Lévy processes and applications - Lévy Processes

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Lévy Processes

Definition

Let $X = (X(t); t \geq 0)$ be a stochastic process. We say that X has independent increments if for each $n \in \mathbb{N}$ and each sequence $0 \leq t_1 < t_2 < \ldots < t_{n+1} < \infty$, the random variables $(X(t_{j+1}) - X(t_j); 1 \leq j \leq n)$ are independent and X has stationary increments if $X(t_{j+1}) - X(t_j) \stackrel{d}{=} X(t_{j+1} - t_j) - X(0)$.

Definition

We say that X is a Lévy process if
(1) $X(0) = 0$ (a.s),
(2) X has independent and stationary increments,
(3) X is stochastically continuous, i.e. for all $a > 0$ and for all $s \geq 0$,

$$\lim_{t \to s} P(|X(t) - X(s)| > a) = 0.$$

Lévy Processes

- Conditions (1) and (2) imply that (3) is equivalent to $\lim_{t \searrow 0} P(|X(t)| > a) = 0$.
- The sample paths (trajectories) X are the maps $t \to X(t)(\omega)$ from $\mathbb{R}^+$ to $\mathbb{R}^d$ for each $\omega \in \Omega$.

Proposition

If X is a Levy process, then $X(t)$ is infinitely divisible for each $t \geq 0$.

Proof: For each $n \in \mathbb{N}$, $X(t) = Y_1^{(n)}(t) + \cdots + Y_n^{(n)}(t)$, where $Y_j^{(n)}(t) = X\left(\frac{jt}{n}\right) - X\left(\frac{(j-1)t}{n}\right)$. By condition (2), these $Y_j^{(n)}(t)'s$ are iid r.v. and therefore, $X(t)$ is infinitely divisible. ■

Lévy Processes

Theorem

If X is a Lévy process, then

$$\phi_{X(t)}(u) = e^{t\eta(u)},$$

for each $u \in \mathbb{R}^d$, where η is the characteristic exponent (or Lévy symbol) of $X(1)$.

L-K formula for Lévy Processes

- Exercise: Prove that if X is stochastically continuous , then the map $t \to \phi_{X(t)}(u)$ is continuous for each $u \in \mathbb{R}^d$ (Hint: see Applebaum, pages 43-44).
- L-K formula for a Lévy Process $X = (X(t); t \geq 0)$:

$$\phi_{X(t)}(u) = E\left[e^{i(u,X(t))}\right] = \exp\left\{t\left[i(b,u) - \frac{1}{2}(u,Au) + \right.\right.$$
$$\left.\left. + \int_{\mathbb{R}^d - \{0\}} \left[e^{i(u,x)} - 1 - i(u,x)\mathbf{1}_{|x|<1}(x)\right]\nu(dx)\right]\right\}, \quad (1)$$

for each $t \geq 0$ and $u \in \mathbb{R}^d$. The characteristics (b, A, ν) are the characteristics of $X(1)$.

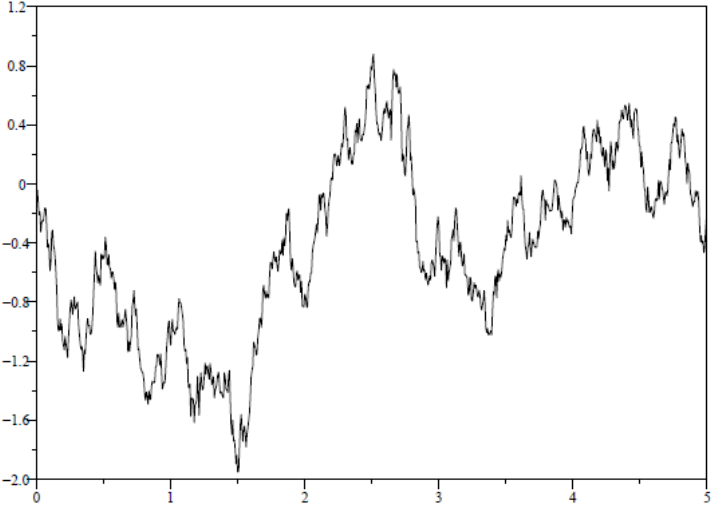
Lévy processes - Brownian motion

- A standard Brownian motion in $\mathbb{R}^d$ is a Lévy process B for which
  (1) $B(t) \sim N(0, tI)$.
  (2) B has continuous sample paths.
- From (1) we obtain

$$\phi_{B(t)}(u) = \exp\left\{ -\frac{1}{2} t |u|^2 \right\}.$$

Lévy processes - Brownian motion

- Simulated path of standard Brownian motion:



