Instituto Superior de Economia e Gestão
UNIVERSIDADE TÉCNICA DE LISBOA

Mathematical Economics
Exam – 14/01/08
Duration: 3h00

NOTE: Answer each group in separate sheets. Justify clearly all answers.

I

1. **(2.0)** Consider the following functions:

$$\begin{aligned} \Phi(u,v) &= \left(u-v, f\left(u^2\right)\right) \\ \Omega\left(s,t\right) &= \left(s, s+t\right), \end{aligned}$$

where f is C^2 in R

(a) Write the Jacobian Matrices of Φ and Ω.

(b) Now, consider the composition of the two functions $w\left(u,v\right) = \left(\Omega \circ \Phi\right)\left(u,v\right)$. Construct explicitly the function $w = \left(w_1, w_2\right)$.

2. **(2.0)** Consider the following function

$$f\left(x\right) = \left(x+1\right)e^{-x}.$$

Compute the integral using integration by parts,

$$\int f\left(x\right)dx.$$

3. **(2.0)** Without using the definition of homogeneity show that the function $f\left(x,y\right) = \frac{x^2}{y}$ is homogeneous and indicate the degree of homogeneity.

II

1. **(3.5)** Consider the following correspondence $\varphi(x)$ where x is a real number and $\varphi\left(x\right)$ is a set of real numbers.

a) $0 \leq x < 1$, $\varphi(x) = \{x+2\}$

b) $x = 1$, $\varphi(x) = [0,5]$

c) $5 \geq x > 1$, $\varphi(x) = \{x-1\}$

Can the Theorem of Kakutani be used to assure that the correspondence has a fixed point? Justify your answer. Now consider that a) and c) remain the same and

b) $x = 1$, $\varphi(x) = \{0, 4, 5\}$. Can the Theorem of Kakutani be used to assure that the correspondence has a fixed point? Justify.

2. **(3.5)** Consider the following excess demand functions in a Walrasian model with two goods 1 and 2. p_1 and p_2 are respectively the prices of good 1 and 2.

$$\begin{aligned} Z_1(p_1, p_2) &= p_1 - p_2^2 - ap_1p_2 \\ Z_2(p_1, p_2) &= 3p_1^2 + p_1p_2 - \frac{p_1^2}{p_2} \end{aligned}$$

   (a) Without computing the equilibrium prices, indicate the value of a which guarantees the existence of those prices and explain why.

   (b) After answering the previous question, compute the equilibrium prices.

III

1. **(2.5)** Consider a continuous time version of a two-state Markov process $\dot{y} = My$, where the transition matrix is

$$M = \begin{pmatrix} -\pi_1 & \pi_1 \\ \pi_2 & -\pi_2 \end{pmatrix}$$

for $0 < \pi_1 < 1$ and $0 < \pi_2 < 1$

   (a) solve the differential equation;

   (b) let $y(0) = (0, 1)$. Solve the initial value problem;

   (c) draw the phase diagram associated to the initial value problem.

2. **(2.5)** Assume that that a consumer has an endowment denoted by W_t at time $t \in \{0, 1, \ldots, T\}$. The horizon T is finite. The endowment evolves over time as $W_{t+1} = (1 + r)W_t - C_t$, where C_t is the amount of the endowment consumed at time t and $r > 0$ is a parameter. Assume that $W_0 = \phi > 0$ and that the consumer wants to have $W_T = \phi$. The consumer has a psychological discount factor $0 < \beta < 1$ and a static logarithmic utility function.

   (a) Transform the problem into a calculus of variations problem and determine the Euler-Lagrange equation.

   (b) Solve the problem.Consider a continuous time version of a two-state Markov process $\dot{y} = My$, where the transition matrix is

$$M = \begin{pmatrix} -\pi_1 & \pi_1 \\ \pi_2 & -\pi_2 \end{pmatrix}$$

   for $0 < \pi_1 < 1$ and $0 < \pi_2 < 1$

(c) solve the differential equation;

(d) let $y(0) = (0, 1)$. Solve the initial value problem;

(e) draw the phase diagram associated to the initial value problem.

3. **(2.0)** Assume that that a consumer has an endowment denoted by W_t at time $t \in \{0, 1, \ldots, T\}$. The horizon T is finite. The endowment evolves over time as $W_{t+1} = (1 + r)W_t - C_t$, where C_t is the amount of the endowment consumed at time t and $r > 0$ is a parameter. Assume that $W_0 = \phi > 0$ and that the consumer wants to have $W_T = \phi$. The consumer has a psychological discount factor $0 < \beta < 1$ and a static logarithmic utility function.

   (a) Transform the problem into a calculus of variations problem and determine the Euler-Lagrange equation.

   (b) Solve the problem.

Instituto Superior de Economia e Gestão
UNIVERSIDADE TÉCNICA DE LISBOA

Mathematical Economics

Exam – 30/01/08 Duration: 3h00

NOTE: Answer each group in separate sheets. Justify clearly all answers

I

1. **(2,0)** Study the definiteness of matrix A:

$$A = \begin{bmatrix} 1 & 4 & -1 \\ -1 & 8 & 3 \\ -1 & 4 & 2 \end{bmatrix}$$

2. **(2,0)** Using the chain rule, compute $\frac{\partial z}{\partial t}$, at $t = 0$ for:

$$z(t, x, w) = \frac{5t^2 + 3x}{2w^2}, x(t) = t^2 + 1, w(t) = e^t + 1.$$

3. **(3,0)** Consider the following problem:

$$\min_{x,y} \; U = x^2 + (y - x)^2$$
$$s.t. \; x - 2y = b$$

   (a) Solve the minimization problem.

   (b) Construct the function $U^*(b)$ consisting on the maximum value of U for each b.

   (c) Compute $U^{*\prime}(b)$ and relate this value to the Lagrange multiplier. Justify carefully your answer.

II

1. **(3.5)** Consider the following two sets A and B of points of R^2

$$A = [01] \times [a\ 5]$$
$$B = \{(x, y) : x^2 + y^2 < 1\}$$

("$\times$" means Cartesian product of the two intervals and a is unknown)

   (a) Find the set of values of a that assures the existence of a hyperplane separating A and B. Justify

   (b) Choose one value for a and find one of those hyperplanes for that value of a.

2. **(3.5)** Consider the following correspondence, where $\varphi(x)$ are sets corresponding to each x.

$$a)\ -1 \leq x < 0\ \varphi(x) = \{(x+2)/3\}$$
$$b)\ x = 0\ \varphi(x) = [-13]$$
$$c)\ 0 < x \leq 4\ \varphi(x) = \{(x-1)/3\}$$

Show that the theorem of Kakutani can be used to assure the existence of a fixed point. Find a fixed point.

III

1. **(2.0)** Consider the ode $\dot{y}\hat{E} = -1 + \lambda y$ where $\lambda > 0$.

   (a) Solve the differential equation.

   (b) Consider the terminal condition $\lim_{t\to\infty} e^{-\lambda t} y(t) = 0$. Solve the terminal value problem.

2. **(2.0)** Let $y_t \in \mathbb{R}^2$ and consider the planar difference equation $y_{t+1} = Ay_t + B$ for $A = \begin{pmatrix} -1 & 1/2 \\ 1/2 & -1 \end{pmatrix}, whereB = (1,0)$.

   (a) Solve the difference equation;

   (b) Assume that $y_{1,0} = 3/15$ and that $\lim_{t\to\infty} y_{2t} = \overline{y}_{2t}$, where $\overline{y}_{2t}$ is the steady state level for y_{2t}. Determine the solution of the initial-terminal value problem.

3. **(2.0)** Assume that a consumer has an endowment $W(t)$ at time $t \in [0, T]$, where T is finite. He/she wants to consume it totally until time t, such that $W(T) = 0$. The endowment accumulates according to the equation $\dot{W} = C(t) - rW(t)$ where $r > 0$ and is constant. Initially $W(0) = \phi > 0$. The consumer has a psychological rate of time preference $\rho > 0$ and a static logarithmic utility function.

   (a) Determine the first order conditions from the Pontryiagin's maximum principle.

   (b) Solve the problem.

Instituto Superior de Economia e Gestão
UNIVERSIDADE TÉCNICA DE LISBOA

Economia Matemática

Exame – 07/01/09   Duração: 2h30

NOTA: Responda a cada grupo em **folhas separadas**. Justifique claramente todas as respostas.

Grupo **I** (**5 val**)

1. Considere a matriz $B = \begin{bmatrix} 1 & -2 \\ -3 & 2 \end{bmatrix}$.

   (a) (2.0 val.) Calcule a inversa da matriz B.

   (b) (3.0 val.) Proceda à diagonalização da matrix B. Calcule B^3.

Grupo **II** (**7 val**)

1. (4.0 val) Considere a seguinte correspondência de [0 5] em [0 5] :

$$\begin{aligned} 0 &\leq x < 2 \quad \varphi(x) = \{x + 0,2; x + 0,4\} \\ x &= 2 \quad \varphi(x) = [a\ b] \\ 2 &< x \leq 5 \quad \varphi(x) = \{x - 1\} \end{aligned}$$

   (a) (2.0 val) Indique, **justificando** a resposta, dois valores possíveis para a e b que garantam que a correspondência é semicontínua superior em todos os pontos de [0 5].

   (b) (2.0 val) Nesse caso pode ser utilizado o teorema de Kakutani para provar a existência de um ponto fixo? **Justifique**.

   NOTE BEM : O símbolo $\{u; v\}$ representa o conjunto de dois elementos, u e v e não o intervalo de extremidades u e v.

2. (3.0 val) Considere um espaço R^2 e os seguintes conjuntos A, B, C e D

$$\begin{aligned} A &= [1\ 3] \\ B &= (0\ 5) \\ C &= \{(x, y) \text{ de } R^2 \text{ tais que } (x-4)^2 + (y-3)^2 \leq r^2\} \\ D &= A \times B \end{aligned}$$

   (a) (1.5 val) Indique um valor de r que permita garantir que existe um hiperplano a separar C de D.

(b) (1.0 val) Indique as razões que lhe permitem justificar a resposta à questão anterior.

(c) (0.5 val) Apresente a equação de um hiperplano separador.

NOTE BEM: O símbolo "×" representa o produto cartesiano de conjuntos e $(a\ b)$ representa o intervalo aberto de extremidades a e b.

Grupo III (8 val.)

1. (2.0 val) A taxa de rendimento de uma acção, é igual à taxa de variação da sua cotação, $\dot{p}/p$, mais o ratio entre o dividendo e a cotação , $d/p(t)$. Em equilíbrio, com ausência de oportunidades de arbitragem e previsão perfeita, a taxa de rendimento da acção deverá ser igual à taxa de juro do mercado, r, que se admite constante.

   (a) Escreva e resolva a equação diferencial ordinária para a cotação da acção. Forneça uma ilustração geométrica.

   (b) Excluem-se bolhas especulativas se se admitir que $\lim_{t\to\infty} p(t)e^{-rt} = 0$. Qual seria a expressão para a cotação, em função de $t \in [0, \infty)$, se aquela hipótese se verificar ? Interprete os resultados obtidos.

2. (3.0 val) Considere a equação às diferenças planar $y_{t+1} = Ay_t + B$, em que

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \ B = \begin{pmatrix} b \\ 0 \end{pmatrix}.$$

   (a) Resolva a equação às diferenças (sugestão: Considere separadamente os casos $b = 0$ e $b \neq 0$).

   (b) Desenhe o diagrama de fases.

3. (3.0 val) Admita que um consumidor tem uma dotação, cuja quantidade no início do período $t \in \{0, 1, \ldots, T\}$ é designada por W_t, com T finito. A dotação evolui de acordo com a equação $W_{t+1} = (1 + r)W_t - C_t$, em que C_t é a quantidade consumida ao longo do período t, e $r > 0$ é um parâmetro. Admita que $W_0 = \phi > 0$ e que o consumidor pretende ter a dotação final $W_T = \phi$. O consumidor quer determinar uma trajectória óptima para para a dotação, usando uma função de utilidade intertemporalmente aditiva, em que a função de utilidade para o período t é $\ln(C_t)$, e há um factor de desconto psicológico igual a $\beta \in (0, 1)$.

   (a) Exprima o problema como um problema de cálculo das variações, e escreva as condições de primeira ordem.

   (b) Determine a solução do problema.

Instituto Superior de Economia e Gestão
Mestrado em Economia Monetária e Financeira e Mestrado em Economia

Economia Matemática
Exame – 27/01/09 Duração: 2h30

NOTA: Responda a cada grupo em folhas separadas. Justifique claramente todas as respostas.

