University of Lisbon – ISEG
Departament of Mathematics

Probability Theory and Stochastic Processes

EXAM January 15, 2016

Time limit: 2 hours

Each question: 2.5 points

(1) Consider a set Ω, a function $f\colon \Omega \to \Omega$ and

$$\mathcal{F} = \{A \subset \Omega\colon f^{-1}(A) = A\}.$$

(a) Show that $(\Omega, \mathcal{F})$ is a measurable space.

(b) Consider a measure μ on $(\Omega, \mathcal{F})$ and $A, B \in \mathcal{F}$ disjoint sets. Find

$$\int_{f^{-1}(B)} \mathcal{X}_A \circ f \, d\mu,$$

where $\mathcal{X}_A$ is the indicator function for the set A.

(2) Compute

$$\lim_{n\to+\infty} \int_0^1 \frac{1}{\sqrt{t}} e^{-t/n} \, dt.$$

(3) Given a sequence of i.i.d. random variables $X_1, X_2 \ldots$ with uniform distribution on $[0,1]$, determine

$$\lim_{n\to+\infty} \sqrt[n]{X_1 \ldots X_n}$$

with probability 1.

(4) Let $(\Omega, \mathcal{B}, m)$ be a probability space, where $\Omega = [0,1]$, $\mathcal{B}$ is the Borel σ-algebra of Ω and m is the Lebesgue measure on Ω. Given the random variables $X(\omega) = \omega$ and

$$Y(\omega) = \begin{cases} 2\omega, & 0 \leq \omega \leq \frac{1}{2} \\ 2\omega - 1, & \frac{1}{2} < \omega \leq 1, \end{cases}$$

compute $E(X|Y)$.

(5) On the finite state space $S = \{1, 2, \ldots, a\}$ consider a homogeneous Markov chain X_n on S with probabilities

$$P(X_1 = j | X_0 = i) = \begin{cases} \frac{1}{2}, & j = i \\ \frac{1}{2}, & j = i + 1 \text{ or } (i, j) = (a, 1). \end{cases}$$

(a) Classify the states of the chain and determine their periods.
(b) If possible, find the stationary distributions and the mean recurrence time of each state.

(6) Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{F}_n$ a filtration. Suppose that $(X_n, \mathcal{F}_n)$ and $(Y_n, \mathcal{F}_n)$ are martingales and T is a stopping time with respect to $\mathcal{F}_n$ and $X_T = Y_T$. Is

$$Z_n = \begin{cases} X_n, & n < T \\ Y_n, & n \geq T \end{cases}$$

a martingale with respect to $\mathcal{F}_n$?

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Departament of Mathematics

Probability Theory and Stochastic Processes

EXAM February 1, 2016

Time limit: 2 hours

Each question: 2.5 points

(1) Let $(\Omega, \mathcal{F}, P)$ be a probability space.

   (a) Let $A, B \in \mathcal{F}$. If $P(A) = 1$, find $P(B) - P(B \cap A)$.

   (b) Consider a random variable X that can only take two values $a, b \in \mathbb{R}$. Write $\sigma(X)$.

   (c) Consider a function $g \colon \Omega \to \mathbb{R}$ and σ-algebras $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}$ such that g is $\mathcal{F}_2$-mensurable. Is g also $\mathcal{F}_1$-mensurable?

(2) Compute

$$\lim_{n \to +\infty} \int_0^n \sin(e^{-x}) e^{-nx} \, dx$$

(3) Consider a homogeneous Markov chain with transition matrix given by

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

   (a) Classify the states of the chain and determine their periods.

   (b) If possible, find the stationary distributions and the mean recurrence time of each state.

(4) Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X_1, X_2, \ldots$ a sequence of iid random variables with distribution

$$P(X_n = 1) = \frac{1}{2} \quad \text{and} \quad P(X_n = -1) = \frac{1}{2}.$$

Consider the stopping time

$$\tau = \min\{n \in \mathbb{N} \colon X_n = 1\}$$

with respect to the filtration $\sigma(X_1, \ldots, X_n)$.

