$f_X(x) = \sum_{y \in D_y} P(X = x, Y = y) = \begin{cases} 0.35, & x = 0 \\ 0.65, & x = 1 \\ 0, & \text{otherwise} \end{cases}$$

$$f_Y(y) = \sum_{x=0}^{1} P(X = x, Y = y) = \begin{cases} 0.4, & x = -1, 1 \\ 0.2, & x = 3 \\ 0, & \text{otherwise} \end{cases}$$

$$E(X) = \sum_{x=0}^{1} x \cdot P(X = x) = 0 \times 0.35 + 1 \times 0.65 = 0.65$$

$$E(Y) = \sum_{y \in D_y} y \cdot P(Y = y) = -1 \times 0.4 + 1 \times 0.4 + 3 \times 0.2$$

$$= 0.6$$

b) Compute $Cov(X,Y)$.

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

$$E(XY) = \sum_{x=0}^{1} \sum_{X \in D_y} xy\, P(X=x, Y=y)$$

$$= 1\times(-1)\times 0.3 + 1\times 1\times 0.2 + 1\times 3\times 0.15$$

$$= 0.35$$

$$Cov(X,Y) = 0.35 - 0.6\times 0.65$$

$$= -0.04$$

c) Find a and b such that $M_{a+bY}(t) = 0.4 + 0.4e^{2t} + 0.2e^{4t}$.

$$M_{a+by}(t) = E(e^{(a+by)t}) = e^{at} E(e^{bty}) = e^{at} M_y(bt)$$

$$M_y(bt) = E(e^{bty}) = 0.4e^{-bt} + 0.4e^{bt} + 0.2e^{3bt}$$

Therefore, $M_{a+by}(t) = 0.4e^{(a-b)t} + 0.4e^{(a+b)t} + 0.2e^{(a+3b)t}$

which allows us to get

$$\begin{cases} a-b=0 \\ a+b=2 \\ a+3b=4 \end{cases} \Leftrightarrow \begin{cases} a=b \\ 2b=2 \\ 4b=2 \end{cases} \Leftrightarrow \begin{cases} a=1 \\ b=1 \end{cases}$$

3. Assume that Brand A sells, on average, 4 cars per month with a standard deviation of 2 cars.

a) If the number of cars sold per month follows a Poisson process, compute:

(i) the probability that more than 1 car is sold in a month.

X_t represents the number of cars sold by Brand A in t months

$X_t \sim Poi(\lambda t)$ and $E(X_1) = 4 \Rightarrow \lambda = 4$

$$P(X_1 > 1) = 1 - P(X_1 \leq 1) = 1 - (P(X_1 = 0) + P(X_1 = 1))$$

$$= 1 - \frac{e^{-4}4^0}{0!} - \frac{e^{-4}4^1}{1!} \simeq 0.908$$

(ii) the probability that Brand A sells 10 cars in 3 months when it sold 2 cars in the first month.

$$P(X_3 = 10 \mid X_1 = 2) = \frac{P(X_3 = 10,\ X_1 = 2)}{P(X_1 = 2)}$$

$$= \frac{P(X_3 - X_1 = 10-2,\ X_1 = 2)}{P(X_1 = 2)} = \frac{P(X_3 - X_1 = 8)\,P(X_1 = 2)}{P(X_1 = 2)}$$

$$= P(X_2 = 8) = \frac{e^{-8}\,8^8}{8!} \simeq 0.1396$$

$X_2 \sim Poi(8)$

b) Compute the probability that, in 3 years, Brand A sells more than 140 cars, assuming that there is no dependence between the number of cars sold in each month.

- Consider the sequence $Y_1, Y_2, \dots, Y_{36}$ where Y_i represents the number of cars sold by Brand A in month i, with $i = 1, 2, \dots, 36$
- $E(Y_i) = var(Y_i) = 4$
- All the random variables are independent and identically distributed
- By the central limit theorem

$$T_{36} = \sum_{i=1}^{36} X_i \overset{a}{\sim} N(36 \times 4,\ 36 \times 4)$$

Therefore, $Z = \dfrac{T_{36} - 144}{12} \overset{a}{\sim} N(0,1)$

$$P(T_{36} > 140) = P\left(Z > \frac{140-144}{12}\right) = P(Z > -1/3)$$

$$= P(Z < 1/3) = \Phi(1/3) \simeq 0.6293$$

4. Let X be a geometric random variable with parameter $p \in]0,1[$, $X \sim Geo(p)$ (equivalently, X is a negative binomial random variable with parameters $k = 1$ and $p \in]0,1[$, $X \sim NB(1,p)$).

