Probability Theory and Stochastic Processes

LIST 5

Markov chains

(1) Prove that the following propositions are equivalent[1]
   (a) P is a stochastic $n \times n$ matrix (i.e. all its coefficients are in $[0, 1]$ and their sum over each row is equal to 1).
   (b) For any vector $v \geq 0$ in $\mathbb{R}^n$ we have $Pv \geq 0$ and $P(1, \dots, 1) = (1, \dots, 1)$.
   (c) For any probability vector $v \in \mathbb{R}^n$ (i.e. $v \geq 0$ and $\sum_{i=1}^{r} v_i = 1$), we have that vP is also a probability vector.

(2) Show that the product of two stochastic matrices is also a stochastic matrix.

(3) (Chapman-Kolmogorov equations) Consider a homogeneous Markov chain with transition matrix P. Show that the n-step transition matrix is P^n. *Hint*: Recall the following facts: $P(A \cap B|C) = P(A|B \cap C)P(B|C)$ and $P(A|B) = \sum_n P(A \cap C_n|B)$ where $C_1, C_2, \dots$ is a sequence of disjoint sets whose union has probability 1.

(4) Consider a homogeneous Markov chain on the state space $S = \mathbb{N}$ given by the transition probabilities:

$$P(X_1 = i|X_0 = i) = r, \qquad i \geq 2,$$
$$P(X_1 = i-1|X_0 = i) = 1 - r, \qquad i \geq 2,$$
$$P(X_1 = j|X_0 = 1) = \frac{1}{2^j}, \qquad j \geq 1.$$

   Classify the states of the chain and find their mean recurence times:
   (a) using the stationary distribution.
   (b) * computing the probability of first return after n steps.

(5) Let $0 < p < 1$. Classify the states of the Markov chains given by the transition matrices below, and determine the mean recurrence times of each state.

[1]Given a vector $v \in \mathbb{R}^n$ we use the notation $v \geq 0$ to mean that each coordinate v_i of v is ≥ 0.

(a)

$$\begin{bmatrix} 1-2p & 2p & 0 \\ p & 1-2p & p \\ 0 & 2p & 1-2p \end{bmatrix}$$

(b)

$$\begin{bmatrix} 0 & p & 0 & 1-p \\ 1-p & 0 & p & 0 \\ 0 & 1-p & 0 & p \\ p & 0 & 1-p & 0 \end{bmatrix}$$