Lévy processes - Brownian motion

- Given a non-negative definite symmetric $d \times d$ matrix, let σ be the square root of A (in the sense: $\sigma\sigma^T = A$) with σ a $d \times m$ matrix. Let $b \in \mathbb{R}^d$ and let B be a standard Brownian motion in $\mathbb{R}^m$.
- The process C defined by

$$C(t) = bt + \sigma B(t) \tag{2}$$

  is a Lévy process that satisfies $C(t) \sim N(tb, tA)$. Moreover, C is also a Gaussian process (all finite dimensional distributions are Gaussian).
- The process C is called Brownian motion with drift. The characteristic exponent (or Lévy symbol) of C is

$$\eta_C(u) = i(b, u) - \frac{1}{2}(u, Au).$$

- A Lévy process has continuous sample paths if and only if it is of the form (2).

Lévy processes - Poisson Process

- $N(t) \sim Po(\lambda t)$ is a process taking values in $\mathbb{N}_0$:

$$P[N(t)=n] = \frac{(\lambda t)^n}{n!} e^{-\lambda t}.$$

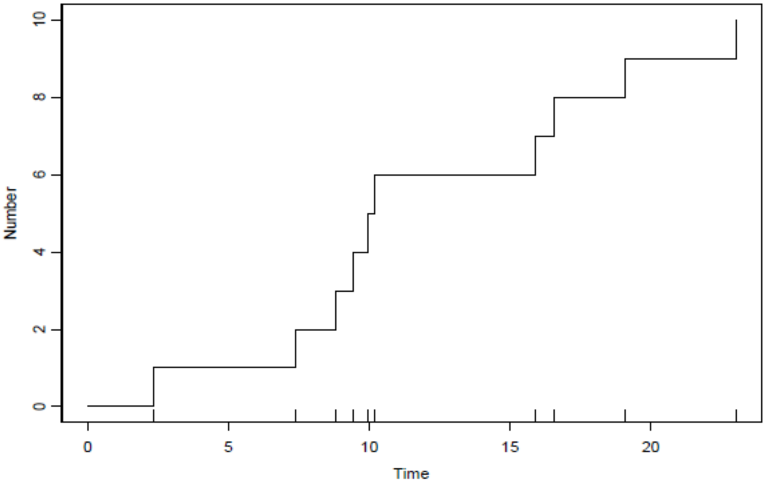
- Let us define the non-negative r.v. $\{T(n), n \in \mathbb{N}_0\}$ (waiting times), $T(0) = 0$,

$$T(n) = \inf\{t \geq 0 : N(t) = n\}.$$

  The r.v. $T(n)$ has a gamma distribution and the inter-arrival times $T(n) - T(n-1)$ are iid with exponential distribution (with mean $1/\lambda$).

- Compensated Poisson process: $\widetilde{N} = \left(\widetilde{N}(t), t \geq 0\right)$ where $\widetilde{N}(t) = N(t) - \lambda t$. Note: $E\left[\widetilde{N}(t)\right] = 0$ and $E\left[\left(\widetilde{N}(t)\right)^2\right] = \lambda t$.

Lévy processes - Poisson Process



Lévy processes - Compound Poisson Process

- Sequence of iid r.v. $\{Z(n), n \in \mathbb{N}\}$ with values in $\mathbb{R}^d$ with law μ_Z. Let N be a Poisson process with intensity λ and independent of the $Z(n)'$ s.
- Compound Poisson process

$$Y(t) = \sum_{n=1}^{N(t)} Z(n),$$

and $Y(t) \sim \pi(\lambda t, \mu_Z)$.

