ISEG - Lisbon School of Economics and Management

Statistics I

2nd Semester of 2018/2019

Exam 2

19 June 2019 Duration: 70 minutes

Name: ______________________ Number: ________

Please tick the boxes below to assure you understand the following:

- ☐ Remaining in the exam room during the second block of 70 minutes means that students desire to be evaluated without the continuous evaluation and, thus, must deliver Exam 1 & 2.
- ☐ Those who want to proceed with their continuous evaluation must deliver Exam 2 during the first block of 70 minutes.
- ☐ Justify your answers carefully and present all the calculations you consider necessary.

Q	1a	1b	2a	2b	2c	3a	3b	3c	4a	4b
Val	3	1.5	2	2	2	1	2	3	2	1.5

1. Let (X, Y) be a two-dimensional continuous random variable with the following joint probability density function:

$$f_{X,Y}(x,y) = \begin{cases} x-y, & 1<x<2,\ 0<y<1 \\ 0, & \text{otherwise} \end{cases}.$$

a) Compute $E(X)$ and $E(Y)$.

$$E(X) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x \cdot f_{X,Y}(x,y)\,dx\,dy = \int_0^1\int_1^2 x(x-y)\,dx\,dy$$

$$= \int_0^1 \left[\frac{x^3}{3} - \frac{x^2}{2}y\right]_1^2 dy = \int_0^1 \frac{7}{3} - \frac{3}{2}y\,dy = \left[\frac{7}{3}y - \frac{3}{4}y^2\right]_0^1$$

$$= \frac{7}{3} - \frac{3}{4} - 0 = \frac{19}{12}$$

$$E(Y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} y\, f_{X,Y}(x,y)\,dx\,dy = \int_0^1\int_1^2 y(x-y)\,dx\,dy =$$

$$= \int_0^1 \left[y\frac{x^2}{2} - y^2x\right]_1^2 dy = \int_0^1 \frac{3}{2}y - y^2\,dy = \left[\frac{3}{4}y^2 - \frac{y^3}{3}\right]_0^1$$

$$= \frac{3}{4} - \frac{1}{3} = \frac{5}{12}$$

b) Compute $Cov(X, Y)$.

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

$$E(XY) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy(x-y)\,dx\,dy = \int_0^1\int_1^2 xy(x-y)\,dx\,dy$$

$$= \int_0^1 \left[\frac{x^3}{3}y - \frac{x^2y^2}{2}\right]_1^2 dy = \int_0^1 \frac{7}{3}y - \frac{3}{2}y^2\,dy$$

$$= \left[\frac{7}{6}y^2 - \frac{1}{2}y^3\right]_0^1 = 7/6 - 1/2 = 2/3$$

$$Cov(X,Y) = E(XY) - E(X)E(Y) = \frac{2}{3} - \frac{19}{12} \times \frac{5}{12}$$

$$= \frac{1}{144}$$

2. Let (X, Y) be a two-dimensional discrete random variable, such that its joint probability function is represented by

X/Y	0	1
0	0.1	0.4
1	0.3	0.2

$$P(x=x) = \sum_{y=0}^{1} P(X=x, y=y)$$

then $P(X=0) = P(X=1) = 0.5$

a) Find the conditional probability function of Y given $X = x$, for $x = 0, 1$.

$$f_{Y|X=x}(y) = \frac{P(Y=y, X=x)}{P(X=x)}, \quad \text{for } x = 0, 1$$

If $x = 0$

$$f_{Y|X=0}(y) = \begin{cases} \frac{0.1}{0.5}, & y=0 \\ \frac{0.4}{0.5}, & y=1 \\ 0, & \text{otherwise} \end{cases} = \begin{cases} 1/5, & y=0 \\ 4/5, & y=1 \\ 0, & \text{otherwise} \end{cases}$$

If $x = 1$

$$f_{Y|X=1}(y) = \begin{cases} \frac{0.3}{0.5}, & y=0 \\ \frac{0.2}{0.5}, & y=1 \\ 0, & \text{otherwise} \end{cases} = \begin{cases} 3/5, & y=0 \\ 2/5, & y=1 \\ 0, & \text{otherwise} \end{cases}$$

b) Compute $E(Y|X = x)$, for $x = 0, 1$.

$$E(Y|X=x) = \sum_{y=0}^{1} y \cdot P(Y=y|X=x) \quad \text{for} \quad x=0,1$$

If $x = 0$

$$E(Y|X=0) = 1/5 \times 0 + 4/5 \times 1 = 4/5$$

If $x = 1$

$$E(Y|X=1) = 3/5 \times 0 + 2/5 \times 1 = 2/5$$

c) Find the probability function of $W = E(Y|X)$.