Grupo I (5.0 valores)

1. (2.0 val.) Considere a função $f(x,y) = \frac{x^2}{y}$. Suponha que x e y são função de t: $x(t) = \frac{1}{2}t^2$, $y(t) = ln(t)$. Calcule a derivada $\frac{\partial f}{\partial t}$.

2. (3.0 val.) Determine se a função $f(x,y) = x^2y$ é côncava ou convexa no domínio $\{x > 0, y > 0\}$.

Grupo II (7.0 valores)

1. (4.0 val.) Considere o seguinte intervalo A de R, $A = [0\ 1]$ e também a função real definida sobre A, $f(x) = ax/(x+1)$

   (a) (3.0 val.) Determine o conjunto de todos os valores de a que permitem garantir, através da aplicação do teorema do ponto fixo de Brouwer, a existência de um ponto fixo de f em A

   (b) (1.0 val.) Suponha agora que $A = [0 \quad 0,5] \cup [0,7 \quad 1]$ e que a toma o valor $a = 0,5$. Continuará a ser aplicável o teorema de Brouwer? Justifique.

2. (3.0 val) Sejam os seguintes conjuntos A e B de R^2

$$\begin{aligned} A &= \{(x,y) : 2x + 3y \leq 1\} \\ B &= \{(x,y) : (x-1)^2 + (y-1)^2 \leq 1\} \end{aligned}$$

   Diga, justificando, se podemos usar o Teorema do Hiperplano Separador para provar a existência de um hiperplano a separar A de B.

Grupo III (8.0 valores)

1. (3.0 val.) Considere a equação diferencial ordinária planar $\dot{y} = Ay + B$, em que

$$A = \begin{pmatrix} -3 & 2 \\ -1 & -6 \end{pmatrix}, \ B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

   (a) Determine a solução da equação diferencial.

   (b) Desenhe o diagrama de fases e discuta o resultado obtido.

   (c) Seja $y(0) = (0,0)$. Resolva o problema de valor inicial.

2. (2.0 val.) Considere a equação às diferenças $y_{t+1} = -3/2y_t - 1/2$.

   (a) Determine a sua solução e caracterize-a.

   (b) Seja $y_0 = 0$. Resolva o problema de valor inicial. Desenhe o diagrama das iterações (*iteration map*).

3. (2.0 val.) Admita que um consumidor tem um recurso, cuja quantidade no início do período $t \in \{0, 1, \ldots, \infty\}$ é designada por W_t. A dotação é consumida na quantidade C_t ao longo do período t. A dotação inicial é $W_0 = \phi > 0$. O consumidor quer determinar uma trajectória óptima para a dotação, que se admite assumir valores não negativos assintoticamente, usando um função de utilidade intertemporalmente aditiva, em que a função de utilidade para o período t é isoelástica, $\frac{1}{1-\sigma}(C_t)^{1-\sigma}$, com $\sigma > 0$ e um factor de desconto psicológico igual a $\beta \in (0, 1)$.

   (a) Exprima o problema como um problema de controle óptimo e escreva as condições de óptimo de primeira ordem segundo o princípio de Pontryagin's.

   (b) Determine a solução do problema.

Exame da Época Normal 2009/2010
Economia Matemática
Mestrado em Economia Monetária e Financeira

Duração: 2h30

Responda a cada grupo em folhas de ponto separadas. Não são permitidas calculadoras gráficas, nem telemóveis.

Bom Trabalho

Grupo I

1. (4 valores) Seja $f : R^n \to R$ dada por:

$$f(x_1, x_2) = log\,(x_1^\alpha x_2^\alpha)$$

com $\alpha > 0$.

(a) (1 valor) Como se define, sem recurso a diferenciabilidade, uma função côncava?

(b) (3 valores) Mostre que a função f é côncava.

2. (3 valores) Determine o máximo e o mínimo de $f(x, y) = x^2 - y^2$ no círculo unitário, $x^2 + y^2 = 1$,usando o método de Lagrange. Resolva o mesmo problema usando o método de substituição. Obtém os mesmos resultados? Porquê, ou porque não?

Grupo II

1. (3,5 valores) Considere a seguinte correspondência φ definida em $S = [2\ 10] \subset R$:

$$\begin{aligned} 2 &\leq x < 5\varphi(x) = \{x + 2\} \\ x &= 5\varphi(x) = [a\ b] \cup [c\ 8] \\ 5 &< x \leq 10\varphi(x) = [x - 3\ x - 1] \end{aligned}$$

(a) (3 valores) Indique três valores, um para cada um dos números a, b e c, que permitam aplicar o teorema do ponto fixo de Kakutani . Justifique.

(b) (0,5 valores) No caso anterior, calcule um ponto fixo.

2. (3,5 valores) Considere os seguintes conjuntos A e B de R^2:

$$\begin{aligned} A &= [0\ 2] \cup (a\ 4] \times [1\ 3] \\ B &= \{(x, y) \in R^2 : y \geq b - x\} \end{aligned}$$

(o símbolo "×" representa o produto cartesiano de conjuntos)

(a) (3 valores) Indique o menor valor de a e um valor para b que permitam garantir que existe uma recta a separar os conjuntos A e B. Justifique.

(b) (0,5 valores) No caso anterior, apresente a equação de uma dessas rectas.

Grupo III

1. (1 valor) Considere a equação $y_{t+1} = \alpha y_t - 1$, para $\alpha > 0$.

   (a) (0,5 valores) Determine a solução da equação às diferenças para os diferentes valores de α.

   (b) (0,5 valores) Admita a condição terminal $\lim_{t\to+\infty} \alpha^{-t} y_t = 0$. Discuta a existência e unicidade de soluções para o problema de valor terminal. Determine a solução do problema, caso exista.

2. (2 valores) Considere a equação planar

$$\begin{aligned} k_{t+1} &= (1+\alpha)k_t - \alpha h_t + c + (1-\gamma)k_t \\ h_{t+1} &= -\beta k_t + (1+\beta)h_t + c + (1-\gamma)h_t \end{aligned}$$

   em que $c > 0$, $\gamma > 0$, $0 < \alpha < 1$ e $0 < \beta < 1$.

   (a) (1 valor) Faça uma representação matricial e determine os valores próprios da matriz dos coeficientes de (k_t, h_t).

   (b) (1 valor) Obtenha o diagrama de bifurcação no espaço $(\gamma, \alpha + \beta)$, indicando os tipos de diagramas de fases que poderão existir.

3. (3 valores) Considere o problema de controle óptimo: $\max_{\{u\}} \sum_{t=0}^{3} y_t - (2 - u_t)^2$ sujeito a $y_{t+1} = 1/2(y_t - u_t)$ e a $y_0 = 0$ e $y_4 = 45/2$.

   (a) (1 valor) Escreva as condições de primeira ordem segundo o princípio de Pontriyagin.

   (b) (2 valores) Resolva o problema, ou seja, obtenha as sequências óptimas $\{y_t^*\}_{t=0}^4$ e $\{u_t^*\}_{t=0}^4$

Exame da Época de Recurso 2009/2010
Economia Matemática
Mestrado em Economia Monetária e Financeira

Duração: 2h30

Responda a cada grupo em **folhas de ponto separadas**. Exames que não respeitam esta condição **não serão corrigidos.** Não são permitidas calculadoras gráficas, nem telemóveis.

Bom Trabalho!

Grupo I

1. (4 valores) Considere o problema de maximização:

$$\max_{x,y} f(x,y) = x^3 + y^3 \quad s.a. \quad x + y = 1$$

   (a) (2 valores) Mostre que o problema não tem solução e discuta este resultado à luz do Teorema de Weierstrass.

   (b) (2 valores) Mostre que, se o Metodo de Lagrange fosse utilizado, os pontos críticos da Lagrangeana teriam uma solução única. Determine se este ponto seria um máximo ou um mínimo global.

2. (3 valores) Considere o seguinte problema de minimização:

$$\begin{aligned} &\min_{x,y}(x-1)^2 + (y-\ 2)^2, \ s.a. \\ 4 &\geq 2y + x, \\ 20 &\geq 3y + 10x, \\ x, y &\geq 0 \end{aligned}$$

   Verifique que no óptimo existe apenas uma restrição activa, nomeadamente a primeira.

Grupo II

1. (3,5 valores) Considere uma economia competitiva em que se trocam dois bens, 1 e 2 e para os quais se conhecem as respectivas funções de procura (D_i) e oferta (S_i):

   Bem 1:

$$\begin{aligned} D_1 &= p_2 - p_1^2 p_2 \\ S_1 &= \alpha p_1 p_2^2 - p_2^2 + p_1 p_2 \end{aligned}$$

   Bem 2:

$$\begin{aligned} D_2 &= p_1^3 - p_1 p_2 - p_1 \\ S_2 &= 3p_1^2 p_2 - p_1^2 \end{aligned}$$

   (a) (2 valores). Determine o valor de α que permite calcular o vector de preços de equilíbrio.

   (b) (1,5 valores) Verifique que, para esse valor, $p_1 = 0,842$ e $p_2 = 0,158$ são, aproximadamente, preços de equilíbrio e calcule o valor do erro de aproximação para cada um dos mercados.

2. (3,5 valores) Considere a seguinte correspondência φ definida no intervalo $[0,5 \;\; 2]$ de $\mathbb{R}$:

$$\begin{aligned} 0.5 &\leq x < 1, \; \varphi(x) = \{1, 5x\} \\ x &= 1, \; \varphi(1) = [a \;\; b] \\ 1 &< x \leq 2, \; \varphi(x) = [x - 0.5 \quad x - 0.4] \end{aligned}$$

   (a) (3 valores) Indique um valor para a e outro para b que permitam aplicar o Teorema de Kakutani para provar a existência de um ponto fixo da correspondência.

   (b) (0,5 valores) Com esses valores, calcule um ponto fixo.

Grupo III

1. (1 valor) Considere a equação $y_{t+1} = -1/2y_t + 3/2$.

   (a) (0,5 valor) Determine a solução da equação às diferenças e caracterize-a.

   (b) (0,5 valor) Seja $y_0 = -1$. Resolva o problema de valor inicial. Desenhe o diagrama das iterações (*iteration map*).

2. (2 valores) Considere a equação às diferenças planar $\mathbf{y}_{t+1} = \mathbf{A}\mathbf{y}_t$, em que

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$$

   (a) (1,5 valores) Determine a solução da equação às diferenças. Caracterize-a qualitativamente.

   (b) (0,5 valor) Suponha que $\mathbf{y}_0 = (1, -1)^\top$. Obtenha a solução do problema de valor inicial.

3. (3 valores) Considere o seguinte problema de investimento óptimo para uma empresa: determinação da sequência de investimento, $\{I_t\}_{t=0}^{\infty}$, que maximiza o funcional objectivo $\sum_{t=0}^{\infty}(1+r)^{-t}\pi_t$, onde $r > 0$ é taxa de juro de mercado. O *cash flow* no período t é denotado por $\pi_t = AK_t - I_t(1 + \xi I_t)$, em que K_t representa o stock de capital, e $A > 0$ e $\xi > 0$ são parâmetros de produtividade e de custo de investimento, respectivamente. O problema tem como restrições, a equação de acumulação do stock de capital $K_{t+1} = I_t + (1-\delta)K_t$, em que $\delta \in [0, 1)$ é a taxa de depreciação do capital, e o stock de capital inicial é dado por $K_0 = \phi > 0$. Suponha que $A > r + \delta$.

   (a) (1 valor) Exprima o problema como um problema de cálculo de variações e determine as condições de primeira ordem de óptimo.

   (b) (2 valores) Determine a solução do problema como uma função explícita para K_t. Justifique e forneça uma intuição económica para a solução que obteve.

Economia Matemática
Ano Lectivo de 2010/2011 – Exame da Época Normal
Duração: 2h30

Antes de iniciar o teste, tenha em atenção os seguintes aspectos:

- Não é permitida a consulta de qualquer material de apoio, nem de calculadoras gráficas;
- Desligue e arrume o telemóvel;
- Responda a cada um dos 3 **grupos** de questões em folhas separadas e correctamente identificadas;
- Apresente todos os cálculos que efectuar e não apenas os resultados finais;
- Justifique todas as suas respostas

Grupo I

1. (6,5 valores) Seja $W(x,y,y)$ a função que representa a relação entre a produção de x, y e z e o bem estar social. O objectivo deste problema é obter o bem estar social máximo dadas as restrições existentes na economia para a produção de x, y e z. Nomeadamente:

$$\begin{aligned} \max_{x,y,z} W(x,y,z) &= a\log x + b\log y + c\log z, \text{ s.a.} \\ & 2x + y + 3z \leq 600 \\ & x + 2y + z \leq 550 \\ 1 &\leq x, 1 \leq y \text{ e } 1 \leq z \end{aligned}$$

Sendo os parâmetros $a, b, c > 0$.