- The characteristic exponent is

$$\eta_Y(u) = \int_{\mathbb{R}^d} \left(e^{i(u,x)} - 1 \right) \lambda \mu_Z(dx).$$

- The sample paths of Y are piecewise constant with jumps at times $T(n)$, but now the jump sizes are random and the jump at $T(n)$ can be any value in the range of the r.v. $Z(n)$.

Lévy processes - Compound Poisson Process

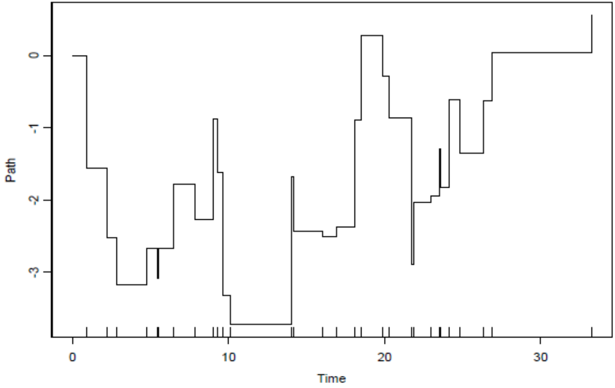


Figure 3. Simulation of a compound Poisson process with $N(0,1)$ summands$(\lambda = 1)$.

Lévy processes - Stable Lévy processes

- A stable Lévy process is a Lévy process X with characteristic exponent ($\sigma > 0$, $-1 \leq \beta \leq 1$ and $\mu \in \mathbb{R}$) (each $X(t)$ is a stable random variable):

Theorem

1. *when* $\alpha = 2$,
$$\eta_X(u) = i\mu u - \frac{1}{2}\sigma^2 u^2;$$
2. *when* $\alpha \neq 1, 2$
$$\eta_X(u) = i\mu u - \sigma^\alpha |u|^\alpha \left[1 - i\beta \operatorname{sgn}(u) \tan\left(\frac{\pi\alpha}{2}\right)\right]$$
3. *when* $\alpha = 1$,
$$\eta_X(u) = i\mu u - \sigma |u| \left[1 + i\beta \frac{2}{\pi} \operatorname{sgn}(u) \log(|u|)\right]$$

Lévy processes - Stable Lévy processes

- Important case (rotationally invariant stable Lévy processes):

$$\eta_X(u) = -\sigma^\alpha |u|^\alpha, \qquad 0 < \alpha \leq 2.$$

- Why are these process important? they are self-similar!
- A process $Y = (Y(t), t \geq 0)$ is self-similar with Hurst index $H > 0$ if $(Y(at), t \geq 0)$ and $\left(a^H Y(t), t \geq 0\right)$ have the same finite dimensional distributions for all $a \geq 0$.
- By examining the characteristic functions, we can prove that a rotationally invariant stable Lévy process is self-similar with $H = 1/\alpha$.
- It can be proved that a Lévy process X is self-similar if and only if each $X(t)$ is strictly stable.

Lévy processes - Subordinators

- A subordinator is a one-dimensional Lévy process wich is increasing a.s.
- Subordinator≈ random model of time evolution: If $T = (T(t), t \geq 0)$ is a subordinator then $T(t) \geq 0$ a.s. and $T(t_1) \leq T(t_2)$ a.s. if $t_1 \leq t_2$.

Theorem

If T is a subordinator then its charact. exponent has the form

$$\eta_T(u) = i(bu) + \int_{(0,\infty)} \left(e^{iux} - 1\right) \lambda(dx), \tag{3}$$

where $b \geq 0$, and the Lévy measure λ satisfies: $\lambda(-\infty, 0) = 0$ and $\int_{(0,\infty)} (x \wedge 1) \lambda(dx) < \infty$.
Conversely, any mapping $\eta : \mathbb{R} \to \mathbb{C}$ of the form (3) is the charact. exponent of a subordinator.