(a) Decide if $X_{\tau \wedge n}$ is a martingale, where $\tau \wedge n = \min\{\tau, n\}$.

(b) Let $S_n = \sum_{i=1}^{n} 2^i X_i$. Compute $E(S_{\tau-1})$.

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Departament of Mathematics

Probability Theory and Stochastic Processes

EXAM January 18, 2017

Time limit: 2 hours

Each question: 2.5 points

(1) Consider the probability space $(\mathbb{R}, \mathcal{P}, \delta_a)$, where δ_a is the Dirac measure on $\mathbb{R}$ at $a = 2$, and a random variable $X(x) = \sqrt{|x|}$.
  (a) Find the distribution and characteristic functions of X.
  (b) Write an example of a random variable Y with the same distribution of X.

(2) For each $n \in \mathbb{N}$ consider a random variable X_n with distribution function

$$F_n(x) = \begin{cases} 0, & x \leq 0 \\ nx, & 0 < x \leq \frac{1}{n} \\ 1, & x > \frac{1}{n}. \end{cases}$$

Find the limit in distribution of X_n as $n \to +\infty$.

(3) Consider a homogeneous Markov chain with transition matrix given by

$$P = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & 0 & \frac{3}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

  (a) Classify the states of the chain.
  (b) Determine the period of each state.
  (c) If possible, find the stationary distributions and the mean recurrence time of each state.

(4) Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X_1, X_2, \dots$ a sequence of iid random variables with distribution

$$P(X_n = 1) = \frac{1}{2} \quad \text{and} \quad P(X_n = -1) = \frac{1}{2}.$$

Consider the stopping time

$$\tau = \min\{n \in \mathbb{N}\colon X_n = 1\}$$

with respect to the filtration $\sigma(X_1, \dots, X_n)$.

(a) Decide if $X_{\tau \wedge n}$ is a martingale, where $\tau \wedge n = \min\{\tau, n\}$.

(b) Let $S_n = \sum_{i=1}^{n} 2^i X_i$. Compute $E(S_{\tau-1})$.

University of Lisbon – ISEG

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Probability Theory and Stochastic Processes

EXAM February 3, 2017

Time limit: 2 hours

Each question: 2.5 points

(1) Consider the probability space $(\mathbb{R}, \mathcal{B}, m)$, where m is the Lebesgue measure on $[0,1]$, and the random variable $X(x) = 2x$.

(a) Find the distribution and characteristic functions of X.

(b) Write an example of a random variable Y with the same distribution of X.

(2) Let δ_a be the Dirac measure on $\mathbb{R}$ at a. Consider the sequences

$$a_n = \frac{1-(-1)^n}{2}, \quad n \in \mathbb{N}.$$

and

$$\mu_n = \frac{1}{n}\sum_{i=1}^{n} \delta_{a_i}, \quad n \in \mathbb{N}.$$

Show that μ_n is a probability measure for each $n \in \mathbb{N}$ and compute

$$\lim_{n\to+\infty} \int \mathcal{X}_{\{0\}}\, d\mu_n$$

where $\mathcal{X}_{\{0\}}$ is the indicator function at 0.

(3) Consider a homogeneous Markov chain with transition matrix given by

$$P = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}.$$

(a) Classify the states of the chain.

(b) Determine the period of each state.

(c) If possible, find the stationary distributions and the mean recurrence time of each state.

(4) Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X_1, X_2, \ldots$ a sequence of iid random variables with distribution

$$P(X_n = 1) = \frac{2}{3},$$
$$P(X_n = -1) = \frac{1}{3}.$$

Consider the sum

$$S_n = \sum_{i=1}^{n} X_i.$$

(a) Determine if $Y_n = 2^{-S_n}$ is a martingale with respect to the filtration $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$.

