ISEG - Lisbon School of Economics and Management

Statistics I

2[nd] Semester of 2018/2019

Resit Exam

3 July 2019 Duration: 120 minutes

Name: ______________________________ Number: __________

Justify your answers carefully and present all the calculations you consider necessary.

Q	1	2a	2b	2c	2d	2e(i)	2e(ii)	3a	3b	3c	3b	4a	4b	4c	4d	5a	5b
Val	1	0.5	1	1	0.5	1	0.5	1	1	1	2	1	1	1	2	2	2.5

1. Let A and B be two events of a sample space S such that A and B are independent events. Prove that $\overline{A}$ and B are also independent events.

$P(B \cap \bar{A}) = P(B \setminus A) = P(B) - P(A \cap B)$

$= P(B) - P(A) \times P(B)$

because A and B are independent

$= P(B)(1 - P(A)) = P(B) \times P(\bar{A})$

then B and $\bar{A}$ are independent

2. Assume that, in a country, 51% of adults are males. Additionally, it is known that 80% of males use credit cards, whereas 90% of females use credit cards. One adult is randomly selected for a survey involving credit card usage.

   (a) Present the events and probabilities mentioned in the question regarding the experiment described.

M: "An adult is a male" $P(M) = 0.51$

C: "An adult use credit card" $P(C \mid M) = 0.8$

$\bar{M}$: "An adult is a female" $P(C \mid \bar{M}) = 0.9$

(b) Compute the probability that (i) the adult is a male that uses credit cards ~~and~~ (ii) the adult is a female that uses credit cards (iii) the adult is a male that does not use credit cards?

(i) $P(M \cap C) = P(C|M) \times P(M) = 0.8 \times 0.51 = 0.408$

(ii) $P(\bar{M} \cap C) = P(C|\bar{M}) \times P(\bar{M}) = 0.9 \times 0.49 = 0.441$

(iii) $P(M \cap \bar{C}) = P(M) - P(M \cap C) = 0.51 - 0.408 = 0.102$

(c) What is the probability that the randomly selected adult does not use credit cards?

$P(\bar{C}) = 1 - P(C) = 0.151$

$P(C) = P(C \cap M) + P(C \cap \bar{M})$

$= 0.408 + 0.441 = 0.849$

(d) Knowing that the adult uses credit cards, what is the probability that this adult is a male?

$$P(M|C) = \frac{P(M \cap C)}{P(C)} = \frac{0.408}{0.849} = 0.480565$$

(e) Consider the additional information: 50% of males that use credit cards have personal loans, and 10% of males that do not use credit cards have personal loans.

i) Compute the probability that the randomly selected adult is a male that has personal loans.

L: "An adult has personal loans"

$P(L|M\cap C) = 0.5 \quad P(L|M\cap \bar{C}) = 0.1$

$$P(M\cap L) = P(L\cap M\cap C) + P(L\cap M\cap \bar{C})$$
$$= P(L|M\cap C)\times P(M\cap C) + P(L|M\cap \bar{C})\times P(M\cap \bar{C})$$
$$= 0.5\times 0.408 + 0.1\times 0.102$$
$$= 0.2142$$

ii) Compute the probability that the adult has personal loans given that the adult is a male.

$$P(L|M) = \frac{P(M\cap L)}{P(M)} = \frac{0.2142}{0.51} = 0.42$$

3. Assume that X is a continuous random variable, such that its cumulative distribution function is given by

$$F_X(x) = \begin{cases} 0, & x < a \\ 1 - \left(\frac{a}{x}\right)^\beta, & x \geq a \end{cases},$$

with $a, \beta > 0$. We usually write $X \sim Pareto(a, \beta)$.

(a) Find the probability density function of X.

$$f_X(x) = F_X'(x) = \begin{cases} 0, & x < a \\ \beta \cdot \frac{a^\beta}{x^{\beta+1}}, & x \geq a \end{cases}$$

$$\left(1 - \left(\frac{a}{x}\right)^\beta\right)' = -\beta\left(\frac{a}{x}\right)^{\beta-1} \times \left(-\frac{a}{x^2}\right) = +\beta\,\frac{a^\beta}{x^{\beta+1}}$$

(b) Compute $P(X > c \mid X > b)$, for $a < b < c$.

$$P(X > c \mid X > b) = \frac{P(X > c)}{P(X > b)} = \frac{1 - F_X(c)}{1 - F_X(b)}$$

$$= \frac{\left(\frac{a}{c}\right)^\beta}{\left(\frac{a}{b}\right)^\beta} = \left(\frac{b}{c}\right)^\beta$$

(c) Compute the median of X.

$$me(X) = \min\{x > a : F_X(x) \geq 0.5\}$$

Since X is a continuous R.V. we have that

$$1 - \left(\frac{a}{x}\right)^\beta = 0.5 \iff \left(\frac{a}{x}\right)^\beta = 0.5 \iff \frac{a}{x} = 0.5^{1/\beta}$$

$$\iff x = a \times 0.5^{-1/\beta}$$ Then $me(X) = a \times 0.5^{-1/\beta} = a \times 2^{1/\beta}$

(d) Consider the two independent random variables X_1 and X_2 such that

$$X_i \sim Pareto(a, \beta), \text{ with } i = 1, 2.$$

Prove that $Y = \min\{X_1, X_2\} \sim Pareto(a, 2\beta)$.
Hint: Start by computing $P(Y > y)$.