- (b, λ) are called the characteristics of the subordinator T.

Lévy processes - Subordinators

- For each $t \geq 0$, the map $u \to E\left[e^{iuT(t)}\right]$ can be analytically continued to the region $\{iu, u > 0\}$ and we obtain (Laplace transform of the distribution):
$$E\left[e^{-uT(t)}\right] = e^{-t\psi(u)},$$
where
$$\psi(u) = -\eta(iu) = bu + \int_{(0,\infty)} \left(1 - e^{-xu}\right) \lambda(dx). \quad (4)$$
- ψ is called the Laplace exponent of the distribution.

Subordinators - Poisson case

- Poisson processes are subordinators
- Compound Poisson processes are subordinators if and only if the $Z(n)'s$ are positive r.v.

Subordinators -stable subordinators

- It can be proved (using the usual calculus) that (for $0 < \alpha < 1$ and $u \geq 0$)

$$u^{\alpha} = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{\infty} \left(1 - e^{-ux}\right) \frac{dx}{x^{1+\alpha}}.$$

- By (4) and the characteristics of a stable Lévy process, there exists an α-stable subordinator with Laplace exponent $\psi(u) = u^{\alpha}$ and the characteristics of T are $(0, \lambda)$, where $\lambda(dx) = \frac{\alpha}{\Gamma(1-\alpha)} \frac{dx}{x^{1+\alpha}}$.
- When we analytically continue this in order to obtain the Lévy charac. exponent, we obtain $\mu = 0$, $\beta = 1$ and $\sigma^{\alpha} = \cos(\alpha\pi/2)$.
- Exercise: Show that there exists an α-stable subordinator with Laplace exponent $\psi(u) = u^{\alpha}$ and the characteristics of T are $(0, \lambda)$, where $\lambda(dx) = \frac{\alpha}{\Gamma(1-\alpha)} \frac{dx}{x^{1+\alpha}}$.

Subordinators -the Lévy subordinator

- The $\left(\frac{1}{2}\right)$-stable subordinator has a density given by the Lévy distribution (with $\mu = 0$ and $\sigma = \frac{t^2}{2}$):

$$f_{T(t)}(s) = \left(\frac{t}{2\sqrt{\pi}}\right) s^{-\frac{3}{2}} \exp\left(\frac{-t^2}{4s}\right).$$

- It is possible to show directly that

$$E\left[e^{-uT(t)}\right] = \int_0^\infty e^{-us} f_{T(t)}(s)\, ds = e^{-tu^{\frac{1}{2}}}.$$

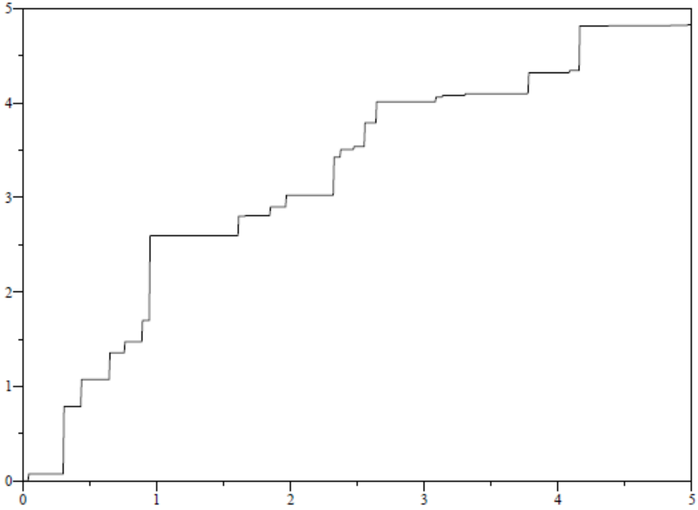
- This subordinator can be represented by a hitting time of the Bm:

$$T(t) = \inf\left\{s > 0 : B(s) = \frac{t}{\sqrt{2}}\right\}. \tag{5}$$

Gamma subordinators

- Let $T(t)$ be a Gamma process with parameters $a, b > 0$ such that $T(t)$ has a density
$$f_{T(t)}(x) = \frac{b^{at}}{\Gamma(at)} x^{at-1} e^{-bx}, \quad x \geq 0.$$
- Using some calculus, we can show that
$$\int_0^\infty e^{-ux} f_{T(t)}(x)\, dx = \exp\left(-t \int_0^\infty \left(1 - e^{-ux}\right) a x^{-1} e^{-bx} dx\right).$$
- Therefore, by (4), $T(t)$ is a subordinator with $b = 0$ and $\lambda(dx) = a x^{-1} e^{-bx} dx$

