Lévy processes and applications - Part 2

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Characteristic function

Definition

The characteristic function of the random variable X (with values in $\mathbb{R}^d$), and distribution μ, is the function $\phi_X : \mathbb{R}^d \to \mathbb{C}$, defined by

$$\phi_X(u) = \mathbb{E}\left[e^{i(u \cdot X)}\right] = \int_{\mathbb{R}^d} e^{i(u \cdot x)} \mu(dx), \quad u \in \mathbb{R}^d.$$

- The characteristic function of a random variable completely characterizes its distribution, so we can write $\phi_X = \phi_\mu$.
- Properties of a characteristic function ϕ :
  1. $\phi(0) = 1$
  2. $|\phi(u)| \leq 1, \ \forall u \in \mathbb{R}^d$.
  3. ϕ is uniformly continuous
- The moments of a random variable are related to the derivatives at zero of its characteristic function - see Cont and Tankov, page 30.
- Exercise: Prove property 2.

Infinite divisibility

Definition

A probability distribution μ on $\mathbb{R}^d$ is said to be infinitely divisible if for any $n \in \mathbb{N}$, there exist n i.i.d. random variables $Y_1^{(n)}, Y_2^{(n)}, \ldots, Y_n^{(n)}$ such that $Y_1^{(n)} + Y_2^{(n)} + \cdots + Y_n^{(n)}$ has distribution μ.

Definition

A r.v. X is infinitely divisible if its distribution μ is infinitely divisible. This means that

$$X \stackrel{d}{=} Y_1^{(n)} + Y_2^{(n)} + \cdots + Y_n^{(n)},$$

where $Y_1^{(n)}, \ldots, Y_n^{(n)}$ are i.i.d., for each $n \in \mathbb{N}$.

Theorem

The distribution μ is infinitely divisible iff for all $n \in \mathbb{N}$, exists μ_n with charact. func. ϕ_n:

$$\phi_\mu(u) = (\phi_n(u))^n$$

for all $u \in \mathbb{R}^d$.

- idea of the proof: Let X be a r.v. with distribution μ and characteristic function ϕ_μ. Taking the i.i.d. $Y_1^{(n)}, Y_2^{(n)}, \ldots, Y_n^{(n)}$ such that $X \overset{d}{=} Y_1^{(n)} + Y_2^{(n)} + \cdots + Y_n^{(n)}$, by the independence of the $Y_i^{(n)}$,

$$\mathbb{E}\left[e^{iuX}\right] = \left(\mathbb{E}\left[e^{iuY_1^{(n)}}\right]\right)^n = (\phi_n(u))^n,$$

where $\phi_n(u)$ is the charact. function of $Y_1^{(n)}, Y_2^{(n)}, \ldots, Y_n^{(n)}$.

- Exercise: Let $\alpha > 0$, $\beta > 0$. Show that the gamma-(α, β) distribution

$$\mu_{\alpha,\beta}(dx) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} dx, \quad \text{with } x > 0,$$

with characteristic function $\left(1 - \frac{iu}{\beta}\right)^{-\alpha}$, is an infinitely-divisible distribution.

- For a table with examples of characteristic functions, see Cont and Tankov, page 33.

Infinite divisibility - Examples

- In each example, we will find iid $Y_1^{(n)}, \ldots, Y_n^{(n)}$ such that $X \stackrel{d}{=} Y_1^{(n)} + Y_2^{(n)} + \cdots + Y_n^{(n)}$.

Example

(Gaussian random variable) Let X be Gaussian random variable, with density:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-m)^2}{2\sigma^2}\right), \quad x \in \mathbb{R}^d.$$

$X \sim N(m, \sigma^2)$.
One can show that

$$\phi_X(u) = \exp\left(imu - \frac{1}{2}\sigma^2 u^2\right).$$

Infinite divisibility - Examples

Example

(continued) Therefore:

$$(\phi_X(u))^{\frac{1}{n}} = \exp\left(i\frac{m}{n}u - \frac{1}{2}\frac{\sigma^2}{n}u^2\right).$$

and X is inf. divis. with $X \overset{d}{=} Y_1^{(n)} + Y_2^{(n)} + \cdots + Y_n^{(n)}$ and

$$Y_j^{(n)} \sim N\left(\frac{m}{n}, \frac{\sigma^2}{n}\right).$$

Infinite divisibility - Examples

Example

(Poisson r.v.) Let $d = 1$ and $X : \Omega \to \mathbb{N}_0$ with $X \sim Po(\lambda)$, i.e.