(a) (0,5 valor) Defina as funções $h_i(x,y,z), i = 1,..5$ que representam as restrições deste problema para que o Teorema de Kuhn-Tucker possa ser utilizado.

(b) (2 valores) Seja o conjunto D definido por

$$D = \{(x,y,z) : h_i(x,y,z) \geq 0 \ , i = 1,..,5\}.$$

i. O conjunto D é compacto? Justifique.

ii. A função W tem um máximo no conjunto D? E um mínimo? Justifique.

(c) (1 valor) Escreva as condições de primeira ordem do teorema de Kuhn-Tucker que têm de ser resolvidas para que se obtenha um ponto de máximo de W no conjunto definido pelas restrições.

(d) (1 valor) Escreva as condições de complementariedade do Teorema de Kuhn-Tucker deste problema.

(e) (2 valores) Suponha que o ponto óptimo do problema ocorre quando (x, y, z) = (50, 200, 100). Mostre que o teorema de Kuhn-Tucker se pode aplicar neste caso.

Grupo II

1. (2,5 valores) Considere a seguinte economia de troca em que existem só dois bens:1 e 2. As funções de oferta (S) e procura (D) de cada bem são, respectivamente:

   Para o bem 1:

$$\begin{aligned} S_1 &= 4p_1p_2^2 - p_1g(p_2) \\ D_1 &= 4p_1p_2^2 - 3p_1^2p_2 \end{aligned}$$

   Para o bem 2:

$$\begin{aligned} S_2 &= -p_1p_2 + p_1^2p_2 \\ D_2 &= -p_1p_2 + 3p_1^3 \end{aligned}$$

   Encontre a expressão analítica da função $g(p_2)$ que permite garantir a existência de um vector de preços de equilíbrio de Walras e calcule esse vector.

2. (4 valores) Seja a seguinte correspondência:

$$\varphi : [0 \quad 1] \to 2^{[0 \quad 1]}$$

$$\varphi(x) = \begin{cases} \frac{(2-x)}{2}, & \text{Para } 0 \leq x < 0,5 \\ [a \quad b], & \text{Para } x = 0,5 \\ \frac{1}{(0,7+x)}, & \text{Para } 0,5 < x \leq 1 \end{cases}$$

   Indique o maior valor de a e o menor valor de b que permitem garantir a existência de um ponto fixo através do teorema de Kakutani. Determine esse ponto e prove que, neste exercício, ele é único.

Grupo III

1. Considere uma economia descrita pelas seguintes equações: (1) uma função de produção $Y_t = AK_t$, em que Y_t é produção, K_t é o stock de capital e A é um parâmetro de produtividade; (2) uma função poupança Keynesiana, $S_t = sY_t$, em que $0 < s < 1$ é a propensão marginal a poupar; (3) uma função investimento, $I_t = K_{t+1} - (1-\delta)K_t$ em que $0 < \delta < 1$ é a taxa de depreciação do capital; e (4) a equação de equilíbrio $S_t = I_t$. Admita que o nível inicial do stock de capital, $K_0 > 0$, é conhecido.

(a) Obtenha uma equação às diferenças escalar em K_t.

(b) Resolva o problema de valor inicial associado.

(c) Caracterize a solução. Faça uma representação geométrica.

2. Considere a equação às diferenças planar $y_{t+1} = Ay_t$ com

$$A = \begin{pmatrix} 1 & 0 \\ -1 & a \end{pmatrix}, 0 < a < 1$$

(a) Determine a solução geral da equação planar

(b) Desenhe o diagrama de fases. Comente os resultados obtidos.

3. Considere o problema de cálculo das variações $\max_y \sum_{t=0}^{T} -(y_{t+1} - 2y_t)^2$ com $y_0 = 1$ e $\lim_{t\to\infty} y_t = 0$.

(a) Obtenha a condição de Euler-Lagrange;

(b) Obtenha a solução do problema.

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Economia Matemática
Ano Lectivo de 2010/2011 – Exame da Época de Recurso
Duração: 2h30

Antes de iniciar o teste, tenha em atenção os seguintes aspectos:

- Não é permitida a consulta de qualquer material de apoio, nem de calculadoras gráficas;
- Desligue e arrume o telemóvel;
- Responda a cada um dos 3 **grupos** de questões em folhas separadas e correctamente identificadas;
- Apresente todos os cálculos que efectuar e não apenas os resultados finais;
- Justifique todas as suas respostas

Grupo I

1. (6,5 valores) Considere a função

$$f(x,y) = \frac{x+y}{1+x^2+y^2}$$

definida em $\mathbb{R}^2_+ = \{(x,y) : x, y \geq 0\}$.

(a) (1,5 valor) Determine o(s) ponto(s) críticos da função no interior do domínio definido. (dica: no óptimo, teremos $x^* = y^*$)

(b) (0,5 valor) Determine o(s) ponto(s) crítico(s) da função $g(x) = f(x,0)$, $x \geq 0$, i.e.ao longo da fronteira $y = 0$. Classifique o(s) ponto(s) crítico(s) de $g(x)$.

(c) (0,5 valor) Determine e classifique o(s) ponto(s) crítico(s) de $h(y) = f(0,y)$, $y \geq 0.$, i.e.ao longo da fronteira $x = 0$. Classifique o(s) ponto(s) crítico(s) de $h(y)$.

(d) (1,5 valores) Compare as soluções obtidas em **b**, **c** e **d** em termos do valor da função. Analise ainda o ponto $f(0,0)$. Explique a necessidade desta análise.

(e) (3,0 valores) Considere agora o seguinte problema:

$$\max_{x,y} f(x,y) \text{ s.a. } x \geq 0, y \geq 0 \text{ e } x + y \leq \frac{3}{4}$$

i. (1,0 valor) Apresente as condições de primeira ordem e de complementariedade do teorema de Kuhn-Tucker aplicado a este problema.

ii. (1,5 valor) Verifique se existe uma solução em que apenas a restricção $x + y \leq \frac{3}{4}$ é activa.

Grupo II

1. (2,5 valores) Considere a seguinte economia de troca em que existem só dois bens:1 e 2. As funções de oferta (S) e procura (D) de cada bem são, respectivamente:

   Para o bem 1:

$$\begin{aligned} S1 &= f(p_1, p_2) + p_1 g(p_1, p_2) \\ D1 &= f(p_1, p_2) + 2p_1 p_2 \end{aligned}$$

   Para o bem 2:

$$\begin{aligned} S_2 &= h(p_1, p_2) + 2p_1^2 \\ D_2 &= h(p_1, p_2) + p_1 \end{aligned}$$

   Encontre as condições que permitem garantir a existência de um vector de preços de equilíbrio de Walras e calcule esse vector.

2. (4 valores) Seja um espaço $\mathbb{R}^2$ e os seguintes conjuntos do espaço:

$$\begin{aligned} A &= \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}, \\ B &= [0,5 \quad 1,5] \times [0,5 \quad 1,5] \end{aligned}$$

   em que $\times$ é o símbolo de produto cartesiano de conjuntos, e

$$C = \{(x,y) \in \mathbb{R}^2 : k - y \leq x\}.$$

   Considere ainda o conjunto

$$D = A \cap B$$

   Indique, caso exista, um valor de k, que seja menor que 2 e que permita garantir a existência de um hiperplano a separar D e C. Justifique detalhadamente.

   Sugestão: represente graficamente os conjuntos envolvidos.

Grupo III

1. (1,5 valores) Considere a equação $y_{t+1} = -\frac{1}{2}y_t + 2$.

   (a) (0,5 valor) Determine a solução da equação às diferenças.

   (b) (0,5 valor) Desenhe o gráfico de iterações.

   (c) (0,5 valor) Admita a condição terminal $\lim_{t\to+\infty} y_t = \overline{y}$ em que $\overline{y}$ é o equilíbrio estacionário. Discuta a existência e unicidade de soluções para o problema de valor terminal. Determine a solução do problema, caso exista.

2. (2 valores) Considere a equação às diferenças planar $y_{t+1} = Ay_t$ em que

$$A = \begin{pmatrix} 4/3 & 0 \\ 0 & 1/3 \end{pmatrix}$$

   (a) (1 valor) Determine a solução da equação às diferenças.

   (b) (1 valor) Desenhe o diagrama de fases e caracterize o comportamento dinâmico do modelo.

3. (3,5 valores) Seja o problema do consumidor com a função objectivo

$$\max_{C_t} \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}}{1-\sigma}, \text{ sujeito a,}$$

$$W_{t+1} = (1-\delta)W_t - Ct,\ W_0 = \phi > 0,\ \lim_{t\to+\infty} W_t \geq 0$$

   Suponha que $\sigma > 0$, $< \beta < 1$ e $0 < \delta < 1$ e que $\beta^{\frac{1}{\sigma}} < (1-\delta)^{1-1/\sigma}$.

   (a) (1 valor) Escreva as condições de primeira ordem segundo o princípio de Pontriyagin. Represente o sistema canónico como uma equação às diferenças planar em $(C; W)$.

   (b) (2,5 valores) Resolva o problema, ou seja, obtenha as sequências óptimas $\{W_t^*\}_{t=0}^{\infty}$ e $\{C_t^*\}_{t=0}^{\infty}$. Comente os resultados obtidos.

Universidade Técnica de Lisboa – ISEG

Departamento de Economia

Economia Matemática

1º Semestre 2011/2012

EXAME DE ÉPOCA NORMAL

3 Janeiro 2012

Duração máxima: 2 horas

Resolva cada parte do exame numa folha separada

PARTE I

(1) Suponha uma economia em que se trocam dois bens (bem 1 e bem 2) com, respectivamente, as seguintes funções de procura e oferta dependentes dos preços:

Bem 1

$$D_1 = a(p_1/p_2)^{1/2} + p_1^2 \qquad S_1 = -p_1p_2^2 + p_1^2$$

Bem 2

$$D_2 = bp_1(p_1/p_2)^{1/2} + p_2 \qquad S_2 = p_2 + p_1^2p_2.$$

Sabendo que $a-b = 0,8$, calcule os valores de a e de b que permitem, a priori, garantir que existe um vector de preços de equilíbrio no sentido de Walras. (2 valores)

(2) Considere um mercado onde existem n agentes ligados em rede e em que cada agente comunica com todos os outros, seja directamente seja indirectamente (isto é, através de outro agente). Se cada agente comunica com outro, esse outro comunica com o primeiro. Se um agente i comunica directamente com um outro agente j diz-se que deu um passo na comunicação. Por convenção, o número de passos de i para i é 0.

(a) Prove que a função $d(i,j) = N$, sendo N o número mínimo de passos que o agente i dá para comunicar com o agente j, é uma distância definida no conjunto de todos os pares (i,j) para todos os agentes i e j. (2 valores)

(b) Se a rede é tal que cada agente só comunica directamente com cinco outros agentes, diga quais são os elementos de cada esfera aberta de raio 2 e centro em cada agente i. (1 valor)

PARTE II

(1) Considere o seguinte problema:

$$\max f(x,y) = 2x^2 + 3y^2$$

$$\text{sujeito a } x + 2y \leq 11$$

$$x, y \geq 0$$

(a) Enuncie o teorema de Weierstrass e explique se esse teorema pode ser usado na resolução do problema. (1 valor)

(b) Resolva o prolema utilizando o teorema de Kuhn-Tucker. Explicite claramente o seu raciocínio e todos os passos que efectuar. (4 valores)

PARTE III

(1) Considere a equação diferencial

$$\dot{x} = x - x^3.$$

(a) Encontre todos os seus pontos de equilíbrio. ($\frac{1}{2}$ valor)

(b) Determine se cada ponto de equilíbrio é estável, asimptoticamente estável ou instável. (1 valor)

(c) Desenhe o retrato de fases da equação. ($\frac{1}{2}$ valor)

(2) Considere a matriz

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(a) Calcule a forma normal de Jordan J de A. (1 valor)

(b) Determine a matriz exponential e^{tJ}. ($\frac{1}{2}$ valor)

(c) Desenhe o retrato de fases da equação linear $\dot{y} = Jy$. (1 valor)

(d) Encontre os pontos de equilíbrio de $\dot{x} = Ax$, e determine a sua estabilidade. ($\frac{1}{2}$ valor)

PARTE IV

(1) Considere o problema de um agente cujo objectivo é maximizar a utilidade total descontada obtida a partir do seu consumo durante o intervalo de tempo $[0,T]$. Seja $K(t)$ o capital acumulado por esse agente no instante $t \in [0,T]$ e $C(t)$ o seu consumo nesse mesmo instante de tempo. Suponha que:

- o horizonte temporal $T > 0$ é finito,
- o agente tem um capital acumulado inicial $K(0) = K_0$,
- o capital acumulado por esse agente satisfaz a equação diferencial

$$\dot{K}(t) = K(t) - C(t) \ ,$$

- as preferências do agente relativamente ao consumo são descritas pelo funcional

$$J[C(t)] = \int_0^T 2\exp(-t)\sqrt{C(t)}\ \mathrm{d}t \ .$$

Justifique convenientemente a sua resposta às seguintes questões.