(b) Let τ be the stopping time given by

$$\tau = \min\{n \geq 1\colon S_n \in \{-1, 2\}\}.$$

Compute the expected value of Y_τ, the probability of $Y_\tau = 1/4$ and the probability of $S_\tau = 2$.

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Probability Theory and Stochastic Processes

EXAM January 17, 2018

Time limit: 2 hours

Each question: 2.5 points

(1) Consider the probability space $([0,1], \mathcal{B}([0,1]), P)$, where

$$P(A) = \int_A 2x\, dx, \quad A \in \mathcal{B}([0,1]),$$

and the random variable $X(x) = x^2 - 1$.

(a) Find the distribution of X and its characteristic function.

(b) Write an example of a random variable Y with the same distribution of X.

(2) Given $a \in \mathbb{R}$, consider the Dirac measure on $\mathbb{R}$:

$$\delta_a(A) = \begin{cases} 1, & a \in A \\ 0, & a \notin A \end{cases}$$

for any $A \subset \mathbb{R}$, and $\mu = \frac{1}{2}(\delta_1 + \delta_2)$

(a) Show that $\mu = \frac{1}{2}(\delta_1 + \delta_2)$ is a probability measure and that

$$\int f\, d\mu = \frac{1}{2}\left(\int f\, d\delta_1 + \int f\, d\delta_2\right)$$

for any function $f \colon \mathbb{R} \to \mathbb{R}$.

(b) Compute the expected value of $X(x) = 1/x$ with respect to μ.

(3) Consider a homogeneous Markov chain with states $\{1, 2, 3, 4\}$ and transition matrix

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} \end{bmatrix}.$$

(a) Classify the states of the chain and determine their periods.

(b) If possible, find the stationary distributions and the mean recurrence time of each state.

(c) Compute

$$\lim_{n\to+\infty} P(X_n = 1 | X_0 = 2).$$

(4) Let X_n be a martingale with respect to the filtration $\mathcal{F}_n$ and τ is a stopping time. Determine $E(X_{\tau\wedge n})$, where $\tau \wedge n = \min\{\tau, n\}$.

University of Lisbon – ISEG
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Probability Theory and Stochastic Processes

EXAM February 2, 2018

Time limit: 2 hours

Each question: 2.5 points

(1) (a) Let Ω be an infinite set and $\mathcal{A}$ the collection of all finite subsets of Ω. Is $\mathcal{A}$ a σ-algebra?
(b) Let Ω be any set and $\mathcal{A} = \{\{x\}: x \in \Omega\}$. Determine the σ-algebra generated by $\mathcal{A}$.

(2) Let $(\Omega, \mathcal{F}, P)$ be a probability space and X, Y independent random variables. Show that:
(a) $E(XY) = E(X)E(Y)$.
(b) $Var(X+Y) = Var(X) + Var(Y)$.

(3) Consider a homogeneous Markov chain with states $\{1, 2, 3, 4\}$ and transition matrix

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) Classify the states of the chain and determine their periods.
(b) If possible, find the stationary distributions and the mean recurrence time of each state.
(c) Compute

$$\lim_{n\to+\infty} P(X_n = 1 | X_0 = 2).$$

(4) Let X_n be a martingale with respect to the filtration $\mathcal{F}_n$ and τ is a stopping time. Determine $E(X_{\tau\wedge n})$, where $\tau \wedge n = \min\{\tau, n\}$.

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Probability Theory and Stochastic Processes

EXAM January 21, 2019

Time limit: 2 hours

Each question: 2.5 points

(1) Consider a measure space $(\Omega, \mathcal{F}, \mu)$ and a σ-subalgebra $\mathcal{A} \subset \mathcal{F}$. Let f, g, h be $\mathcal{F}$-measurable functions and h be also $\mathcal{A}$-measurable. Are the following propositions true? If not, write examples that contradict the statements.

