Lecture 1: Cash-in-Advance Model

Cole, Chapters 2, 3 and 4

Bernardino Adão

ISEG, Lisbon School of Economics and Management

September 12, 2023

First Step

Putting Money in

- To introduce money into a model you need a friction. This is because if barter is possible and efficient, it will generally be preferred to money.
- The simplest friction is to require that one have cash in order to buy goods.
- Assume that when people meet to exchange goods, they do so in a goods market where it is hard to track people's identities; therefore, one cannot use fancy long-term contracts.
- Assume also that the structure of markets is such that exchanges take place through a sequence of meetings and that these meetings always involve people who want to buy a particular good and those who want to sell the same good.
- Hence, they cannot directly exchange goods-for-goods, so barter cannot occur. Instead, exchanges must be financed using a medium of exchange. Call it money.

First Step

- Given these assumptions, one must always use money to buy goods.
- Since we have essentially forced money's use here by assumption, this is a poor model for understanding the role of money. But that is not what we will be using this set-up for.
- We will be using it to understand the impact of changes in the supply of money and other exogenous variables.
- For this, we just need our assumptions about how money is used to be accurate enough to yield interesting predictions.

First Step

Formally, here is what we assume:

- In each period, a household can produce a particular type of good; call it good i, where i indexes all of the different possible goods, $i \in I$.
- The household, which is composed of two agents whom we will call the seller and the buyer, splits up, with the buyer taking the available cash off to the goods markets to buy all the different types of goods that the household does not produce, $\{j \in I : j \neq i\}$.
- At the same time, the seller heads off to the market for good i, where they sell some of their good i, saving the rest to consume themselves.
- We will assume that each of the goods markets $i \in I$ has lots of buyers and sellers, and hence, the markets are competitive.

First Step

- To keep things simple we will assume that the supply of goods is symmetric (i.e., equal for all types i) and that the distribution of possible buyers is also symmetric.
- Hence, in equilibrium, the prices $P(i)$ that clears these markets will all be identical. But we will not impose this right away.
- Besides the goods market, we will also assume that there is an asset market.
- In the asset market, households will exchange money and financial assets. To keep things simple, we will assume that the only type of asset they exchange is a one-period pure discount bond.
- The presence of this bond will give us an interest rate that we can determine.
- It is in the asset market that the government will conduct monetary policy either by directly increasing the money supply (perhaps through transfer payments) or through buying or selling bonds for cash.

First Step

- We will assume that each household produces the good at the beginning of each period using labor. The output of the good is given by

$$y_i = ZL, \tag{1}$$

where Z is the current productivity level of the economy and L is the amount of labor expended by the household.

- We will assume that the households have the following preferences over their joint consumption within the period. They have a concave utility function over a composite of all the possible goods, or

$$u(C) \text{ concave, and } C = \left\{ \frac{1}{\#I} \sum_{i \in I} C(i)^{\rho} \right\}^{1/\rho}, \tag{2}$$

where $\#I$ is the number of different goods and $\rho \in (0, 1)$. Elasticity of substitution $1/(1-\rho)$.

First Step

- We assume that $u.$ is concave: specifically $u' > 0$ and $u'' < 0$.
- The households also have a disutility of labor, so their total payoff for a period is

$$u(C) - v(L). \tag{3}$$

- We will assume that v is convex; specifically that $v' > 0$ and $v'' > 0$. So, the cost of additional effort is positive and increasing in the level of effort.

First Step

- The timing of the model will end up mattering quite a bit.

Timing Within Each Period

1. The household starts a period with m units of money
2. It exerts labor effort L to produce its good
3. The seller and buyer split up and go to their respective markets
4. The seller and buyer come back together in the asset market
5. They jointly consume the consumption good.
6. The period ends.

First Step

- We start with a simple, largely static version of the household's problem before moving on to dynamic versions.
- For now, denote by $V(W)$ the future value of wealth. For simplicity we do not yet distinguish among the types of wealth.
- We will regularly use continuation payoff functions to avoid infinite horizon models. However, we need an infinite horizon because fiat money cannot have value in finite time.
- The household can be thought of as choosing how much to consume of each of the different goods, $\{C(i)\}_{i\in I}$,; how much to work to produce their production good $L(j)$; and how much wealth W for next period in order to maximize its payoff.