Simulation of a Gamma subordinator



Time change

- Important application of subordinators: time change!
- Let X be a Lévy process and let T be a subordinator independent of X. Let

$$Z(t) = X(T(t)).$$

Theorem

Z is a Lévy process

Proof: see Applebaum, pags. 56-58

Proposition

$$\eta_Z = -\psi_T \circ (-\eta_X).$$

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Time change

Proof: Let $p_{T(t)}$ be the distribution associated to $T(t)$. Then

$$\begin{aligned} E\left[e^{t\eta_{Z(t)}(u)}\right] &= E\left(e^{i(u,Z(t))}\right) = E\left(e^{i(u,X(T(t)))}\right) \\ &= \int E\left(e^{i(u,X(s))}\right) p_{T(t)}(ds) \\ &= \int e^{s\eta_X(u)} p_{T(t)}(ds) \\ &= E\left[e^{-(-\eta_X(u))T(t)}\right] \\ &= e^{-t\psi_T(-\eta_X(u))}. \qquad \blacksquare \end{aligned}$$

Brownian motion and 2 alpha stable motion

- Let T be an α-stable subordinator (with $0 < \alpha < 1$) and X be a Brownian motion with covariance $A = 2I$, independent of T. Then

$$\psi_T(s) = s^{\alpha}, \quad \eta_X(u) = -|u|^2$$

and therefore, by the Proposition,

$$\eta_Z(u) = -|u|^{2\alpha}$$

and Z is a 2α stable process.

- If $d = 1$ and T is the Lévy subordinator, then Z is the Cauchy process and each $Z(t)$ has a symmetric Cauchy distribution with $\mu = 0$ and $\sigma = 1$.
- Moreover, by (5), the Cauchy process can be constructed from two indepedent Brownian motions.

The variance gamma process

- Let $Z(t) = B(T(t))$, where T is a gamma subordinator and B is a Brownian motion. Then, the Lévy process Z is called a variance-gamma process.
- we replace the variance of B by a gamma r.v.
- Then, we have

$$\Phi_{Z(t)}(u) = E\left[e^{uiZ(t)}\right] = \left(1 + \frac{u^2}{2b}\right)^{-at},$$

where a and b are the usual parameters determining the gamma process.
- Exercise: Prove this result.

The variance gamma process

- Manipulating characteristic functions, it is possible to show that:

$$Z(t) = G(t) - L(t)$$

where G and L are independent gamma subordinators with parameters $\sqrt{2b}$ and a (difference of independent "gains" and "losses").

- From this representation, it is possible to show that $Z(t)$ has a Lévy density:

$$g_\nu(x) = \frac{a}{|x|^1}\left(e^{\sqrt{2b}x}\mathbf{1}_{(-\infty,0)}(x) + e^{-\sqrt{2b}x}\mathbf{1}_{(0,\infty)}(x)\right),$$

$$a > 0.$$

CGMY model

- The CGMY model (Carr, Geman, Madan and Yor) is a generalization of the variance gamma process, with Lévy density:

$$g_{\nu}(x) = \frac{a}{|x|^{1+\alpha}} \left(e^{b_1 x} \mathbf{1}_{(-\infty,0)}(x) + e^{-b_2 x} \mathbf{1}_{(0,\infty)}(x) \right),$$
$$a > 0,\ 0 \leq \alpha < 2, \quad b_1, b_2 \geq 0.$$

- When $b_1 = b_2 = 0$, we obtain stable Lévy processes.
- The exponential dampens the effects of large jumps.

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