   a) If $p = 0.2$, compute $P(X > 2)$.

$$P(X>2) = 1 - P(X \leq 2) = 1 - P(X=1) - P(X=2)$$
$$= 1 - 0.2(1-0.2)^0 - 0.2(1-0.2)^1$$
$$= 0.64$$

   b) Prove that $E(X) = \sum_{n=0}^{\infty}(1 - F_X(n))$.
   Hint: Firstly, deduce the expression for $(1 - F_X(n))$ and, secondly, compute the series and compare with the expected value of a geometric distribution.

It is known that $E(X) = 1/p$

$$(1 - F_X(n)) = P(X > n) = P(X = n+1) + P(X = n+2) + P(X = n+3) + \cdots$$
$$= p(1-p)^n + p(1-p)^{n+1} + \cdots = p(1-p)^n \left(1 + (1-p) + \cdots\right)$$
$$= p(1-p)^n \sum_{i=0}^{\infty}(1-p)^i = p(1-p)^n \times \frac{1}{1-(1-p)} = (1-p)^n$$

$\downarrow$

$|1-p| < 1$ then the geometric series is convergent

$$\sum_{n=0}^{\infty}(1-p)^n = 1/p$$ by the calculations above which finish the proof.

This page can be used to complete your answers.

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Exam 1

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1. Let A, B and C be three events of a sample space S.

   (a) Compute $P(C|\, A \cap B)$ when $P(A \cap B \cap C) = 0.2$, $P(A|B) = 0.8$ and $P(B) = 0.5$.

$$P(C|A\cap B) = \frac{P(A\cap B\cap C)}{P(A\cap B)}$$

$$= \frac{P(A\cap B\cap C)}{P(A|B)P(B)} = \frac{0.2}{0.8\times 0.5} = 1/2$$

(b) Assume that $C \subset A$ and A and B are independent.

i) Prove that

$$P(C|A \cap B) = \frac{P(C|B)}{P(A)}.$$

Since $C \subset A \Rightarrow C \cap A = C$

A is independent of $B \Rightarrow P(A \cap B) = P(A)P(B)$

$$P(C|A \cap B) = \frac{P(C \cap A \cap B)}{P(A \cap B)} = \frac{P(C \cap B)}{P(A)P(B)}$$

$$= \frac{P(C \cap B)/P(B)}{P(A)} = \frac{P(C|B)}{P(A)}$$

ii) Taking into account the previous question, compute $P(A \cup B)$ and $P(C \cap B)$ if

$$P(C|A \cap B) = \frac{1}{4}, \quad P(A) = \frac{1}{2}, \quad P(B) = \frac{2}{5}.$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= P(A) + P(B) - P(A)P(B) = 1/2 + 2/5 - 1/5$$

$$= 7/10$$

$$1/4 = P(C|A \cap B) = \frac{P(C|B)}{P(A)} = \frac{P(C|B)}{1/2}$$

$$\Leftrightarrow P(C|B) = 1/8$$

$$P(C \cap B) = P(C|B) \times P(B) = 1/8 \times 2/5 = 1/20$$

2. A goldsmith buys gold every morning to sell throughout the day. The goldsmith's daily balance (profit or loss) is given by the difference between the amount received with the gold sold and the amount paid to buy the gold. Let X be the random variable that represents the cost of the gold bought at the beginning of a day, and Y be the random variable that represents the total amount received with the gold sold during the same day. Both random variables are measured in hundreds of euros. It is known that

$$f_X(x) = \begin{cases} \frac{1}{8}, & 2 < x < 10 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad f_{Y|X=x}(y) = \begin{cases} \frac{y}{2x^2}, & 0 < y < 2x \\ 0, & \text{otherwise} \end{cases}$$

a) Compute the cumulative distribution function of X.