First Step

This leads to the following formalization of the optimization problem for a household whose production type is j.

$$\max_{\{C(i)\}_{i\in I}, L(j), W} \; u(C) - v(L(j)) + V(W) \text{ subject to}$$

$$\begin{aligned} M &\geq \sum_{i\in I/j} P(i)C(i) \\ W &\leq P(j)\left[ZL(j) - C(j)\right] + \left[M - \sum_{i\in I/j} P(i)C(i)\right]. \end{aligned}$$

The first condition is the cash-in-advance condition which states that the household can only spend as much as it has in cash to buy goods. The notation I/j means the set I less element j.

The second constraint is its budget constraint, which says that the household's net period wealth is whatever it has left out of its money holdings and the proceeds of what it sells in the goods market.

First Step

This is a complicated multi-dimensional maximization problem. To address it, we form the Lagrangian, which is given by

$$\begin{aligned}\mathcal{L} \quad = \quad & \max_{\{C(i)\}_{i\in I}, L(j), W} \; \min_{\lambda,\mu} \; u(C) - v(L(j)) + V(W) \\ & +\lambda \left\{ M - \sum_{i\in I/j} P(i)C(i) \right\} \\ & +\mu \left\{ P(j)\left[ZL(j) - C(j)\right] + \left[M - \sum_{i\in I/j} P(i)C(i) \right] - W \right\}.\end{aligned} \tag{4}$$

In this Lagrangian, the multipliers λ and μ are the "penalty prices" that we attach to violations of the constraint. The impact of the penalties comes through violations of the conditions, and can be minimized by setting them so there is no violation.

In this maximization we are simultaneously trying to minimize the impact of these penalty prices on the overall objective and maximize the value of the objective in terms of the direct choice variables.

First Step

The first-order conditions will include the consumption conditions for each type of consumption, or

$$u'(C)\left\{\frac{1}{\#I}\sum_i C(i)^\rho\right\}^{(1-\rho)/\rho}\frac{1}{\#I}C(i)^{\rho-1}=(\lambda+\mu)\,P(i) \tag{5}$$

for each $i\neq j$. For their own good j, the condition is

$$u'(C)\left\{\frac{1}{\#I}\sum_i C(i)^\rho\right\}^{(1-\rho)/\rho}\frac{1}{\#I}C(j)^{\rho-1}=\mu P(j).$$

Note that this implies that the household will consume more of its own good to the extent that the cash-in-advance constraint binds and $\lambda>0$.
Arrises because holding money has a cost due to its poor return.

First Step

The optimal condition for labor effort, or

$$v'(L(j)) = \mu P(j) Z. \tag{6}$$

The optimal choice of wealth, or

$$V'(W) = \mu.$$

We can use this last result to replace μ in the consumption and labor conditions.

First Step

Going forward, we will not be interested in the extent to which households buy more of their own good because they don't need to use cash.

Focus a model on what you want to study. Detail for details sake is bad.

Simplifying Modification:
To keep things simple and symmetric, we will now assume that you have to use cash even to buy your own consumption good. So, the constraints become

$$M \geq \sum_{i \in I} P(i)C(i)$$

and

$$W \leq P(j)ZL(j) + \left[M - \sum_{i \in I} P(i)C(i) \right].$$

First Step

- If the prices of the consumption good are the same $P(i) = \tilde{P}$ and $C(i) = C$, the l.h.s. of (5) is given by

$$
\begin{aligned}
u'(C)\left\{\frac{1}{\#I}\sum_i C(i)^{\rho}\right\}^{(1-\rho)/\rho}\frac{1}{\#I}C(i)^{\rho-1} &= \\
u'(C)\left\{\frac{1}{\#I}\#I\right\}^{(1-\rho)/\rho}\{C^{\rho}\}^{1-\rho}\frac{1}{\#I}C^{\rho-1} &= \frac{u'(C)}{\#I}.
\end{aligned}
$$

- We can rewrite (5)

$$
\begin{aligned}
\frac{u'(C)}{\#I} &= (\lambda+\mu)\,\tilde{P} \text{ or} \\
u'(C) &= (\lambda+\mu)\,\#I * \tilde{P}
\end{aligned}
$$