The random variable W can take 2 values $\{4/5, 2/5\}$. The first one when $X=0$ and the second one when $X=1$. Then

x	0	1	otherwise
W	$4/5$	$2/5$	otherwise
$P(W=w)$	$P(X=0)$	$P(X=1)$	0

$$P(W=w) = \begin{cases} 0.5, & w = 2/5 \\ 0.5, & w = 4/5 \\ 0, & \text{otherwise} \end{cases}$$

3. A new factory will start selling industrial vacuum cleaners on the 1^{st} of January of 2020. According to the market analysis developed by the experts of the factory, the number of industrial vacuum cleaners that the factory will be able to sell per month is a random variable X with the following probability function

$$f_X(x) = \frac{1}{11}, \quad x = 0, 1, 2, \cdots, 10.$$

Remark: Note that $E(X) = 5$ and $Var(X) = 10$.

a) What is the probability that, in a random month, this factory will sell more than 4 industrial vacuum cleaners?

$$P(X>4) = 1 - P(X \leq 4) = 1 - F_X(4) = 1 - \sum_{x=0}^{4} f_X(x)$$

$$= 1 - 5/11 = 6/11$$

b) What is the probability that this factory has to wait until March (3 months) to finally sell more than 4 industrial vacuum cleaners in a single month? Assume independence when you think it is necessary.

Y: number of months until the first one when we finally sell more than 4 industrial vacuum cleaner in a single month.

$Y \sim Geo(p)$, $p = P(X > 4) = \frac{6}{11}$

$$P(Y=3) = \left(1-\frac{6}{11}\right)^2 \cdot \frac{6}{11}$$

(c) What is the probability that this factory will ~~send~~ sell more than 185 industrial vacuum cleaners in the next three years? Assume independence when you think it is necessary.

- X_i : number of industrial vacuum cleaners that the factory will sell in month i , with $i = 1, 2, \cdots, 36$
- $T_{36} = \sum_{i=1}^{36} X_i$ represents the number of industrial vacuum cleaner that the factory will sell in the next 3 years.
- Assuming that X_i are independent and identically distributed to $X \sim$ DiscreteUniform$\{0, 1, 2, \cdots, 10\}$

$T_{36} \overset{a}{\sim} N(\mu, \sigma^2)$ according with the central limit theorem where $\mu = E\left(\sum_{i=1}^{36} X_i\right) = 36 \times 5$

and $\sigma^2 = Var\left(\sum_{j=1}^{36} X_i\right) = 36 \times 10$

Then,

$$P(T_{36} > 185) = P\left(\frac{T_{36} - 180}{\sqrt{360}} > \frac{185-180}{\sqrt{360}}\right) \simeq 1 - \Phi\left(\frac{5}{6\sqrt{10}}\right)$$

$$\simeq 1 - \Phi(0.26) = 0.3974$$

4. Assume that $X \sim Exp(\lambda)$.

a) Prove that $W = 2\lambda X$ follows a $\chi^2_{(2)}$.
Hint: Start by computing the moment-generating function of W.

$$M_W(t) = E(e^{tW}) = E(e^{2\lambda t X}) = E(e^{(2\lambda t) X})$$

$$= M_X(2\lambda t) = (1 - 2\lambda t/\lambda)^{-1}, \quad 2\lambda t < \lambda$$

$$= (1 - 2t)^{-1}, \quad t < 1/2$$

Therefore $W \sim Game(1, 2)$

but, we know that $Game(1, 2) = Game(\frac{2}{2}, 2)$
$= \chi^2_{(2)}$

Therefore
$W \sim \chi^2_{(2)}$

b) Compute $P(W < 0.102587)$

$$P(W < 0.102587) = 1 - P(W \geq 0.102587)$$
$$= 1 - 0.950 = 0.05$$

This page can be used to complete your answers.

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Statistics I

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Exam 1

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Name: ______________________ Number: __________

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Q	1a	1b	2a	2b	3a	3b	3c	3d	4a	4b	4c	4d
Val	2	2	1	1.5	1.5	1.5	2	1.5	2	1	2	2

1. Let A, B and C be three events of a sample space S such that A and B are disjoint events.

   (a) Prove that
$$P(A|B \cup C) = \frac{P(A \cap C)}{P(B \cup C)}.$$

$$P(A|B\cup C) = \frac{P(A\cap(B\cup C))}{P(B\cup C)} = \frac{P(A\cap C)}{P(B\cup C)}$$

$$P(A\cap(B\cup C)) = P(A\cap B \cup A\cap C)$$
$$= P(A\cap B) + P(A\cap C) - P(A\cap B\cap C)$$

since $A\cap B = \emptyset$
$$= P(A\cap B) + P(A\cap C) = P(A\cap C)$$

(b) Assume that B and C are independent. Prove that

$$P(A|B \cup C) = \frac{P(A \cap C)}{1 - P(\overline{B})P(\overline{C})}.$$

$P(B \cup C) = P(B) + P(C) - P(B \cap C)$
$= P(B) + P(C) - P(B) \times P(C)$
$= P(B)(1 - P(C)) + P(C) = P(B)P(\overline{C}) + P(C)$
$= P(B)P(\overline{C}) + 1 - P(\overline{C})$
$= 1 + P(\overline{C})(P(B) - 1) = 1 - P(\overline{C})P(\overline{B})$

By using question 1a) we get the result

2. A large box contains two small boxes inside, one of them numbered with 1 and the other with 2 (box 1 & box 2). Inside box 1, there are 4 green balls and 2 blue balls. Inside box 2, there are 1 green ball and 5 blue balls. Consider the following experiment: one puts his/her hand inside the large box, chooses randomly one of the small boxes, and, finally, picks randomly one ball from the small box previously chosen.