$$P(X = n) = \frac{\lambda^n}{n!} e^{-\lambda}.$$

It is well known that $E[X] = Var[X] = \lambda$ and it is easy to verify that

$$\phi_X(u) = \exp\left[\lambda\left(e^{iu} - 1\right)\right].$$

Therefore

$$(\phi_X(u))^{\frac{1}{n}} = \exp\left[\frac{\lambda}{n}\left(e^{iu} - 1\right)\right].$$

and X is inf. divis. with $X \stackrel{d}{=} Y_1^{(n)} + Y_2^{(n)} + \cdots + Y_n^{(n)}$ and

$$Y_j^{(n)} \sim Po\left(\frac{\lambda}{n}\right).$$

Example

(Compound Poisson r.v.) Let $\{Z(n), n \in \mathbb{N}\}$ be a sequence of iid r.v. with law μ_Z. Let $N \sim Po(\lambda)$ and independent of the $Z(n)'s$. Define

$$X = Z(1) + Z(2) + \cdots + Z(N) = \sum_{n=0}^{N} Z(n).$$

Let us prove that, for each $u \in \mathbb{R}^d$,

$$\phi_X(u) = \exp\left[\int_{\mathbb{R}^d} \left(e^{i(u,y)} - 1\right) \lambda \mu_Z(dy)\right]. \quad (1)$$

$$\begin{aligned}\phi_X(u) &= E\left[e^{i(u,X)}\right] = \sum_{n=0}^{\infty} E\left[e^{i(u,Z(1)+Z(2)+\cdots+Z(N))} | N = n\right] P[N = n] \\ &= \sum_{n=0}^{\infty} E\left[e^{i(u,Z(1)+Z(2)+\cdots+Z(n))}\right] \frac{\lambda^n}{n!} e^{-\lambda} = e^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda \phi_Z(u))^n}{n!} \\ &= \exp\left[\lambda(\phi_Z(u) - 1)\right].\end{aligned}$$

Infinite divisibility - Examples

Example

(Continued) Therefore, with $\phi_Z(u) = \int_{\mathbb{R}^d} e^{i(u,y)} \mu_Z(dy)$, we obtain (1). We denote the Compound Poisson by $X \sim Po(\lambda, \mu_Z)$. We have

$$(\phi_X(u))^{\frac{1}{n}} = \exp\left[\frac{\lambda}{n}(\phi_Z(u) - 1)\right]$$

and X is inf. divis. with $X \overset{d}{=} Y_1^{(n)} + Y_2^{(n)} + \cdots + Y_n^{(n)}$ and

$$Y_j^{(n)} \sim Po\left(\frac{\lambda}{n}, \mu_Z\right).$$

- Exercise: Let $d = 1$. Show that if $X \sim Po(\lambda)$ then $\phi_X(u) = \exp\left[\lambda\left(e^{iu} - 1\right)\right]$.

The Lévy measure

Definition

Let ν be a Borel measure defined on $\mathbb{R}^d - \{0\}$. We say that ν is a Lévy measure if

$$\int_{\mathbb{R}^d-\{0\}} \left(|x|^2 \wedge 1\right) \nu(dx) < \infty \qquad (2)$$

- Note that $\varepsilon^2 \leq |x|^2 \wedge 1$ when $0 < \varepsilon \leq 1$ and $|x| \geq \varepsilon$. Therefore, by (2), we have that
$$\nu\left[(-\varepsilon, \varepsilon)^c\right] < \infty, \qquad \text{for all } \varepsilon > 0.$$
- Note: Condition (2) is equivalent to
$$\int_{\mathbb{R}^d-\{0\}} \frac{|x|^2}{1+|x|^2} \nu(dx) < \infty.$$
- Note: one can assume that $\nu(\{0\}) = 0$ and then ν is defined on $\mathbb{R}^d$.
- Exercise: Show that $\nu\left[(-\varepsilon, \varepsilon)^c\right] < \infty, \quad$ for all $\varepsilon > 0$.
- Exercise: Show that Condition (2) is equivalent to $\int_{\mathbb{R}^d-\{0\}} \frac{|x|^2}{1+|x|^2} \nu(dx) < \infty$.

Lévy-Khintchine formula

Theorem

(Lévy-Khintchine): A distribution μ on $\mathbb{R}^d$ is infinitely divisible if exists a vector $b \in \mathbb{R}^d$, a $d \times d$ positive definite symmetric matrix A and a Lévy measure ν on $\mathbb{R}^d - \{0\}$ such that, for all $u \in \mathbb{R}^d$,

$$\phi_\mu(u) = \exp\left\{ i(b,u) - \frac{1}{2}(u, Au) + \right. \\ \left. + \int_{\mathbb{R}^d - \{0\}} \left[e^{i(u,x)} - 1 - i(u,x)\mathbf{1}_{|x|<1}(x) \right] \nu(dx) \right\}. \tag{3}$$

Conversely, any mapping of the form (3) is the characteristic function of an inf. divis. probability measure on $\mathbb{R}^d$.