To characterize Y one needs to compute the CDF of Y.

$$F_Y(y) = P(Y \leq y) = 1 - P(Y > y)$$

$$P(Y > y) = P(\min\{X_1, X_2\} > y) = P(X_1 > y, X_2 > y)$$

$$= P(X_1 > y)\, P(X_2 > y) = (1 - F_{X_1}(y))(1 - F_{X_2}(y))$$

due to the independence

$$= \left(\frac{a}{y}\right)^\beta \left(\frac{a}{y}\right)^\beta = \left(\frac{a}{y}\right)^{2\beta}, \quad \text{if } y > a$$

Then $$F_Y(y) = \begin{cases} 1 - \left(\frac{a}{y}\right)^{2\beta}, & y > a \\ 0, & y \leq a \end{cases}$$

consequently $Y \sim Pareto(a, 2\beta)$

4. Let (X, Y) be a two dimensional random variable, such that its set of discontinuities is $D_{X,Y} = \{(0,1), (0,2), (1,0), (1,1), (1,2)\}$ and its density probability function is

$$f_{X,Y}(x,y) = \begin{cases} \frac{x+y}{a}, & (x,y) \in D_{X,Y} \\ 0, & \text{otherwise} \end{cases}.$$

(a) Find a and represent $f_{X,Y}$ by filling a suitable table.

$f_{X,Y}(x,y) \geq 0 \quad \forall (x,y) \in D_{X,Y} \quad \Rightarrow \quad a > 0$

$\sum_{(x,y) \in D_{X,Y}} f_{X,Y}(x,y) = 1 \iff \frac{9}{a} = 1 \iff a = 9$

X \ Y	0	1	2
0	0	1/9	2/9
1	1/9	2/9	3/9

(b) Compute $E(X)$ and $E(Y)$.

$E(X) = \sum_{x=0}^{1} \sum_{y=0}^{2} x P(X=x, Y=y) = 1 \times (1/9 + 2/9 + 3/9)$

$= 6/9 = 2/3$

$E(Y) = \sum_{x=0}^{1} \sum_{y=0}^{2} y P(X=x, Y=y) = 1 \times (1/9 + 2/9) + 2(2/9 + 3/9)$

$= 13/9$

(c) Calculate $Cov(2X, 3Y)$.

$Cov(2X, 3Y) = 2 \times 3 \, Cov(X, Y)$

$Cov(X, Y) = E(XY) - E(X)E(Y)$

$E(XY) = \sum_{x=0}^{1} \sum_{y=0}^{2} xy \, P(X=x, Y=y)$

$= 1 \times 1 \times 2/9 + 1 \times 2 \times 3/9 = 8/9$

$Cov(2X, 3Y) = 6 \, Cov(X, Y) = 6\left(8/9 - 2/3 \times 13/9\right)$

$= -4/9$

(d) Compute $f_{X|Y=1}(x)$ and $E(X|Y=1)$.

$$f_{X|Y=1}(x) = \frac{P(X=x, Y=1)}{P(Y=1)} = \begin{cases} \frac{1/9}{3/9} & , x=0 \\ \frac{2/9}{3/9} & , x=1 \\ 0 & , \text{otherwise} \end{cases}$$

$$= \begin{cases} 1/3 & , x=0 \\ 2/3 & , x=1 \\ 0 & , \text{otherwise} \end{cases}$$

$$P(Y=1) = \sum_{x=0}^{1} P(X=x, Y=1) = 3/9$$

$$E(X|Y=1) = \sum_{x=0}^{1} x\, P(X=x|Y=1) = 0 \times 1/3 + 1 \times 2/3$$
$$= 2/3$$

5. Assume that, in a country, the number of cruise ship breakdowns per month reported to an insurance company follows a Poisson distribution with expected value 10.

(a) What is the probability that, in a random month, the insurance company will receive 8 reports?

X represents the number of cruise ship breakdowns per month reported to an insurence company

$X \sim \text{Poisson}(\lambda)$ and $\lambda = E(X) = 10$

$$P(X=8) = \frac{e^{-10}\, 10^8}{8!} =$$

(b) What is the probability that, in 4 years, the insurance company will receive more than 500 reports? (If necessary, assume independence and compute an approximated probability.)

X_i represents the number of cruise ship breakdowns in month i, with $i = 1, 2, \dots, 48$

$$T_{48} = \sum_{i=1}^{48} X_i \quad \text{with} \quad X_i \sim \text{Poisson}(10)$$

& independent R.V.

Therefore, T_{48} represents the number of reports received by the insurance company in 4 years

From the central limit theorem:

$$T_{48} \overset{a}{\sim} N(\mu, \sigma^2) \qquad \text{where} \quad \mu = 48 \times 10 = 480$$

$$\sigma^2 = 48 \times 10 = 480$$

Thus,

$$Z = \frac{T_{48} - 480}{\sqrt{480}} \overset{a}{\sim} N(0,1)$$

$$P(T_{48} > 500) \simeq P\left(Z > \frac{500 - 480}{\sqrt{480}}\right)$$

$$\simeq 1 - \Phi(0.91) = 1 - 0.8186$$

$$= 0.1844$$

This page can be used to complete your answers.