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(u)\,du$$

$$= \begin{cases} 0, & x < 2 \\ \frac{x-2}{8}, & 2 \leq x < 10 \\ 1, & x \geq 10 \end{cases}$$

b) Compute the probability that the goldsmith has to pay more than 8 hundred euros in a random day to buy the gold.

$$P(X > 8) = 1 - P(X \leq 8) = 1 - F_X(8)$$

$$= 1 - \frac{8-2}{8} = \frac{2}{8} = \frac{1}{4}$$

c) Compute the expected cost of the gold bought in a random day.

$$E(X) = \int_{-\infty}^{+\infty} x \cdot f_X(x)\,dx = \int_{2}^{10} \frac{x}{8}\,dx$$

$$= \left.\frac{x^2}{16}\right]_2^{10} = \frac{100-4}{16} = 6$$

d) Compute the probability that the goldsmith receives more than 8 hundred euros when the cost of the gold bought was 5 hundred euros.

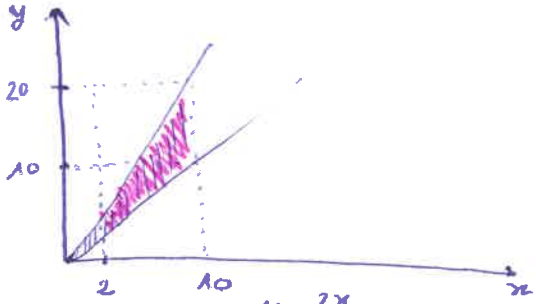
$$P(Y>8 \mid X=5) = \int_8^{+\infty} f_{Y\mid X=5}(y)\, dy$$

$$= \int_8^{10} \frac{y}{2\times 25}\, dy = \int_8^{10} \frac{y}{50}\, dy = \left[\frac{y^2}{100}\right]_8^{10}$$

$$= 1 - \frac{64}{100} = 0.36$$

e) Compute the probability that the goldsmith has profit in a random day.

$$P(Y>X) = ?$$



$$f_{X,Y}(x,y) = f_{Y\mid X=x}(y)\cdot f_X(x)$$

$$= \begin{cases} \dfrac{y}{16x^2}, & 2<x<10,\ 0<y<2x \\ 0, & \text{otherwise} \end{cases}$$

$$P(Y>X) = \int_2^{10}\int_x^{2x} \frac{y}{16x^2}\, dy\, dx =$$

$$= \int_2^{10} \left[\frac{y^2}{32x^2}\right]_x^{2x} dx = \int_2^{10} 4/32 - 1/32\, dx = 3/4$$

3. Let X and Y be two discrete random variables with a joint probability function given by

X/Y	-1	0	1
-1	a	0.1	0.2
1	0.3	0.1	0.05
2	0.05	0.05	b

a) Find a and b such that $P(X=-1) = 0.35$.

$f_{X,Y}$ is a probability function iff

(1) $f_{X,Y}(x,y) \geq 0 \quad \forall (x,y) \in \mathbb{R}^2$

(2) $\sum_{(x,y)\in D_{X,Y}} f_{X,Y}(x,y) = 1$

From (1) we get that $a, b \geq 0$ and from (2) we have that

$a + b + 0.85 = 1 \Leftrightarrow$

$\Leftrightarrow a + b = 0.15$

$$P(X=-1) = \sum_{y=-1}^{1} P(X=-1, Y=y)$$

$$= a + 0.1 + 0.2$$

$$= 0.3 + a \Rightarrow a = 0.05$$

thus,

$$\begin{cases} a+b = 0.15 \\ a = 0.05 \end{cases} \Leftrightarrow \begin{cases} b = 0.1 \\ a = 0.05 \end{cases}$$

b) Find the probability function of $Z = |X| - |Y|$

$D_Z = \{0, 1, 2\}$

$P(Z=0) = P(|X| = |Y|) = P(X=-1, Y=-1) + P(X=-1, Y=1)$
$+ P(X=1, Y=-1) + P(X=1, Y=1) = 0.6$

$P(Z=1) = P(|X| = |Y|+1) = P(X=-1, Y=0)$
$+ P(X=1, Y=0) + P(X=2, Y=-1)$
$+ P(X=2, Y=1) = 0.35$

$P(Z=2) = P(|X| = |Y|+2) = P(X=2, Y=0)$
$= 0.05$

$P(Z=z) = 0$ when $z \notin \{0, 1, 2\}$

This page can be used to complete your answers.