- If $\int_B f\, d\mu = \int_B g\, d\mu$ for every $B \in \mathcal{F}$, then $f = g$ a.e.
- If $\int_A f\, d\mu = \int_A h\, d\mu$ for every $A \in \mathcal{A}$, then $f = h$ a.e.

(2) Given a random variable X with distribution function

$$F(x) = \begin{cases} 0, & x < 0 \\ x/2, & 0 \le x < 1 \\ 1/2, & 1 \le x < 2 \\ 1, & x \ge 2 \end{cases}$$

compute:

(a) $P(1/4 \le X^2 < 4)$

(b) the distribution function of $Y = \sqrt{X}$.

(3) For an iid sequence of random variable $X_1, X_2, \dots$ denote by S_n the sum of the n first terms, i.e.

$$S_n = \sum_{i=1}^{n} X_i.$$

Suppose that the distribution of each X_i is $P(X_i = -1) = p$ and $P(X_i = 1) = 1 - p$ where $0 < p < 1$.

(a) What are the characteristic functions of the random variables S_n, S_n/n? Find also the limit distribution of S_n/n.

(b) Decide if S_n is a martingale with respect to the filtration $\sigma(X_1, \ldots, X_n)$.

(c) Find the expected value of the stopping time

$$\tau = \{n \in \mathbb{N} \colon S_n = 1\}.$$

(d) Compute $P(\tau = 5 | X_2 = 1)$.

(4) Write an example of a finite homogeneous Markov chain with two stationary distributions.

University of Lisbon – ISEG

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Probability Theory and Stochastic Processes

EXAM February 6, 2019

Time limit: 2 hours

Each question: 2.5 points

(1) Let $(\Omega, \mathcal{F}, P)$ be a probability space and X, Y independent random variables. Show that:
 (a) $E(XY) = E(X)E(Y)$.
 (b) $Var(X + Y) = Var(X) + Var(Y)$.

(2) Given a random variable X with distribution function

$$F(x) = \begin{cases} 0, & x < 0 \\ x/6, & 0 \leq x < 3 \\ 1/2, & 3 \leq x < 4 \\ 1, & x \geq 4 \end{cases}$$

and $Y = \sqrt{X}$, compute:
 (a) $P(1/4 \leq X^2 < 16)$, $E(X)$ and $Var(X)$.
 (b) the distribution function of Y.
 (c) $E(XY)$ and $Var(XY)$.

(3) Consider a simplified weather model described in the following way: the probability of a rainy day being followed by a sunny day is 0.5, and the probability of a sunny day being followed by another day with sunshine is 0.7. If today is raining how long should I wait on average in order to have another day with rain?

(4) Let $(\Omega, \mathcal{F}, P)$ be a probability space and X_n a sequence of iid random variables with distribution given by

$$P(X_n = 0) = p, \qquad P(X_n = 1) = 1 - p$$

for some $0 < p < 1$. Consider the stochastic process

$$S_n = \sum_{i=1}^{n} X_i,$$

the filtration $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ and the stopping time

$$\tau = \min\{n \in \mathbb{N} \colon S_n = 10\}.$$

(a) Is S_n a martingale?
(b) Determine $P(\tau = +\infty)$ and $E(\tau)$.

University of Lisbon – ISEG

Department of Mathematics

Probability Theory and Stochastic Processes

EXAM January 9, 2020

Time limit: 2 hours

Each question: 2.5 points

(1) Consider a set Ω, a function $f\colon \Omega \to \Omega$ and

$$\mathcal{F} = \{A \subset \Omega \colon f^{-1}(A) = A\}.$$

(a) Show that $(\Omega, \mathcal{F})$ is a measurable space.

(b) Consider a measure μ on $(\Omega, \mathcal{F})$ and $A, B \in \mathcal{F}$ disjoint sets. Find

$$\int_{f^{-1}(B)} \mathcal{X}_A \circ f \, d\mu,$$

where $\mathcal{X}_A$ is the indicator function for the set A.