Lévy-Khintchine formula

- (b, A, ν) are the characteristics of the inf. divis. distribution μ.
- $\eta := \log(\phi_\mu)$ is the Lévy symbol or characteristic exponent or Lévy exponent:

$$\eta(u) = i(b, u) - \frac{1}{2}(u, Au) + \int_{\mathbb{R}^d - \{0\}} \left[e^{i(u,x)} - 1 - i(u, x) \mathbf{1}_{|x|<1}(x) \right] \nu(dx).$$

- We will not prove the first part of the theorem (difficult, but it can be proved as a by product of the Lévy-Itô decomposition - to be discussed later)
- We prove the second part.

Proof (2nd part)

- We need to prove that the r.h.s of (3) is a characteristic function.
- i) Let $\{U(n), n \in \mathbb{N}\} \subset \mathbb{R}^d$ be a sequence of Borel sets such that $U(n) \searrow 0$ and define

$$\phi_n(u) = \exp\left\{ i\left(b - \int_{U(n)^c \cap \{x:|x|<1\}} x\nu(dx), u \right) - \frac{1}{2}(u, Au) + \right. \\ \left. + \int_{U(n)^c} \left(e^{i(u,x)} - 1 \right) \nu(dx) \right\}.$$

ii) Clearly, ϕ_n is the distribution of a sum of a Normal dist. with an independent compound Poisson dist. Therefore, it is infinit. divis.
iii) Clearly,

$$\phi_\mu(u) = \lim_{n\to\infty} \phi_n(u).$$

Proof (continued)

iv) To prove that ϕ_μ is a charac. func., use Lévy's continuity theorem (Applebaum, p. 18) and we only need to prove that $\psi_\mu(u)$ is continuous at 0:

$$\psi_\mu(u) = \int_{\mathbb{R}^d - \{0\}} \left[e^{i(u,x)} - 1 - i(u,x)\,\mathbf{1}_{|x|<1}(x) \right] \nu(dx)$$
$$= \int_{|x|<1} \left(e^{i(u,x)} - 1 - i(u,x) \right) \nu(dx) +$$
$$+ \int_{|x|\geq 1} \left(e^{i(u,x)} - 1 \right) \nu(dx).$$

v) By Taylor's theorem, the Cauchy-Schwarz inequality and dominated convergence:

$$|\psi_\mu(u)| \leq \frac{1}{2} \int_{|x|<1} |(u,x)|^2 \nu(dx) + \int_{|x|\geq 1} \left| e^{i(u,x)} - 1 \right| \nu(dx)$$
$$\leq \frac{|u|^2}{2} \int_{|x|<1} |x|^2 \nu(dx) + \int_{|x|\geq 1} \left| e^{i(u,x)} - 1 \right| \nu(dx) \to 0 \text{ as } u \to 0.$$

vi) It is now easy to verify directly that μ is infin. divis. ■

Remarks

- Gaussian case: $b = m$ (mean), $A =$covariance matrix, $\nu = 0$.
- Poisson case: $b = 0$, $A = 0$, $\nu = \lambda\delta_1$
- Compound Poisson case: $b = 0$, $A = 0$, $\nu = \lambda\mu$, $\lambda > 0$ and μ a probab. measure on $\mathbb{R}^d$

Stable random variables

- The set of stable distributions is an important subclass of the set of inf. divis. distributions
- Let $d = 1$ and $\{Y_n, n \in \mathbb{N}\}$ be a sequence of iid r.v. We consider the general central limit problem. Define the rescaled partial sums sequence:

$$S_n = \frac{Y_1 + \cdots + Y_n - b_n}{\sigma_n}.$$

  where $\{b_n, n \in \mathbb{N}\}$: sequence of real numbers; $\{\sigma_n, n \in \mathbb{N}\}$: sequence of positive numbers.
- Problem: When exists a r.v. X such that

$$\lim_{n\to\infty} P(S_n \le x) = \lim_{n\to\infty} P(X \le x) \quad ? \tag{4}$$

  In that case S_n converges in distribution to X.
- Usual central limit theorem: if $b_n = nm$, $\sigma_n = \sqrt{n}\sigma$. Then $X \sim N(0,1)$.