(a) Represente o problema descrito acima como um problema de cálculo de variações. (1 valor)

(b) Determine uma condição necessária para a existência de uma solução C^2 para o problema da alínea (a). (2 valores)

(c) Assuma que existe uma solução C^2 para a condição obtida na alínea (b). Mostre que tal solução é um maximizante do problema de cálculo de variações da alínea (a). (2 valores)

Universidade Técnica de Lisboa – ISEG

Departamento de Economia

Economia Matemática

1º Semestre 2011/2012

EXAME DE ÉPOCA DE RECURSO

24 Janeiro 2012

Duração máxima: 2 horas

Resolva cada parte do exame numa folha separada

PARTE I

(1) Considere a seguinte correspondência definida no intervalo [0 4]:

$$\begin{array}{rl} 0 \leq x < 1 & \varphi(x) = \{2x^2\} \\ x = 1 & \varphi(1) = [0,5\ m) \cup [1\ 2] \\ 1 < x < 4 & \varphi(x) = \{\sqrt{x}\} \\ x = 4 & \varphi(4) = [1\ 3] \end{array}$$

(a) Determine o conjunto de todos os valores de m que permitem que se verifiquem as condições do teorema do ponto fixo de Kakutani e mostre que estão também reunidas todas as outras condições. (2 valores)

(b) Escolha um dos valores de m e calcule todos os pontos fixos da correspondência. (1 valor)

(2) Considere uma economia em que se produzem n bens.

Os vectores X dos bens procurados satisfazem a condição $X \geq A$ em que A é um vector não negativo. Os vectores Y dos bens produzidos satisfazem a condição $0 \leq Y \leq B$ em que 0 é o vector nulo, B é um vector não negativo e $A \geq B$. Existe pelo menos uma componente i de B, b_i tal que $b_i < a_i$.

Mostre que existe um vector de preços não negativo tal que, com esses preços, o valor total das quantidades produzidas nunca é superior ao valor total das quantidades procuradas, quaisquer que sejam as quantidades produzidas e procuradas. (2 valores)

(*Sugestão*: para demonstrar que o vector de preços é não negativo proceda por absurdo, supondo que o preço de um dos bens é negativo).

PARTE II

(1) Utilizando o teorema de Kuhn-Tucker, resolva o problema seguinte. Explicite claramente o seu raciocínio e todos os passos que efectuar.

$$\max f(x,y) = x + y$$
$$\text{sujeito a } 2x + y \leq 8$$
$$x, y \geq 0$$

(5 valores)

PARTE III

(1) Considere a equação às diferenças:

$$x_{n+1} = 4x_n(1 - x_n).$$

(a) Encontre todos seus pontos fixos. ($\frac{1}{2}$ valor)

(b) Determine se cada ponto fixo é estável, asimptoticamente estável ou instável. (1 valor)

(c) Trace os *stair-step diagrams* com quatro iterações e condições iniciais: $x_0 = -1$ e $x_0 = 0,25$. ($\frac{1}{2}$ valor)

(2) Considere a matriz

$$A = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}.$$

(a) Calcule a forma normal de Jordan J de A. (1 valor)

(b) Determine a matriz exponential e^{tJ}. ($\frac{1}{2}$ valor)

(c) Desenhe o retrato de fases da equação linear $\dot{y} = Jy$. (1 valor)

(d) Encontre os pontos de equilíbrio de $\dot{x} = Ax$, e determine a sua estabilidade. ($\frac{1}{2}$ valor)

PARTE IV

(1) Considere o problema de um agente cujo objectivo é maximizar a utilidade total descontada obtida a partir do seu consumo durante o intervalo de tempo $[0, T]$. Seja $K(t)$ o capital acumulado por esse agente no instante $t \in [0, T]$ e $C(t)$ o seu consumo nesse mesmo instante de tempo. Suponha que:

- o horizonte temporal $T > 0$ é finito,
- o agente tem um capital acumulado inicial $K(0) = K_0$ e quer atingir o horizonte temporal T com um capital acumulado nulo,
- o capital acumulado pelo agente satisfaz a equação diferencial

$$\dot{K}(t) = K(t) - C(t) \ ,$$

- as preferências do agente relativamente ao consumo são descritas pelo funcional

$$J[C(t)] = \int_0^T 2\exp(-t)\sqrt{C(t)}\,\mathrm{d}t \ .$$

Justifique convenientemente a sua resposta às seguintes questões.

(a) Represente o problema descrito acima como um problema de controlo óptimo. (1 valor)

(b) Utilize o princípio do máximo de Pontryagin para caracterizar o par óptimo para o problema da alínea (a). (2 valores)

(c) Utilize a condição do máximo obtida na alínea anterior para determinar o consumo óptimo em função da variável de estado e da variável adjunta. (1 valor)

(d) Assuma que existe uma solução para o sistema Hamiltoniano alargado obtido na alínea (b). Mostre que tal solução determina um maximizante para o problema de controlo óptimo da alínea (a). (1 valor)

Universidade Técnica de Lisboa – ISEG

Economics Departament

Mathematical Economics

1^{st} Semester 2012/2013

FIRST EXAM

8 January 2013

Maximum time length: 2 hours

Solve each part of the exam on a separable sheet

PART I

(1) Consider an exchange economy with two goods, 1 and 2 and the following demand (D_i) and supply (S_i) functions with normalized prices $p_1 + p_2 = 1$,

$$D_1 = (p_1)^{-1} p_2 + k\, p_1 p_2 \qquad S_1 = 2p_1 + p_1 p_2$$

$$D_2 = 2p_1^2 (p_2)^{-1} \qquad S_2 = 3p_1^2 p_2 + 1.$$

Find the value of k as a function of p_2 for which we can be sure of the existence of equilibrium prices. (2 points)

(2) Let A and B be two sets of $\mathbb{R}^2$ such that $A = C \cap D$ with

$$C = \{(x, y)\colon y = 2x + 1\}, \quad D = \{(x, y)\colon x^2 + y^2 \leq 4\}$$

and $B = [2\ 3] \times [5\ 8]$ where $\times$ denotes the "Cartesian product". Is there at least a hyperplane separating A and B? Why? Find one such hyperplane. (3 points)

PART II

(1) Consider the following problem:

$$\max f(x,y) = (2x+y)^2$$
$$\text{such that } x^2 + y \leq 16$$
$$x, y \geq 0$$

(a) State the Weierstrass theorem and explain whether it can be used to help solve the problem above. (1 point)

(b) Solve the problem above using the Kuhn-Tucker theorem. Explain carefully all the steps in your reasoning. (4 points)

PART III

(1) Suppose that $1 < \lambda < 3$, and let $F(x) = \lambda x(1-x)$. Consider the difference equation

$$x_{n+1} = F(x_n).$$

(a) Compute the fixed points of F and plot them on the graph of F. (0.5 points)

(b) Determine the stability of the fixed points. (1 point)

(c) Pick a point x_0 in the interval $(0,1)$ and let x_n be the solution of the difference equation above with initial condition x_0. Does the sequence x_n converge? if the answer is positive, what is its limit? Justify your answers. (1 point)

(2) Consider the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}.$$

(a) Compute the Jordan Normal Form J of A. (0.5 points)

(b) Compute the exponential matrix e^{tJ}. (1 point)

(c) Sketch the phase portrait of the linear equation $\dot{y} = Jy$. (1 point)

PART IV

(1) A firm wants to maximise the present value of its cash-flow selecting the optimal path of investment $I = \{I_t\}_{t=0}^{T-1}$ by solving the problem:

$$\max_I \sum_{t=0}^{T-1} \left(\frac{1}{1+r}\right)^t \left(pK_t - (I_t)^2\right) \quad \text{subject to} \quad K_{t+1} = I_t - K_t$$

and $K_0 = \phi > 0$ is given, where K_t is the stock of capital. The interest rate r and the output price p are positive parameters.

(a) Transform into a calculus of variations problem and determine the first order conditions. (1 point)

(b) Solve the problem[1]. (2.5 points)

(2) Consider the problem for an agent that wants to find the optimal path of consumption, $(C(t))_{t\in[0,\infty)}$, and financial wealth, $(W(t))_{t\in[0,\infty)}$, by solving the problem:

$$\max_C \int_0^\infty \left(B - \zeta e^{-\frac{C(t)}{\zeta}}\right) e^{-\rho t} dt, \text{ subject to } \dot{W} = Y + rW(t) - C(t)$$

given $W(0) = W_0$ and $\lim_{t\to\infty} e^{-rt} W(t) \geq 0$.

(a) Write the Hamilton Jacobi Bellman (HJB) equation. (1 point)

(b) Determine the optimal policy function and find the equivalent HJB equation. (0.5 points)

[1] The general solution for the equation $x_{t+2} = (1+a)x_{t+1} - ax_t + b$ is $x_t = k_1 + k_2 a^t + b((1-a)t - b)/((a-1)^2)$.

Universidade Técnica de Lisboa – ISEG
Economics Department

Mathematical Economics

1st Semester 2012/2013

SECOND EXAM

January 28, 2013

Maximum duration: 2 hours

Answer each part in separate sheets

PART I

(1) Let

$$f : x \mapsto f(x) = 0.4\,(x^2 - 2x + 1)$$

be a real function of real variable with domain $S = [0\ 1]$. Knowing that any closed subset of a complete metric space is a complete metric space prove that there is one and only one fixed-point of f belonging to the set S. (2.5 points)

(2) Consider the following correspondence defined on the interval $[0\ 2]$ of $\mathbb{R}$:

$$0 \leq x < 1 \qquad \varphi(x) = [x + 0.1\ \ x + 0.3]$$

$$1 \leq x < 1.5 \qquad \varphi(x) = [0.6\ \ k]$$

$$1.5 \leq x \leq 2 \qquad \varphi(x) = [x - 1\ \ x]$$

Find the smallest value of k for which we can use the Kakutani theorem to prove the existence of at least one fixed-point of the correspondence. Find one fixed-point. (2.5 points)

PART II

(1) Consider the following problem:

$$\max f(x,y) = [10-(x+y)](x+y) - ax - (y+y^2)$$

$$\text{such that } x, y \geq 0$$

where a is a positive parameter.

(a) Solve the problem using the Kuhn-Tucker theorem. Explain carefully all the steps in your reasoning. (4.5 points)

(b) What economic decision making problem can the problem above represent? (0.5 points)

PART III

(1) Take your pocket calculator and perform the following experiment. Type any positive number $x_0 > 0$ you like, and then press the button $\sqrt{\ }$. Of course, you get $\sqrt{x_0}$. If you keep pressing $\sqrt{\ }$, you will obtain a sequence of positive numbers $x_0, x_1, x_2, \dots$. Does the sequence x_n for $n = 0, 1, 2, \dots$ have a limit? what is its value? Now, repeat the experiment with a different x_0. What do you get? Try to explain the outcome of your experiment by studying the difference equation $x_{n+1} = F(x_n)$ with $F(x) = \sqrt{x}$ and the initial condition x_0.

   (a) Why does the sequence x_n have a limit? (1 point)
   (b) Compute the value of the limit. (0.5 points)
   (c) Explain why the limit does not depend on the choice of x_0. (1 point)

(2) Consider the matrix

$$A = \begin{pmatrix} 3 & -1 \\ 2 & 4 \end{pmatrix}.$$

   (a) Compute the Jordan Normal Form J of A. (0.5 points)
   (b) Compute the exponential matrix e^{tJ}. (1 point)
   (c) Sketch the phase portrait of the linear equation $\dot{y} = Jy$. (1 point)

PART IV

(1) Consider the following endogenous growth model:

$$\max_C \int_0^\infty \frac{1}{1-\sigma} C(t)^{1-\sigma} e^{-\rho t} dt, \text{ subject to } \dot{K} = Y(t) - C(t)$$

together with $K(0) = K_0$ given and $\lim_{t\to\infty} e^{-At}K(t) \geq 0$. The production function is linear $Y(t) = AK(t)$ and the parameters verify: $\rho > 0$, $\sigma > 1$ and $A > 0$.