(2) Consider a probability space $(\Omega, \mathcal{F}, P)$ and a sequence of iid random variables X_n with Poisson distribution[1] given by

$$P(X_n = k) = \frac{\mu^k}{k!} e^{-\mu}, \quad k \in \{0, 1, 2, \dots\},$$

where $\mu > 0$. Let $Y_0 = 0$ and

$$Y_n = Y_{n-1} + X_n - 1, \quad n \in \mathbb{N}.$$

(a) Compute $E(Y_n)$, $E(2^{Y_n})$ and $P(Y_2 = 1 | X_1 = 0)$.

(b) Determine if Y_n and 2^{Y_n} are martingales with respect to the natural filtration $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$.

(c) Let $\mu = 1$ and consider the stopping time

$$\tau = \min\{n \in \mathbb{N} \colon Y_n \in \{-1, 2\}\}.$$

Compute $P(\tau < +\infty)$ and $E(Y_\tau)$.

[1] Recall that for any $x \in \mathbb{R}$,

$$e^x = \sum_{k=0}^{+\infty} \frac{x^k}{k!}$$

(3) Consider the Markov chain with the following transition probabilities matrix

$$T = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}.$$

(a) For which values of a and b is the chain aperiodic? And to possess an absorving state?

(b) For which values of a and b does the chain have at least one stationary distribution? And to have exactly one stationary distribution?

(4) Prove that for an irreducible Markov chain with N states it is possible to go from any state to any other state in at most $N-1$ steps.

University of Lisbon – ISEG

Department of Mathematics

Probability Theory and Stochastic Processes

EXAM February 4, 2020

Time limit: 2 hours

Each question: 2.5 points

(1) Give an example of a function $f\colon \mathbb{R} \to \mathbb{R}$ that is measurable with respect to the σ-algebra $\mathcal{F} = \{\emptyset, \mathbb{R}, \mathbb{R}_0^+, \mathbb{R}^-\}$.

(2) Given the probability space $([0,1], \mathcal{B}, m)$ where $\mathcal{B}$ is the Borel σ-algebra on $[0,1] \subset \mathbb{R}$ and m is the Lebesgue measure, take the sequence of random variables $X_n\colon [0,1] \to \mathbb{R}$,

$$X_n(x) = \begin{cases} 0, & x \in \mathbb{Q} \\ 1 - \frac{nx^2}{n^2+1}, & \text{o.c.} \end{cases}$$

Compute the pointwise limit of X_n and the limit of $E(X_n)$.

(3) Consider a probability space $(\Omega, \mathcal{F}, P)$ and a sequence of iid random variables X_n with Poisson distribution[1] given by

$$P(X_n = k) = \frac{\mu^k}{k!} e^{-\mu}, \quad k \in \{0, 1, 2, \ldots\},$$

where $\mu > 0$. Let $Y_0 = 0$ and

$$Y_n = Y_{n-1} + X_n - 1, \quad n \in \mathbb{N}.$$

(a) Compute $E(Y_n)$, $E(2^{Y_n})$ and $P(Y_2 = 1 | X_1 = 0)$.
(b) Determine if Y_n and 2^{Y_n} are martingales with respect to the natural filtration $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$.

[1] Recall that for any $x \in \mathbb{R}$,

$$e^x = \sum_{k=0}^{+\infty} \frac{x^k}{k!}$$

(c) Let $\mu = 1$ and consider the stopping time

$$\tau = \min\{n \in \mathbb{N} \colon Y_n \in \{-1, 2\}\}.$$

Compute $P(\tau < +\infty)$ and $E(Y_\tau)$.

(4) Consider a homogeneous finite Markov chain with the following transition probabilities matrix:

$$T = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

(a) Classify the states of the chain and determine their periods.

(b) If possible, find the stationary distributions and the mean recurrence time of each state.

(c) Compute

$$\lim_{n \to +\infty} P(X_n = 1 | X_0 = 4).$$