Suggestion: Consider the events B_i: "Choose the small box i, $i = 1, 2$.", A: "Pick a green ball" and $\overline{A}$: "Pick a blue ball"

a) Compute the probability that one chooses a green ball given that box 1 was chosen and the probability that one chooses a green ball given that box 2 was chosen.

$P(A|B_1) = \frac{4}{6} = 2/3$

$P(A|B_2) = \frac{1}{6}$

b) Compute the probability that one selected box 1 given that a green ball was picked.

$P(B_1|A) = \frac{P(A|B_1) \times P(B_1)}{P(A)} = \frac{2/3 \times 1/2}{P(A \cap B_1) + P(A \cap B_2)}$

$= \frac{2/3 \times 1/2}{2/3 \times 1/2 + 1/6 \times 1/2} = 4/5$

d) Compute the moment-generating function of X.

$$M_X(t) = E(e^{tX}) = \sum_{x \in D_X} e^{tx} P(X=x)$$

$$= 0.2\,(e^{-2t} + e^{t}) + 0.3\,(e^{-t} + e^{3t})$$

4. Let X and Y be two continuous random variables with a joint probability density function given by

$$f_{X,Y}(x,y) = \begin{cases} 4xy, & 0 < x < 1,\ 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}.$$

a) Find the marginal probability density functions of X and Y.

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dy = \int_0^1 4xy\,dy = \frac{4x\,y^2}{2}\Big|_0^1 = 2x, \quad 0 < x < 1$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx = \int_0^1 4xy\,dx = \frac{4x^2 y}{2}\Big|_0^1 = 2y \quad 0 < y < 1$$

3. Let X be a discrete random variable, such that its cumulative distribution function is given by

$$F_X(x) = \begin{cases} 0, & x < -2 \\ 0.2, & -2 \le x < -1 \\ 0.5, & -1 \le x < 1 \\ 0.7, & 1 \le x < 3 \\ 1, & x \ge 3 \end{cases}$$

a) Find the probability function of X.

$$P(X=x) = F_X(x) - F_X(x^-) = \begin{cases} 0.2, & x = -2 \\ 0.3, & x = -1 \\ 0.2, & x = 1 \\ 0.3, & x = 3 \\ 0, & \text{otherwise} \end{cases}$$

b) Find the probability function of $Y = X^2$.

X	-2	-1	1	3
$Y = X^2$	4	1	1	9
$P(X=x)$	0.2	0.3	0.2	0.3

$$P(Y=y) = \begin{cases} 0.5, & y = 1 \\ 0.2, & y = 4 \\ 0.3, & y = 9 \\ 0, & \text{otherwi} \end{cases}$$

$$P(Y=y) = \begin{cases} 0.5, & y = 1 \\ 0.2, & y = 4 \\ 0.3, & y = 9 \\ 0, & \text{otherwise} \end{cases}$$

c) Compute $P\left(X > -\frac{1}{2} \,|\, X \le 3\right)$.

$$P(X > -1/2 \mid X \le 3) =$$

$$= \frac{P(-1/2 < X \le 3)}{P(X \le 3)}$$

$$= \frac{F_X(3) - F(-1/2)}{F_X(3)} = \frac{1 - 0.5}{1} = 0.5$$

b) Are the random variables independent? Justify your answer.

Yes, because

$$f_X(x) \times f_Y(y) = \begin{cases} 4xy, & 0<x<1 \\ & 0<y<1 \\ 0, & \text{otherwise} \end{cases} = f_{X,Y}(x,y) \quad \forall x,y$$

c) Compute $f_{X|Y=y}(x)$ and $F_{X|Y=y}(x)$.

Since X and Y are independent, then

$f_{X|Y=y}(x) = f_X(x) = 2x, \quad 0<x<1$

$$\int_{-\infty}^{x} f_X(u)\,du = \int_0^x 2u\,du = u^2 \Big|_0^x = x^2,$$

$$F_X(x) = \begin{cases} 0, & x<0 \\ x^2, & 0 \leq x < 1 \\ 1, & x \geq 1 \end{cases}$$

d) Calculate the median and the quantile of order 0.75 ($q_{0.75}$) of X.

$me(X) = \min\{x \in \mathbb{R} : F_X(x) \geq 0.5\}$

$F_X(me(X)) = 0.5 \Leftrightarrow x^2 = 0.5 \Leftrightarrow x = \sqrt{0.5}$ because $0<x<1$

$q_{0.75} = \min\{x \in \mathbb{R} : F_X(x) \geq 0.75\}$

$F_X(x) = 0.75 \Leftrightarrow x^2 = 0.75 \Leftrightarrow x = \sqrt{3/4}$ since $0<x<1$

This page can be used to complete your answers.