(a) Write the first order conditions according to the maximum principle of Pontriyagin. (2 points)

(b) Solve the problem[1]. Under which conditions the solution displays unbounded growth? (2 points)

(2) Consider the problem for an agent who wants to find the optimal path of consumption, $\{C_t\}_{t=0}^\infty$, and financial wealth, $\{W_t\}_{t=0}^\infty$, by solving the problem: $\max_C \sum_{t=0}^{T-1} \left(B - \zeta e^{-\frac{C_t}{\zeta}}\right) \beta^t$ subject to $W_{t+1} - W_t = rW_t - C_t$, given $W(0) = W_0$.

(a) Write the Hamilton-Jacobi-Bellman equation. (0.5 points)

(b) Determine the optimal policy function. (0.5 points)

[1] Auxiliary results: the solution of differential equation $\dot{x} = \lambda x(t) + f(t)$ is

$$x(t) = ke^{\lambda t} + \int_0^t e^{\lambda(t-s)} f(s) ds$$

where k is an arbitrary constant.

Universidade de Lisboa – ISEG

Economics Department

Mathematical Economics

FIRST EXAM

January 6, 2014

Maximum duration: 2 hours

Solve each part of the exam on a separate sheet

PART I

(1) Consider the following function $f\colon \mathbb{R} \to \mathbb{R}$ with $f(x) = Kx^2 + M$, $K > 0$. Using the Banach fixed point theorem find one value for K and one value for M such that when K and M take those values the equation $Kx^2 - x + M = 0$ has one and only one solution in the interval [1 2]. (2.5 points)

(2) Consider a Walras economy where two commodities, 1 and 2 are traded and the respective demand (D_i, $i = 1, 2$) and supply functions (S_i, $i = 1, 2$) are

$$D_1 = \alpha p_1 p_2 + p_1 p_2^{1/2} \qquad S_1 = \alpha^2 p_1 p_2^2 + p_1 p_2^{1/2}$$

$$D_2 = \alpha^2 p_1^2 p_2 + p_1^3 p_2 \qquad S_2 = p_1^3 p_2 + 0.5 p_1$$

Prices are normalized by the condition $p_1 + p_2 = 1$. Find the value of α as a function of prices that ensures the existence of a vector of equilibrium prices and calculate those prices. (2.5 points)

PART II

(1) Let $A = \{(x, y, z) \in \mathbb{R}^3 \colon xyz > 0\}$ and $f \colon A \to \mathbb{R}$ given by

$$f(x, y, z) = \ln(xyz).$$

Consider the set

$$D = \left\{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 + z^2 \geq 1\right\}.$$

(a) Find the local extreme points of f. (1 point)

(b) Find and classify the local extreme points of f on the boundary of D. (2 points)

(c) Find the local and global extreme points of f on D. (2 points)

PART III

(1) Consider the differential equation $\dot{x} = x^2 - x^3$.

(a) Plot the graph of $f(x) = x^2 - x^3$, and find the equilibrium points of the equation. (1 point)

(b) Determine the stability of each equilibrium point (0.5 points)

(c) Let $x(t, x_0)$ be the solution of the equation with initial condition x_0. Compute

$$\lim_{t\to+\infty} x(t, x_0)$$

for i) $x_0 = -1$, ii) $x_0 = 1/2$ and iii) $x_0 = 2$. (1 point)

(2) Consider the planar linear differential equation

$$\dot{x} = Ax, \qquad A = \begin{pmatrix} 5 & -4 \\ 4 & -5 \end{pmatrix},$$

and let

$$y = Px, \qquad P = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

(a) Derive the differential equation $\dot{y} = Jy$, and compute explicitly the matrix J. (1 point)

(b) Compute the general solution of $\dot{y} = Jy$. (0.5 points)

(c) Use the answer to part (b) to derive the general solution of $\dot{x} = Ax$. (1 point)

PART IV

(1) Consider the calculus of variations problem:

$$\max_{y_{t=0}^{T}} \sum_{t=0}^{T-1} -(y_{t+1} - y_t - 1)^2, \text{ subject to } y_0 = 1,\ y_T = 1 + T$$

for $T > 0$ and finite.

(a) Write the first order conditions (0.5 points).

(b) Solve the problem (1.5 points).

(2) A representative consumer wants to maximize the intertemporal utility functional $\int_0^\infty e^{-\rho t} \ln(C(t))dt$, where $\rho > 0$, by using consumption $C(\cdot)$ as a control variable. She/he has initial wealth $A(0) = A_0$, and the instantaneous budget constraint is $\dot{A}(t) = (1 - \tau)(Y + rA(t)) - C(t)$, where income Y is constant and positive, and the income tax rate verifies $0 < \tau < 1$. The non-Ponzi game condition $\lim_{t\to\infty} e^{-rt}A(t) \geq 0$ holds.

(a) Write the first order optimality conditions from the Pontryagin's maximum principle. (0.75 points)

(b) Solve the problem, and supply an intuition for your results. (2.25 points)

Universidade de Lisboa – ISEG

Economics Department

Mathematical Economics

SECOND EXAM

28 January 2014

Maximum duration: 2 hours

Solve each part of the exam on a separate sheet

PART I

(1) Consider the following correspondence defined on the interval [0 3]

$$0 \leq x < 1 \qquad \varphi(x) = \{1.5 - x\}$$
$$1 \leq x \leq 2 \qquad \varphi(x) = [0.5x\ 0.7x]$$
$$2 < x \leq 3 \qquad \varphi(x) = \{3 - x\}$$

Verify if the conditions of the Kakutani theorem are met and calculate the fixed points of the correspondence (if any). (3.5 points)

(2) Consider the three subsets of $\mathbb{R}^2$:

$$A = B \cap C$$
$$B = [0\ 1] \times [1\ 3]$$
$$C = \{x = (x_1, x_2) \in \mathbb{R}^2 \colon x_2 = 2x_1 + 1\}$$
$$D = [0.5\ 1] \times [1\ 1.5]$$

Verify if the conditions of the separating hyperplane theorem for the sets A and D are met and independently of this being the case, say if there is a hyperplane separating A from D. (1.5 points)

PART II

(1) Let $f\colon \mathbb{R}^3 \to \mathbb{R}$ given by

$$f(x, y, z) = y(x^2 + y^2 + z^2)$$

and

$$D = \left\{ (x, y, z) \in \mathbb{R}^3 \colon x^2 + 5y = \frac{11}{2}, z = \frac{\sqrt{2}}{2} \right\}.$$

(a) Find and classify the local extreme points of f. (2 points)

(b) Find and classify the local extreme points of f on D. (3 points)

PART III

(1) Consider the difference equation $x_{n+1} = F(x_n)$ with $F(x) = x^2$.
   (a) Plot the graph of F. (1 point)
   (b) Find the fixed points of F and determine their stability. (0.5 points)
   (c) Compute $\lim_{n\to+\infty} x_n$ for every $0 \leq x_0 < 1$. Explain your answer. (1 point)

(2) Consider the matrix

$$A = \begin{pmatrix} 1 & 1 \\ -1 & -3 \end{pmatrix}.$$

   (a) Compute the Jordan Normal Form J for A. (0.5 points)
   (b) Compute the exponential matrix e^{tJ}. (1 point)
   (c) Find the general solution of $\dot{y} = Jy$, and sketch its phase portrait. (1 point)

PART IV

(1) Consider the calculus of variations problem:

$$\max_{y(\cdot)} \int_0^T -(\dot{y}(t) - y(t))^2 \, dt, \text{ subject to } y(0) = 1$$

for T finite and known.

(a) Write the first order conditions (1 point).

(b) Solve the problem (1 point).

(2) Find the optimal investment sequence, $\{I_t\}_{t=0}^T$, that maximizes the value functional

$$\sum_{t=0}^{T} \left(\frac{1}{1+r}\right)^t (pK_t - I_t(1 + (1/2)I_t))$$

where K_t is the capital stock, $r > 0$ is the market interest rate, and $p > 0$ is a productivity parameter. The restrictions of the problem are: the accumulation equation is $K_{t+1} = I_t + (1-\delta)K_t$, where δ is the rate of depreciation of capital, and the initial and terminal capital stock is given by $K_0 = K_T = \phi > 0$. Assume that $p > r + \delta$ and $\delta \in [0, 1)$.

(a) Write the problem as a optimal control problem and determine the optimality conditions from the Pontryagin's maximum principle. (1.5 points)

(b) Find an explicit solution for K_t. Justify it and give an intuition for your results. (1.5 points)

Universidade de Lisboa – ISEG

Economics Department

Mathematical Economics

FIRST EXAM

January 5, 2015

Maximum duration: 2 hours

Solve each part of the exam on a separate sheet

PART I

(1) Consider the following Walras economy with two goods, 1 and 2. D_i, S_i and p_i are respectively the demand functions, supply functions and prices for each good i. Prices belong to the unit simplex set in $\mathbb{R}^2$ and g and h are unknown constants.

$$D_1 = p_1p_2 + gp_2 \qquad S_1 = 4p_1p_2 + p_2^2$$

$$D_2 = 3p_1^2 + p_2 + p_1p_2 \qquad S_2 = h^2p_1 + p_2$$

Find the relation between g and h that has to be verified in order to guarantee the existence of positive equilibrium prices. (2.5 points)

(2) Consider the following correspondence φ defined from the set $A = [-2\ 2]$ to $2^{[-2\ 2]}$:

$$\text{For } x \in [-2\ 0) \qquad \varphi(x) = \{0.8x + \beta\}$$

$$\text{For } x = 0 \qquad \varphi(0) = [0.5\ 1]$$

$$\text{For } x \in (0\ 2] \qquad \varphi(x) = \{0.5x + 1\}$$

Find the values of β that make the correspondence upper semicontinuous. Find a fixed point of the correspondence. (2.5 points)

PART II

(1) Let $f(x, y) = \ln(xy)$ with $x, y > 0$.

   (a) Show that f is a strictly concave function on its domain. (1 point)

   (b) Compute the local optimal points of f on its domain. (1 point)

   (c) Find the global maximizer of f on

$$D = \left\{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 \leq 1, x, y > 0\right\}.$$

   (3 points)

PART III

(1) Consider the difference equation $x_{n+1} = F(x_n)$ with $F(x) = \frac{1}{4}(x + x^2)$.

(a) Plot the graph of F. (1 point)

(b) Find the fixed points of F and determine their stability. (0.5 points)

(c) Compute $\lim_{n\to\infty} x_n$ for every $-1 < x_0 < 1$. Explain your answer. (1 point)

(2) Consider the matrix

$$A = \begin{pmatrix} 1 & 1 \\ -1 & -3 \end{pmatrix}$$

(a) Compute the Jordan Normal Form J of A and the corresponding change of variables P. (1 point)

(b) Compute the exponential matrix e^{tJ}. (0.5 points)

(c) Find the general solution of $\dot{X} = JX$, and sketch its phase portrait. (1 point)

PART IV

(1) Consider the problem for a government which wants to control the level of debt over GDP, b_t, by solving the problem:

$$\max_{\{\tau_t\}_{t=0}^{T-1}} \sum_{t=0}^{T-1} \beta^t(-\tau_t^2)$$

subject to the budget constraint $b_{t+1} = (1+r)b_t - \tau_t$ and the initial and terminal values $b_0 = \phi > 0$ and $b_T = 0$. Assume that $0 < \beta < 1$, $r > 0$ and $T > 0$ and is finite.

(a) Write the problem as a calculus of variations problem and derive the first order conditions (0.75 points).

(b) Solve the problem and provide an intuition to your results (1.75 points).

(2) Assuming that $x(\cdot)$ is a state variable and $u(\cdot)$ is a control variable, consider the optimal control problem

$$\max_{(u(t))_{t=0}^{\infty}} \int_0^{\infty} (x(t)^2 + u(t)^2)e^{-\rho t}dt$$

subject to $\dot{x} = \alpha(x - u)$ and $x(0) = \phi$ and $\lim_{t\to\infty} x(t)e^{-\rho t} = 0$. Assume that $0 < \rho < 2\alpha$ and that $\phi > 0$

(a) Determine the optimality conditions from the Pontryiagin's maximum principle. (0.75 points)

(b) Find an explicit solution for the optimal state variable $x(\cdot)$. Justify. (1.75 points)

Universidade de Lisboa – ISEG

Economics Department

Mathematical Economics

FIRST EXAM
January 5, 2015
Maximum duration: 2 hours
Solve each part of the exam on a separate sheet

PART I

(1) Consider the following Walras economy with two goods, 1 and 2. D_i, S_i and p_i are respectively the demand functions, supply functions and prices for each good i. Prices belong to the unit simplex set in $\mathbb{R}^2$ and g and h are unknown constants.

$$D_1 = p_1 p_2 + g p_2 \qquad S_1 = 4p_1 p_2 + p_2^2$$
$$D_2 = 3p_1^2 + p_2 + p_1 p_2 \qquad S_2 = h^2 p_1 + p_2$$

Find the relation between g and h that has to be verified in order to guarantee the existence of positive equilibrium prices. (2.5 points)

(2) Consider the following correspondence φ defined from the set $A = [-2\ 2]$ to $2^{[-2\ 2]}$:

$$\text{For } x \in [-2\ 0) \qquad \varphi(x) = \{0.8x + \beta\}$$
$$\text{For } x = 0 \qquad \varphi(0) = [0.5\ 1]$$
$$\text{For } x \in (0\ 2] \qquad \varphi(x) = \{0.5x + 1\}$$

Find the values of β that make the correspondence upper semicontinuous. Find a fixed point of the correspondence. (2.5 points)

PART II

(1) Let $f(x, y) = \ln(xy)$ with $x, y > 0$.

