ISEG - Lisbon School of Economics and Management

Statistics I

2nd Semester of 2018/2019

Abbreviated Solution of Exam 1

8 April 2019 Duration: 70 minutes

Name: ______________________________ Number: __________

Justify your answers carefully and present all the calculations you consider necessary.

Q	1	2a	2b	3a	3b	3c	3d	4a	4b	4c	4d
Val	1	2	3	1.5	1.5	1	2	2.5	1	2	2.5

1. Let A, B and C be three events of a sample space S such that:

   - A and B are disjoint events;
   - A and C are independent events;
   - B and C are independent events.

   Show that
   $$P(A \cup B \,|\, C) = P(A) + P(B).$$

   Solution: $P(\cdot|C)$ is a probability measure. Therefore,
   $$P(A \cup B \,|\, C) = P(A \,|\, C) + P(B \,|\, C) - P(A \cap B \,|\, C).$$

   Since A and B are disjoint $A \cap B = \emptyset$, then $P(A \cap B \,|\, C) = 0$. Additionally, $P(A \,|\, C) = A$ and $P(B \,|\, C) = B$ because A and C are independent events and B and C are independent events. Consequently.
   $$P(A \cup B \,|\, C) = P(A) + P(B).$$

2. Suppose that a firm specialized in a certain product is facing the following scenarios:

   - the demand can increase, decrease or stay equal;
   - the price can also increase, decrease or stay equal.

   It is known that:

- the probability that the price will increase given that the demand will also increase is 2/5;
- the probability that the demand will increase is 1/4;
- the probability that the price will decrease is 1/5 and
- the probability that the price will stay equal is 1/4.

Suggestion: Consider the events PI: "The price increases", PD: "The price decreases", PE: "The price stays equal", DI: "The demand increases", DD: "The demand decreases", DE: "The demand stays equal".

a) Compute the probability that the demand will increase given that the price will increase.

Solution: From the problem, one obtains the following probabilities:

$$P(PI|DI) = \frac{2}{5}, \quad P(DI) = \frac{1}{4}, \quad P(PD) = \frac{1}{5}, \quad P(PE) = \frac{1}{4}.$$

Additionally, taking into account that PI, PD and PE are a partition of S, we know that

$$\begin{aligned} P(DI \mid PI) =& \frac{P(PI \mid DI)P(DI)}{P(PI)} \\ =& \frac{P(PI \mid DI)P(DI)}{1 - P(PD) - P(PE)} = \frac{2/5 \times 1/4}{11/20} = \frac{2}{11}. \end{aligned}$$

b) Assume, in addition, that the probability that the demand will stay equal given that the price will increase is 3/11 and the probability that the price will increase given that the demand will stay equal is 1/2. Now, compute the probability that

i) the demand will stay equal and the price will increase;

ii) the demand will stay equal.

Solution: Form the question, we can get that

$$P(DE|PI) = \frac{3}{11} \quad P(PI|DE) = \frac{1}{2}.$$

Therefore,

$$\begin{aligned} P(DE \cap PI) &= P(DE|PI) \times P(PI) = \frac{3}{20} \\ P(DE) &= \frac{P(PI \cap DE)}{P(PI|DE)} = \frac{3}{10}. \end{aligned}$$

3. Let Y be a continuous random variable. The probability density function of Y is given by the function

$$f_Y(y) = \begin{cases} be^{-2y}, & y \geq 0 \\ 0, & y < 0 \end{cases}.$$

a) Find the value of b.

Solution: f_Y is a density probability function, thus it satisfies

$$f_Y(y) \geq 0, \forall y \in \mathbb{R} \quad \text{and} \quad \int_{-\infty}^{+\infty} f_Y(y)dy = 1.$$

Form the first condition, we get that $b \geq 0$ and from the second we obtain $\frac{b}{2} = 1$. Therefore, $b = 2$.

b) Compute the cumulative distribution function F_Y.

Solution: The cumulative distribution functions are such that

$$F_Y(y) = P(Y \leq y).$$

Therefore,

$$F_Y(y) = \int_0^y f_Y(u)du = \begin{cases} 0, & y < 0 \\ 1 - e^{-2y}, & y \geq 0 \end{cases}.$$

c) Compute $P(Y > 2 \,|\, Y > 1)$.

$$\begin{aligned} P(Y > 2 \,|\, Y > 1) &= \frac{P(Y > 2, \, Y > 1)}{P(Y > 1)} = \frac{P(Y > 2)}{P(Y > 1)} \\ &= \frac{1 - F_Y(2)}{1 - F_Y(1)} = \frac{e^{-4}}{e^{-2}} = e^{-2}. \end{aligned}$$

d) Compute the median of Y.

Solution: Since $me(Y) = \min\{y \in \mathbb{R} : F_Y(y) \geq 0.5\}$, we calculate

$$F_Y(y) \geq 0.5 \Leftrightarrow 1 - e^{-2y} \geq 0.5 \Leftrightarrow e^{-2y} \leq 0.5 \Leftrightarrow y \geq \frac{\log(2)}{2}.$$

Therefore,

$$me(Y) = \min\{y \in \mathbb{R} : F_Y(y) \geq 0.5\} = \min\left\{y \in \mathbb{R} : y \geq \frac{\log(2)}{2}\right\} = \frac{\log(2)}{2}$$

4. Let X and Y be random variables such that their joint probability distribution is

$Y \setminus X$	0	1	2
0	1/5	1/10	1/10
1	a	b	1/5

a) Given that $f_{X|Y=1}(1) = \frac{1}{6}$, find the values of a and b.
As we have a discrete joint probability distribution,

$$f_{X,Y}(x,y) \geq 0 \quad \text{and} \quad \sum_{x=0}^{2}\sum_{y=0}^{1} f_{X,Y}(x,y) = 1.$$

Therefore, we get that $a \geq 0$, $b \geq 0$ and $a + b = \frac{2}{5}$. Additionally,

$$\frac{P(X=1, Y=1)}{P(Y=1)} = \frac{1}{6} \Leftrightarrow \frac{b}{a+b+1/5} = \frac{1}{6}.$$

Solving the system of two equation we get $a = 3/10$ and $b = 1/10$.

b) Compute the marginal probability function of Y. Using the definition of marginal probability function, we obtain

$$P(Y=y) = \sum_{x=0}^{2} P(X=x, Y=y) = \begin{cases} 2/5, & y=0 \\ 3/5, & y=1 \\ 0, & \text{otherwise} \end{cases}$$

c) Calculate the moment generating function of Y.
Solution: The moment generating function of Y is given by

$$M_Y(t) = E(e^{tY}) = \sum_{y=0}^{1} e^{ty} P(Y=y) = \frac{2}{5} + \frac{3}{5}e^t.$$

d) Compute the variance of $Z = 2Y + 5$.
Solution: From the properties of the variance, we get that

$$Var(Z) = Var(2Y+2) = 4Var(Y) = 4\left(E(X^2) - (E(X))^2\right).$$

By using the moment generating function, we have

$$E(X) = M_X'(0) = \frac{3}{5} \quad \text{and} \quad E(X^2) = M_X''(0) = \frac{3}{5},$$

because $M_X'(t) = M_X''(t) = \frac{3}{5}e^t$. Therefore, $Var(Z) = 4 * \frac{3}{5}\left(1 - \frac{3}{5}\right) = \frac{24}{25}$.