Stable Random variables

- A r.v. is said to be stable if it arises as a limit as in (4).
- This is equivalent to:

Definition

A r.v. X is said to be stable if exist real valued sequences $\{c_n, n \in \mathbb{N}\}$, $\{d_n, n \in \mathbb{N}\}$ with each $c_n > 0$, such that

$$X_1 + \cdots + X_n \overset{d}{=} c_n X + d_n, \tag{5}$$

where $X_1, \ldots, X_n$ are independent copies of X. In particular, it is strictly stable if each $d_n = 0$.

- In fact, it can be proved that if X is stable then $\sigma_n = \sigma n^{\frac{1}{\alpha}}$ with $0 < \alpha \leq 2$. The parameter α is called the index of stability.
- (5) is equivalent to

$$\phi_X (u)^n = e^{iud_n} \phi_X (c_n u) .$$

- All stable random variables are infinitely divisible (trivial consequence of (5)).

Stable Random variables

Theorem

If X is a stable r.v. then:

1. *when* $\alpha = 2$, $X \sim N(b, A)$
2. *when* $\alpha \neq 2$, $A = 0$ *and*

$$\nu(dx) = \begin{cases} \frac{c_1}{x^{1+\alpha}} dx & \text{if } x > 0 \\ \frac{c_2}{|x|^{1+\alpha}} dx & \text{if } x < 0. \end{cases}, \quad \text{where } c_1, c_2 \geq 0 \text{ and } c_1 + c_2 > 0.$$

Proof can be found in the Book of Sato, p. 80.

Stable Random variables

Theorem

A r.v. X is stable if and only if exist $\sigma > 0$, $-1 \leq \beta \leq 1$ and $\mu \in \mathbb{R}$ such that

1. *when $\alpha = 2$,*
$$\phi_X(u) = \exp\left(i\mu u - \frac{1}{2}\sigma^2 u^2\right);$$
2. *when $\alpha \neq 1, 2$*
$$\phi_X(u) = \exp\left(i\mu u - \sigma^\alpha |u|^\alpha \left[1 - i\beta \operatorname{sgn}(u) \tan\left(\frac{\pi\alpha}{2}\right)\right]\right)$$
3. *when $\alpha = 1$,*
$$\phi_X(u) = \exp\left(i\mu u - \sigma |u| \left[1 + i\beta \frac{2}{\pi} \operatorname{sgn}(u) \log(|u|)\right]\right)$$

Proof can be found in Sato, p. 86.

Stable Random variables

- $E\left[X^2\right] < \infty$ if and only if $\alpha = 2$ (only if X is Gaussian).
- $E\left[|X|\right] < \infty$ if and only if $1 < \alpha \leq 2$.
- All stable r.v. X have densities f_X. In general, can be expressed in series form, but in 3 cases, we have a closed form.
- **Normal distribution:** $\alpha = 2$ and $X \sim N\left(\mu, \sigma^2\right)$.
- **Cauchy distribution:** $\alpha = 1$, $\beta = 0$, $f_X\left(x\right) = \frac{\sigma}{\pi\left[(x-\mu)^2+\sigma^2\right]}$.
- **Lévy distribution**: $\alpha = \frac{1}{2}$, $\beta = 1$,

$$f_X\left(x\right) = \left(\frac{\sigma}{2\pi}\right)^{\frac{1}{2}} \frac{1}{\left(x-\mu\right)^{\frac{3}{2}}} \exp\left[\frac{\sigma}{-2\left(x-\mu\right)}\right] \quad \text{for } x > \mu.$$

Stable Random variables

- Exercise: Let X and Y be independent standard normal random variables (with mean 0). Show that Z has a Cauchy distribution, where $Z = X/Y$ if $Y \neq 0$ and $Z = 0$ if $Y = 0$.
- Remark: if X is stable and symmetric then

$$\phi_X(u) = \exp\left(-\rho^\alpha |u|^\alpha\right) \quad \text{for all } 0 < \alpha \leq 2.$$

where $\rho = \sigma$ for $0 < \alpha < 2$ and $\rho = \frac{\sigma}{\sqrt{2}}$ when $\alpha = 2$.

- Important feature of stable laws: when $\alpha \neq 2$ the decay of the tails is polynomial (slow decay $\Longrightarrow$ "heavy tails") -(if $\alpha = 2$ the decay is exponential):

$$P[X > x] \sim \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}x} \text{ as } x \to \infty \text{ if } \alpha = 2,$$

$$\lim_{x \to +\infty} x^\alpha P[X > x] \sim C_\alpha \frac{1+\beta}{2} \sigma^\alpha \text{ if } \alpha \neq 2, \text{ with } C_\alpha > 1,$$

$$\lim_{x \to -\infty} x^\alpha P[X < -x] \sim C_\alpha \frac{1-\beta}{2} \sigma^\alpha \text{ if } \alpha \neq 2, \text{ with } C_\alpha > 1.$$

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