(a) Show that f is a strictly concave function on its domain. (1 point)

(b) Compute the local optimal points of f on its domain. (1 point)

(c) Find the global maximizer of f on

$$D = \left\{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 \leq 1, x, y > 0\right\}.$$

(3 points)

PART III

(1) Consider the difference equation $x_{n+1} = F(x_n)$ with $F(x) = \frac{1}{4}(x + x^2)$.

(a) Plot the graph of F. (1 point)

(b) Find the fixed points of F and determine their stability. (0.5 points)

(c) Compute $\lim_{n\to\infty} x_n$ for every $-1 < x_0 < 1$. Explain your answer. (1 point)

(2) Consider the matrix

$$A = \begin{pmatrix} 1 & 1 \\ -1 & -3 \end{pmatrix}$$

(a) Compute the Jordan Normal Form J of A and the corresponding change of variables P. (1 point)

(b) Compute the exponential matrix e^{tJ}. (0.5 points)

(c) Find the general solution of $\dot{X} = JX$, and sketch its phase portrait. (1 point)

PART IV

(1) Consider the problem for a government which wants to control the level of debt over GDP, b_t, by solving the problem:

$$\max_{\{\tau_t\}_{t=0}^{T-1}} \sum_{t=0}^{T-1} \beta^t(-\tau_t^2)$$

subject to the budget constraint $b_{t+1} = (1+r)b_t - \tau_t$ and the initial and terminal values $b_0 = \phi > 0$ and $b_T = 0$. Assume that $0 < \beta < 1$, $r > 0$ and $T > 0$ and is finite.

(a) Write the problem as a calculus of variations problem and derive the first order conditions (0.75 points).

(b) Solve the problem and provide an intuition to your results (1.75 points).

(2) Assuming that $x(\cdot)$ is a state variable and $u(\cdot)$ is a control variable, consider the optimal control problem

$$\max_{(u(t))_{t=0}^{\infty}} \int_0^{\infty} (x(t)^2 + u(t)^2)e^{-\rho t}dt$$

subject to $\dot{x} = \alpha(x - u)$ and $x(0) = \phi$ and $\lim_{t\to\infty} x(t)e^{-\rho t} = 0$. Assume that $0 < \rho < 2\alpha$ and that $\phi > 0$

(a) Determine the optimality conditions from the Pontryiagin's maximum principle. (0.75 points)

(b) Find an explicit solution for the optimal state variable $x(\cdot)$. Justify. (1.75 points)

Universidade de Lisboa – ISEG

Economics Department

Mathematical Economics

SECOND EXAM
26 January 2015

Maximum duration: 2 hours

Solve each part of the exam on a separate sheet

PART I

(1) Consider the following correspondence φ defined from the set $A = [-1\ 2]$ to $2^{[a\ b]}$:

$$\text{For } x \in [-1\ 1) \qquad \varphi(x) = \{0.5x + \beta\}$$
$$\text{For } x \in [1\ 2] \qquad \varphi(x) = \{0.8x + 0.4\}$$

Indicate values for a, b and β (one for each of these constants) that allow us to use the Kakutani fixed point theorem to asseverate the existence of at least one fixed point of φ and find one such fixed point. (2.5 points)

(2) Consider the following subsets of $\mathbb{R}^2$:

$$A = [0\ 2] \times [1\ 2]$$
$$B = \{(x\ y) \in \mathbb{R}^2 \colon x^2 + y^2 < 4\}$$
$$C = \{(x\ y) \in \mathbb{R}^2 \colon x + y = 2\}$$
$$D = [-1\ 0] \times [0\ 2]$$

where the symbol $\times$ stands for “cartesian product”. Using the Separating Hyperplane Theorem can we asseverate that there is at least one hyperplane separating the sets E and D where E is the set $E = A \cap B \cap C$? Explain why. (2.5 points)

PART II

(1) Consider the function $f(x, y, z) = x + xy + z^2$ defined in $\mathbb{R}^3$.

(a) Find and classify the critical points of f. (2 points)

(b) Determine the local optimal points of f on

$$D = \left\{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 = 1, z = 0\right\}.$$

(3 points)

PART III

(1) Consider the difference equation $x_{n+1} = F(x_n)$ with $F(x) = 2x(1 - x)$.

(a) Plot the graph of F. (1 point)

(b) Find the fixed points of F and determine their stability. (0.5 points)

(c) Compute $\lim_{n\to\infty} x_n$ for every $x_0 < 0$. Explain your answer. (1 point)

(2) Consider the matrix

$$A = \begin{pmatrix} -1 & 3 \\ 0 & 2 \end{pmatrix}$$

(a) Compute the Jordan Normal Form J of A and the corresponding change of variables P. (1 point)

(b) Compute the exponential matrix e^{tJ}. (0.5 points)

(c) Find the general solution of $\dot{X} = JX$, and sketch its phase portrait. (1 point)

PART IV

(1) A representative consumer wants to maximize the intertemporal utility functional $\sum_{t=0}^{\infty} \beta^t \ln\left(C_t^{\alpha} Z_t^{1-\alpha}\right)$, where $0 < \alpha < 1$ and $0 < \beta < 1$, by using consumption C_t as a control variable. The variable Z_t denotes habits and is governed by the difference equation $Z_{t+1} = \delta(Z_t - C_t)$, where $\delta > 0$. The following initial and terminal conditions hold: $Z(0) = Z_0 > 0$, and $\lim_{t\to\infty} \beta^t Z(t) \geq 0$.

(a) Write the first order optimality conditions from the Pontryiagin's maximum principle. (0.75 points)

(b) Solve the problem, and provide an intuition to your results. (1.75 points)

(2) A central bank wants to determine the optimal inflation rate $\pi(\cdot)$ by maximising the objective function $\int_0^T -(u(t)^2 + \pi(t)^2)e^{-\rho t} dt$ where $u(\cdot)$ is the unemployment rate. It also wants to set the terminal variation of the inflation rate to zero, i.e. $\dot{\pi}(T) = 0$. However, it faces the following constraints: $\dot{\pi} = u - u^n$, where u^n is the constant natural unemployment rate, and $\pi(0) = \pi_0$ is given.

(a) Write the problem as a calculus of variations problem and derive the first order conditions (0.75 points).

(b) Determine the optimal inflation rate function, $\pi^*(t)$, and provide an intuition to your results (1.75 points).

Universidade de Lisboa – ISEG

Mathematical Economics

FIRST EXAM

January 7, 2016

Maximum duration: 2 hours

Solve each part of the exam on a separate sheet

PART I

(1) Consider an economy where two goods, 1 and 2 are exchanged such that $p_1 + p_2 = 1$ and the demand (D) and supply (S) functions are respectively:

For good 1:

$$D_1 = 100p_1^2 - p_2^2 + 2p_1p_2 \qquad S_1 = -p_1p_2 + 100p_1^2 - 0.5p_2$$

For good 2:

$$D_2 = -p_1^2p_2 + p_1p_2 \qquad S_2 = 3p_1^2 - p_1^2p_2 + \alpha p_1$$

(a) Find the value of α that guarantees the existence of positive equilibrium prices. (1.5 points)

(b) Using that value calculate the vector of Walras equilibrium prices. (1 point)

(2) Consider the correspondence defined on the closed interval $[0, 2]$ of $\mathbb{R}$:

$\varphi\colon [0, 2] \to 2^{[0,2]}$

$$\begin{aligned} \text{For } 0 \leq x < 1.5 \qquad & \varphi(x) = \{1 - x/2\} \\ \text{For } x = 1.5 \qquad & \varphi(0) = [\alpha, \beta] \\ \text{For } 1.5 < x \leq 2 \qquad & \varphi(x) = \{1 + x/2\} \end{aligned}$$

(a) Find the values of α and β that allows us to use the Kakutani fixed-point theorem and such that $\beta - \alpha$ is smaller than the respective difference for any other closed interval that verifies the theorem. (1.5 points)

(b) For those values of α and β find all the fixed points of the correspondence. (1 point)

PART II

(1) Let $f(x, y, z) = \ln(xyz)$ with $x, y, z > 0$.

(a) Show that f is a strictly concave function on its domain. (2 point)

(b) Find the global maximizer of f on

$$D = \left\{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 + z^2 \leq 1, x, y, z > 0\right\}.$$

(3 points)

PART III

(1) Consider the difference equation $x_{n+1} = F(x_n)$ with $F(x) = 2x - 1$.

  (a) Find the fixed points of F and determine their stability. (0.5 points)

  (b) Compute the general solution. (1 point)

  (c) Compute $\lim_{n\to\infty} x_n$ for every $x_0 \in \mathbb{R}$. Explain your answer. (1 point)

(2) Consider the matrix

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$$

  (a) Compute the Jordan Normal Form J of A. (0.5 points)

  (b) Sketch the phase portrait of $\dot{y} = Jy$ and classify the equilibrium point of the system. (1 point)

  (c) Find the general solution of $\dot{X} = AX$. (1 point)

PART IV

(1) A firm wants to maximize the present value of the cash flow by investing in capital K_t. The problem of the firm is specified as:

$$\max_{\{I_t\}_{t=0}^{T-1}} \sum_{t=0}^{T-1} \left(\frac{1}{1+r} \right)^t \pi_t$$

where $r > 0$ is the constant market interest rate, $\pi_t = A \cdot K_t - (I_t)^2$ is the cash flow for period t where investment is $I_t = K_{t+1} - K_t$, and A is a productivity parameter. The initial capital stock is $K(0) = \phi$ and the terminal time $T > 0$ is known and it is finite.

(a) Write the problem as a calculus of variations problem and derive the first order conditions (1 point).

(b) Solve the problem and provide an intuition to your results (2 points).

(2) Assuming that $x(.)$ is a state variable and $u(.)$ is a control variable, consider the optimal control problem

$$\max_{(u(t))_{t=0}^{T}} \int_0^T \ln(u(t))dt$$

subject to $\dot{x} = x - u$ and $x(0) = x(T) = 1$.

(a) Determine the optimality conditions from the Pontryiagin,'s maximum principle. (0.5 point)

(b) Find the explicit solution to the problem. (1.5 points)

Universidade de Lisboa – ISEG

Mathematical Economics

SECOND EXAM

February 2, 2016

Maximum duration: 2 hours

Solve each part of the exam on a separate sheet

PART I

(1) Consider the following correspondence φ defined on the closed interval $[-1, 2]$ of $\mathbb{R}$
$\varphi\colon [-1, 2] \to 2^{[-1,2]}$:

$$\text{For } x \in [-1, 0) \qquad \varphi(x) = \{\alpha x + \beta\}$$
$$\text{For } x = 0 \qquad \varphi(0) = [0.5, 1]$$
$$\text{For } x \in (0, 2] \qquad \varphi(x) = \{0.5x + 1\}$$

(a) Find a pair of values (α, β) that allow us to use the Kakutani fixed-point theorem. (1.5 points)
(b) For such a pair find the fixed-points of the correspondence. (1 point)

(2) Consider the following subsets of $\mathbb{R}^2$:

$$A = B \cap C$$
$$B = [1, 2] \times [1, 2]$$
$$C = (0.5, 1.5) \times (1, 1.5)$$
$$D = \{x \in \mathbb{R}^2 \colon x_2 = 3 - x_1\}$$

where the symbol $\times$ stands for the cartesian product and x_1 and x_2 are the components of the vector x.
(a) Determine whether the conditions of the separating hyperplane theorem for the sets A and D are met. (1.5 points)
(b) Independently of this being the case, say if there is a hyperplane separating A from D. (1 point)

PART II

(1) Consider the function $f(x, y, z) = x + y + z + xyz$ defined in $\mathbb{R}^3$.

(a) Find and classify the critical points of f. (2 points)

(b) Determine the local optimal points of f on

$$D = \left\{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 = 1, z = 0\right\}.$$

(3 points)

PART III

(1) Consider the difference equation $x_{n+1} = F(x_n)$ with $F(x) = \frac{1}{3}x + 2$.

   (a) Find the fixed points of F and determine their stability. (0.5 points)

   (b) Compute the general solution. (1 point)

   (c) Compute $\lim_{n\to\infty} x_n$ for every $x_0 \in \mathbb{R}$. Explain your answer. (1 point)

(2) Consider the matrix

$$A = \begin{pmatrix} -2 & 1 \\ -4 & -2 \end{pmatrix}$$

   (a) Compute the Jordan Normal Form J of A. (0.5 points)

   (b) Sketch the phase portrait of $\dot{y} = Jy$ and classify the equilibrium point of the system. (1 point)

   (c) Find the general solution of $\dot{X} = AX$. (1 point)

PART IV

(1) Consider the calculus of variations problem

$$\max_{\{x_t\}_{t=0}^{T-1}} \sum_{t=0}^{T-1} -(1+r)^{-t}(x_{t+1} - r x_t)^2,$$

where $r > 0$, $T > 0$ are known and $x_0 = 1$.

(a) Write the first order conditions (0.5 point).

(b) Determine the solution to the problem (1.5 points).

(2) A representative consumer wants to maximize the intertemporal utility functional

$$\int_{t=0}^{\infty} e^{-\rho t} \ln \left(C(t)^{\beta} Z(t)^{1-\beta}\right),$$

where $0 < \beta < 1$ and $\rho > 0$, by using consumption $C(t)$ as a control variable. The variable $Z(t)$ denotes the habits, as a stock variable, and is governed by the differential equation $\dot{Z} = \delta(Z(t) - C(t))$, where $\delta > 0$. The following initial and terminal conditions hold: $Z(0) = Z_0 > 0$, and $\lim_{t\to\infty} e^{-\rho t} Z(t) \geq 0$.

(a) Write the first order optimality conditions from the Pontyiagin's maximum principle. (1 point)

(b) Solve the problem, and provide an intuition to your results. (2 points)

Universidade de Lisboa - ISEG

Mathematical Economics

FIRST EXAM

January 6, 2017

Maximum duration: 2h30m

Solve each part of the exam on a separate sheet

PART I

(1) Consider the following Walras economy with two goods, 1 and 2. D_i, S_i and p_i are respectively the demand functions, supply functions and prices for each good i. Prices belong to the unit simplex set in $\mathbb{R}^2$ and m is an unknown constant.

$D_1 = p_1^2 + mp_2,$ $\qquad S_1 = p_1 p_2 + p_1^2 + 0.3p_2,$

$D_2 = p_1^2 + p_2^2 + 0.8p_1,$ $\qquad S_2 = p_1 + p_2^2.$

Find the value of m that guarantees the existence of positive equilibrium prices and find those prices for that value of m. (2.5 points)

(2) Consider the following subsets of $\mathbb{R}^2$,

$$A = [-1, 1] \times [-1, 1],$$
$$B = \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 < 1\},$$
$$C = \{(x, y) \in \mathbb{R}^2 \colon y < x + 0.5\},$$
$$E = [-1, 0] \times [0.5, 1].$$

Where the symbol $\times$ stands for "Cartesian product". Using the Separating Hyperplane Theorem can we asseverate that there is at least one hyperplane separating the sets D and E where D is the set $D = A \cap B \cap C$? Explain why. (2.5 points)

PART II

(1) Consider the following problem:

$$\max f(x,y) = (x+2y)^2$$
$$\text{subject to } x + y^2 \leq 9$$
$$x, y \geq 0$$

(a) State the Weierstrass theorem and explain whether it can be used to help solve the problem above. (1 point)

(b) Solve the problem above using the Kuhn-Tucker theorem. Explain carefully all the steps in your reasoning. (4 points)

PART III

(1) Consider the difference equation $x_{n+1} = F(x_n)$ with $F(x) = x/5 + 4$.
   (a) Find the fixed points of F and determine their stability. (0.5 points)
   (b) Compute the general solution. (1 point)
   (c) Compute $\lim_{n\to\infty} x_n$ for every $x_0 \in \mathbb{R}$. Explain your answer. (1 point)

(2) Consider the matrix

$$A = \begin{pmatrix} -3 & 2 \\ -4 & 1 \end{pmatrix}$$

   (a) Compute the Jordan Normal Form J of A. (0.5 points)
   (b) Sketch the phase portrait of $\dot{y} = Jy$ and classify the equilibrium point of the system. (1 point)
   (c) Find the general solution of $\dot{X} = AX$. (1 point)

PART IV

(1) Consider the following calculus of variations problem

$$\max_{\{y_t\}_{t=0}^{T-1}} \sum_{t=0}^{T-1} -\left(y_{t+1} - 2y_t - 2\right)^2$$

subject to $y_0 = 1$, where the terminal time $T > 0$ is known and it is finite.

(a) Write the first order conditions (1 point).

(b) Solve the problem and provide an intuition for your results. (1.5 points)

(2) Consider an household that wants to find the optimal path for consumption and financial assets $(C(t), A(t))_{t=0}^{T}$ that solve the problem

$$\max_{(C(t))_{t=0}^{T}} \int_0^T \frac{C(t)^{1-\theta}}{1-\theta} e^{-\rho t} dt, \ \rho > 0, \ \theta > 0$$

subject to $\dot{A} = rA + Y - C$ and $A(0) = A(T) = 0$, where $T > 0$ and finite, $Y > 0$ represents the non-financial income, $r > 0$ is the interest rate.

(a) Determine the optimality conditions according to the Pontryiagin's maximum principle. (1 point)

(b) Find the explicit solution to the problem. Provide an intuition for your results. (1.5 points)

Universidade de Lisboa - ISEG

Mathematical Economics

SECOND EXAM

January 31, 2017

Maximum duration: 2h30m

Solve each part of the exam on a separate sheet

PART I

(1) Consider the following correspondence φ defined from the set $A = [-1, 2]$ to $2^{[-1,2]}$:

$$\varphi(x) = \begin{cases} \{0.8x + m\}, & x \in [-1, 0) \\ [0.5, n], & x = 0 \\ \{0.5x + 1\}, & x \in (0, 2] \end{cases}$$

Find one value for m and one value for n that make the correspondence upper semicontinuous. Find a fixed-point of the correspondence using those values. (2.5 points)

(2) Let A and B be two sets of $\mathbb{R}^2$ such that $A = C \cap D$ with

$$C = \{(x, y)\colon y \leq x + 1\},$$
$$D = \{(x, y)\colon x^2 + y^2 \leq 1\},$$

and $B = [-0.5, 0.5] \times [a, 2]$ where $\times$ denotes "Cartesian product". Find a value for a that guarantees the existence of at least one hyperplane separating A and B. Find one such hyperplane. (2.5 points)

PART II

(1) Let
$$f(x,y) = \frac{x^2 + y^2 + z^2}{2} - \log(xyz)$$
with $x, y, z > 0$.

(a) Show that f is strictly convex on its domain. (1 point)

(b) Find the global minimizer of f on
$$D = \{(x,y,z) \in \mathbb{R}^3 \colon x + y + z \leq 1, \quad x, y, z > 0\}.$$
(4 points)

PART III

(1) Consider the initial value problem

$$\dot{x} = \lambda x(1 - x), \quad x(0) = 2,$$

where $\lambda > 0$.

(a) Find the solution $x(t)$ of the initial value problem and compute $\lim_{t \to +\infty} x(t)$. (1.5 points)

(b) Sketch the phase portrait of the associated ODE. (1 point)

(2) Consider the matrix

$$A = \begin{pmatrix} 2 & 0 \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix}$$

(a) Find the Jordan Normal Form J of A. (0.5 points)

(b) Compute A^n, $n \geq 0$. (1 point)

(c) Find the general solution of the following difference equation

$$X_{n+1} = AX_n + \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

(1 point)

PART IV

(1) A government wants to minimize fluctuation in the unemployment rate, u_t, and in the inflation rate, π_t, by solving the problem

$$\max_{\{x_t\}_{t=0}^{T-1}} \sum_{t=0}^{T-1} -(u_t)^2,$$

subject to $\pi_{t+1} = \pi_t + \mu - u_t$ and $\pi_T = \pi_{T-1}$, where $\mu > 0$ is the natural unemployment rate and $T > 0$. The initial value for the inflation rate is $\pi_0 = \phi$.

(a) Write the first order conditions according to the Pontryiagin's maximum principle. (1 point).

(b) Determine the solution to the problem. Provide an intuition for your results. (1.5 points).

(2) Consider the following calculus of variations problem

$$\max_{y} \int_0^\infty e^{\dot{y}(t)-y(t)} e^{-\rho t} dt, \ \rho > 0$$

subject to the constraints $y(0) = 0$ and $\lim_{t\to\infty} y(t) \geq 0$

(a) Write the first order optimality conditions. (1 point)

(b) Solve the problem. Provide an intuition for your results. (1.5 points)

Universidade de Lisboa - ISEG

Mathematical Economics

FIRST EXAM

January 9, 2018

Maximum duration: 2h30m

Solve part IV on a separate sheet

PART I

(1) Consider the following correspondence $F : [0,4] \rightrightarrows [0,4]$,

$$F(x) = \begin{cases} [\frac{3}{2}x, x+2], & x < 2 \\ [0,a] \cup [4-a,4], & x = 2 \\ [x-2, \frac{3}{2}x - 2], & x > 2 \end{cases}$$

where $a \in [0,4]$.

(a) Determine the values of a such that F has the closed graph property. (1 point)

(b) Determine the values of a such that F satisfies the assumptions of the Kakutani fixed point theorem. Find the fixed points of F. (2 points)

(2) Let A and B denote the following sets

$$A = \{(x,y) \in \mathbb{R}^2 \colon y \geq (x-2)^2 - 1\}$$
$$B = \{(x,y) \in \mathbb{R}^2 \colon x + y \leq -2\}$$

Using the hyperplane separation theorem can we conclude that A and B are separated by a hyperplane? If affirmative, then find one such separating hyperplane. (2 points)

PART II

(1) Determine the values of the real parameter α for which the following function is concave

$$g(x, y) = x^2 + 2y + \alpha(4 - x^2 - y^2)$$

(2 points)

(2) Consider the following problem:

$$\begin{aligned} \text{maximize } & x^2 + 2y \\ \text{subject to } & x^2 + y^2 \leq 4 \\ & x \geq 0 \end{aligned}$$

Solve the problem above using the Kuhn-Tucker theorem. Explain carefully all the steps in your reasoning. (3 points)

PART III

(1) Consider the differential equation

$$x' = x^2 - \alpha x$$

where α is a real parameter.

(a) Assuming $\alpha = 1$, compute the solution of the initial value problem with $x(0) = 1/2$. (1 point)

(b) Determine and classify the equilibrium points according to the values of α. (1 point)

(2) Consider the matrix

$$A = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$$

(a) Find the Jordan normal form of A. (0.5 points)

(b) Sketch the phase portrait of $X' = AX$ and classify the equilibrium point of the system. (1 point)

(c) Find the solution of the initial value problem

$$X' = AX, \quad X(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

(1.5 points)

PART IV

(1) Consider the following calculus of variations problem

$$\max_{y} \int_0^T \left(y(t) - \frac{1}{2}\dot{y}(t) \right)^2 dt$$

subject to $y(0) = 1$, where the terminal time T is finite, positive and fixed, and the terminal level of the state variable is free.

(a) Write the first order conditions (0.5 point).

(c) Prove that the solution of the Euler-Lagrange equation is $y(t) = k_1 e^{2t} + k_2 e^{-2t}$, where k_1 and k_2 are arbitrary constants. (0.5 point)

(c) Solve the problem and provide an intuition for your results (1 point)

(2) The problem for a consumer/saver is

$$\max_{C} \int_0^{\infty} \ln(C(t)) e^{-\rho t} dt, \ \rho > 0,$$

subject to the budget constraint, $\dot{A} = rA + Y - C$, the initial condition for the asset position, $A(0) = 0$, and the non-Ponzi game condition $\lim_{t\to\infty} e^{-rt}A(t) \geq 0$. In the model $Y > 0$ represents the non-financial income, $r > 0$ is the interest rate and $\rho > 0$ is the rate of time preference.

(a) Write the first-order optimality conditions according to the Pontryiagin's maximum principle, and specify the MHDS in (A, C). (1 point)

(b) Find the explicit solution to the problem. (1.5 points)

(c) Draw the phase diagram, assuming that $r > \rho$, and provide an intuition for the behavior of the consumer. (0.5 point)

Universidade de Lisboa - ISEG

Mathematical Economics

FIRST EXAM

January 7, 2019

Maximum duration: 2h30m

Solve part IV on a separate sheet

PART I

(1) Consider the following correspondence $F : [0,3] \rightrightarrows \mathbb{R}$,

$$F(x) = \begin{cases} [x+1, x+2], & x < 1 \\ [-ax+b, -ax+b+1], & 1 \leq x \leq 2 \\ [x-2, x-1], & x > 2 \end{cases}$$

where $a, b \in \mathbb{R}$.

(a) Determine the values of a and b such that F satisfies the assumptions of the Kakutani fixed point theorem. (1.5 points)

(b) Find the fixed points of F for those values of a and b found in (a). In case you did not solve (a), you may take $a = 1$ and $b = 3$. (1.5 points)

(2) Consider the function $f : \{x \in \mathbb{R} : x \geq 1\} \to \mathbb{R}$ defined by

$$f(x) = \sqrt{x} + 1$$

(a) Verify that f satisfies the hypothesis of the Banach fixed point theorem. (1 point)

(b) Find the fixed point of f. (1 point)

PART II

(1) Find and classify the critical points of

$$f(x, y) = -5y^2 - 2x^2 + 2xy + 4y - 4$$

(2 points)

(2) Consider the following problem:

$$\text{maximize } x - y$$
$$\text{subject to } x^2 + z^2 \leq y$$
$$x \geq 0$$

Solve the problem above using the Kuhn-Tucker theorem. Explain carefully all the steps in your reasoning. (3 points)

PART III

(1) Consider the IVP

$$x' = \alpha x^2 + 1, \quad x(0) = 0$$

where α is a real parameter.

(a) Assuming $\alpha = -1$, compute the solution of the IVP. (1.5 points)

(b) Let $\alpha = 1$ and $x(t)$ denote the corresponding solution of the IVP. Compute $\lim_{t\to+\infty} x(t)$. (0.5 points)

(2) Consider the system of differential equations

$$\begin{cases} x' = 7x - 10y \\ y' = 5x - 8y \end{cases}$$

(a) Write the system in matrix notation. Sketch the phase portrait of the associated Jordan normal form and classify the equilibrium point. (1 point)

(b) Find the solution of the IVP with $x(0) = 1$ and $y(0) = -1$. (2 points)

PART IV

(1) Consider the calculus of variations problem

$$\max_{\{x_t\}_{t=0}^{T-1}} \sum_{t=0}^{T-1} -(1+r)^{-t}(x_{t+1} - r\, x_t)^2,$$

where $r > 0$, and $T > 0$ are known, and $x_0 = 1$.

(a) Write the first order conditions. (1 point)

(b) Find the solution to the calculus of variations problem. (1.5 points)

(2) Assuming that capital, $K(.)$, is a state variable, and consumption, $C(.)$, is a control variable, consider the optimal growth problem:

$$\max_{(C(t))_{t=0}^{\infty}} \int_0^{\infty} \ln(C(t)) e^{-\rho t} dt$$

subject to $\dot{K} = A\, K(t) - C(t)$, $K(0) = k_0 > 0$ is known, and $\lim_{t\to\infty} K(t) \geq 0$. Assume $A > \rho > 0$.

(a) Write the optimality conditions from the Pontryiagin,'s maximum principle. (1 point)

(b) Find the explicit solution to the problem. (1.5 points)

Universidade de Lisboa - ISEG

Mathematical Economics

SECOND EXAM

February 1, 2019

Maximum duration: 2h30m

Solve part IV on a separate sheet

PART I

(1) Consider the following correspondence $F : [0,1] \rightrightarrows [0,1]$,

$$F(x) = \begin{cases} 2x, & x \leq 1/2 \\ \left[-a(x-1), -x + \frac{3}{2}\right], & x > 1/2 \end{cases}$$

where $a \in [0,2]$.

(a) Determine the values of a such that F satisfies the hypothesis of the Kakutani fixed point theorem. (1 point)

(b) Assuming that $a = 1$, find the fixed points of F. (1.5 points)

(2) Consider the following matrix

$$A = \begin{pmatrix} 1/2 & 1 & 1 \\ 0 & 0 & 0 \\ 1/2 & 0 & 0 \end{pmatrix}$$

and define the function $f(v) = Av$ where $v \in \Delta^2 = \{(x,y,z) \in \mathbb{R}^3 \colon x + y + z = 1,\, x, y, z \geq 0\}$.

(a) State the Brouwer fixed point theorem. (0.5 points)

(b) Show that f satisfies the hypothesis of the Brouwer fixed point theorem. (1 point)

(c) Compute the fixed points of f explicitly. (1 point)

PART II

(1) Let

$$f(x,y) = -\frac{x^2+y^2}{4} + \log(x^2y^2)$$

with $x, y > 0$.

(a) Show that f is strictly concave on its domain. (2 points)

(b) Consider the following problem:

$$\text{maximize } f(x,y)$$
$$\text{subject to } x+y \leq 1$$

Solve the problem above using the Kuhn-Tucker theorem. Explain carefully all the steps in your reasoning. (3 points)

PART III

(1) Consider the ODE

$$x' + \lambda t x = 4t,$$

where $\lambda > 0$.

(a) Assuming $\lambda = 2$, compute the solution of the initial value problem when $x(0) = 3$. (1 point)

(b) Let $x(t)$ denote the general solution of the ODE. Determine the value of λ such that $\lim_{t\to+\infty} x(t) = 1$. (1 point)

(2) Consider the matrix

$$A = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$$

(a) Find the Jordan normal form J of A. (0.5 points)

(b) Sketch the phase portrait of $X' = JX$ and classify the equilibrium point of the system. (1 point)

(c) Find the solution of the initial value problem

$$X' = AX, \quad X(0) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

(1.5 points)

PART IV

(1) Consider the following optimal control problem, for a representative household

$$\max_{\{C_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln C_t$$

subject to $A_{t+1} = (1+r)A_t - C_t + Y$, $A_0 = \phi$, and $\lim_{t\to\infty} A_t \geq 0$, where A, r and Y are positive constants.

(a) Write the optimality conditions from the Pontryiagin,'s maximum principle. (1 point)

(b) Find the explicit solution to the problem. (1.5 points)

(2) Consider the following problem

$$\max_{(u(t))_{t=0}^{T}} \int_0^T \ln(u(t))dt$$

subject to $\dot{x} = \alpha x - u$ and $x(0) = 1$ and $x(T) = e^{\alpha T}$.

(a) Specify the problem as a calculus of variations problem and find the optimality conditions. (1 point)

(b) Solve the problem. (1.5 points)

Universidade de Lisboa - ISEG

Mathematical Economics

FIRST EXAM

January 9, 2020

Maximum duration: 2h30m

PART I

(1) Consider the following correspondence $F : [0,1] \rightrightarrows \mathbb{R}$,

$$F(x) = \begin{cases} [2x(1-x), 1], & x < 1/2 \\ [a, b], & x = 1/2 \\ \{2(1-x)\}, & x > 1/2 \end{cases}$$

where $a, b \in \mathbb{R}$.

(a) State the Kakutani fixed point theorem. (1 point)

(b) Determine the values of a and b such that F satisfies the assumptions of the Kakutani fixed point theorem. (1 point)

(c) Find the fixed points of F for those values of a and b found in (a). In case you did not solve (a), you may take $a = 0$ and $b = 1$. (1 point)

(2) Consider the function $f : [-1/3, 1/3] \to \mathbb{R}$ defined by

$$f(x) = x^2 + \frac{2}{9}$$

(a) Verify that f satisfies the hypothesis of the Banach fixed point theorem. (1 point)

(b) Find the fixed point of f. (1 point)

PART II

(1) Find and classify the critical points of

$$f(x, y) = x \log(x^2 + y^2), \quad (x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$$

(2 points)

(2) Solve the following problem:

$$\begin{aligned} \text{minimize } & x + 4y \\ \text{subject to } & x + y = 1 \\ & x^2 + z^2 = 25 \end{aligned}$$

Explain carefully all the steps in your reasoning. (3 points)

PART III

(1) Consider the differential equation

$$tx'(t) + x(t) = 1$$

(a) Classify the differential equation and determine its general solution. (1.5 points)

(b) Let $x(t)$ denote the particular solution when $x(1) = 0$. Find $\lim_{t\to+\infty} x(t)$. (0.5 points)

(2) Consider the matrix

$$A = \begin{pmatrix} 0 & -4 \\ -1 & 0 \end{pmatrix}$$

(a) Compute the Jordan Normal Form J of A. (0.5 points)

(b) Sketch the phase portrait of $\dot{X} = AX$ and classify the equilibrium point of the system. (1 point)

(c) Find the general solution of $\dot{X} = AX$. (1.5 points)

PART IV

(1) Consider the calculus of variations problem

$$\max_{x(t)} \int_0^1 (10 - \dot{x}^2 - 2x\dot{x} - 5x^2)e^{-t}\, dt$$

where $x(0) = 0$ and $x(1)$ is free.

(a) Write the corresponding Euler-Lagrange equation. (1 point)

(b) Find the solution to the calculus of variations problem. (1.5 points)

(2) Assuming that capital, $K(t)$, is a state variable, and consumption, $C(t)$, is a control variable, consider the optimal growth problem:

$$\max_{C(t)} \int_0^T \ln(C(t))e^{-\rho t} dt$$

subject to $\dot{K} = \alpha\, K(t) - C(t)$, $K(0) = k_0 > 0$ is known, and $K(T) \geq 0$. Assume $\alpha > \rho > 0$.

(a) Write the optimality conditions from the Pontryiagin's maximum principle. (1 point)

(b) Find the explicit solution to the problem. (1.5 points)

Universidade de Lisboa - ISEG

Mathematical Economics

SECOND EXAM

February 4, 2020

Maximum duration: 2h30m

PART I

(1) Consider the following Walras economy with two goods, 1 and 2. D_i, S_i and p_i are respectively the demand functions, supply functions and prices for each good i. Prices belong to the unit simplex set in $\mathbb{R}^2$ and α is an unknown constant. Suppose that

$$D_1 = p_1^2 + 0.5p_2, \qquad S_1 = p_1p_2 + p_1^2 + 0.3p_2,$$
$$D_2 = p_1^2 + p_2^2 + \alpha p_1, \qquad S_2 = p_1 + p_2^2.$$

(a) Find the values of α that guarantees that this economy satisfies the Walras' law. (1 point)

(b) Determine the equilibrium price of each good. (1.5 points)

(2) Consider the function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = 1 - \frac{1}{2(x+1)}$$

(a) State the Banach fixed point theorem. (0.5 points)

(b) Verify that f satisfies the hypothesis of the Banach fixed point theorem. (1 point)

(c) Find the fixed point of f. (1 point)

PART II

(1) Find and classify the critical points of

$$f(x, y) = xy(y-4)e^{-x}, \quad (x, y) \in \mathbb{R}^2$$

(2 points)

(2) Solve the following problem:

$$\begin{aligned} &\text{minimize } 2x^2 + 3y^2 \\ &\text{subject to } x^2 + y^2 \le 4 \\ &\qquad\qquad x + 2y \ge 2 \end{aligned}$$

Explain carefully all the steps in your reasoning. (3 points)

PART III

(1) Consider the initial value problem

$$tx' + x^2 = 1, \quad x(1) = 0$$

(a) Classify the differential equation. (0.5 points)
(b) Determine the solution of the IVP. (1.5 points)

(2) Consider the matrix

$$A = \begin{pmatrix} -1 & 4 \\ -10 & -5 \end{pmatrix}$$

(a) Compute the Jordan Normal Form J of A. (0.5 points)
(b) Sketch the phase portrait of $\dot{X} = AX$ and classify the equilibrium point of the system. (1 point)
(c) Find the solution of the IVP

$$\dot{X} = AX, \quad X(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(1.5 points)

PART IV

(1) Consider the optimal savings problem

$$\max_{K(t)} \int_0^T U(\alpha K(t) - \dot{K}(t))e^{-rt}\, dt$$

where $\alpha > r > 0$, the utility function U is C^2, $K(0) = K_0$ and $K(T) = K_T$.

(a) Show that the Euler-Lagrange equation is equal to

$$\frac{\dot{C}}{C} = \frac{r - \alpha}{E(C)}$$

where $C = \alpha K - \dot{K}$ is the consumption and $E(C) = \frac{CU''(C)}{U'}$ is the elasticity of marginal utility. (1 point)

(b) Assuming that $U(C) = \sqrt{C}$, find the solution to the optimal savings problem. (1.5 points)

(2) Consider the following optimal control problem

$$\max_{u(t)} \left\{ \int_0^1 -\frac{1}{2}u(t)^2\, dt + \sqrt{x(1)} \right\}$$

where $\dot{x} = x + u$, $x(0) = 0$ and $x(1)$ is free.

(a) Write the optimality conditions from the Pontryiagin's maximum principle. (1 point)

(b) Find the explicit solution to the problem. (1.5 points)