First Step

So, denote the price of a unit of the composite good (which means one of each individual good) by $P = \# I \tilde{P}$. As a result, this, the first-order condition (f.o.c.) becomes simply

$$u'(C) = (\lambda + \mu) P. \tag{7}$$

First Step

If the cash-in-advance constraint doesn't bind, then $\lambda = 0$, and

$$u'(C) = V'(W)P, \tag{8}$$

If the cash-in-advance constraint does bind, then $\lambda > 0$,

$$C = \frac{M}{P}, \tag{9}$$

and λ will be chosen so that

$$\lambda = \frac{u'(C)}{P} - V'(W).$$

First Step

Turn next to the optimal choice of labor. By again using the f.o.c. for wealth, we get that

$$v'(L(j)) = V'(W)P(j)Z, \tag{10}$$

which says that the optimal labor choice is to set the marginal disutility of effort equal to the nominal marginal production of labor, $P(j)Z$, times the marginal value of nominal wealth, $V'(W)$.

We want to rewrite this in terms of the composite price, and this becomes

$$v'(L(j)) = V'(W)PZ/\#I.$$

Then if we change Z to $Z * \#I$, we get

$$v'(L) = V'(W)PZ.$$

This gives us a nice simple condition to work with later.

First Step

The Asset Market:

- We now extend our simple model to incorporate an asset market at the end of the period.
- In the asset market households can exchange money for bonds and vice versa. They can also buy and sell government bonds, which here are just another form of bond.
- Adding the choice between money and bonds means that we now have to distinguish between the two types of wealth that the household can carry out of the period.We denote its new bond and money positions by B' and M' to distinguish them from the household's initial levels.
- Denote by $V(B', M')$ the future payoff to the household if its bond position as it leaves the period is B' and its money position is M'.
- All bonds are pure discount bonds, which means that the payoff is \$ 1 for each unit of the bond and the cost is q per unit today. Note that $1/q$ is the gross interest rate offered by the bond.

First Step

The household faces a budget constraint in the bond market, which we can write as

$$PZL + [M - PC] + B \geq M' + qB'.$$

With these changes we can rewrite its Lagrangian as

$$\begin{aligned} \mathcal{L} \;=\; & \max_{\{C(i)\}_{i \in I}, L(j), M', B'} \min_{\lambda, \mu} u(C) - v(L(j)) + V(M', B') \\ & + \lambda \{M - PC\} \\ & + \mu \{PZL + [M - PC] + B - M' - qB'\}. \end{aligned} \tag{11}$$

The first-order condition for money M' is

$$-\mu + V_1(M', B') = 0.$$

Similarly, the first-order condition for bonds B' is

$$-\mu q + V_2(M', B') = 0,$$

First Step

The first-order condition for money M' is

$$-\mu + V_1(M', B') = 0.$$

Similarly, the first-order condition for bonds B' is

$$-\mu q + V_2(M', B') = 0,$$

These two conditions differ in important ways.

- First, bonds are cheaper per unit of future value to the extent that $q < 1$.
- Second, they contribute future value in different ways to the extent that money and bonds in the future are imperfect substitutes.
- To determine how their future value differs, we need to move on to a genuine dynamic version of the model.

Second Step

- We now extend our model by creating a genuine dynamic model by simply pushing out in time by one period the point at which we use a continuation payoff function, $V(M, B)$, to characterize outcomes.
- Our household takes as given the price in the first and second periods, P_1 and P_2, and the productivity levels, Z_1 and Z_2. It also takes as given its initial money position M_1 and bond position B_1. Finally, it takes as given the payoff from money and bonds going into the third period $V(M_3, B_3)$.
- The household is choosing consumptions C_1 and C_2, labor L_1 and L_2, money holdings M_2 and M_3, and bond holdings B_2 and B_3.
- We want to allow the government to change the overall money supply through taxes and transfers of money in the asset market at the end of the period.
- Denote by T_t the net transfer that the government is making in cash to the household. When $T_t > 0$, the household is receiving cash, and when the reverse is true, it is making a cash payment to the government.

Second Step

The household's problem can be written as

$$\max_{\{C_t, L_t, M_{t+1}, B_{t+1}\}_{t=1,2}} u(C_1) - v(L_1) + \beta\left[u(C_2) - v(L_2)\right] + \beta^2 V(M_3, B_3) \text{ subject to}$$

$$\begin{aligned} M_t &\geq P_t C_t \text{ and} \\ P_t Z_t L_t + \left[M_t - P_t C_t\right] + B_t + T_t &\geq M_{t+1} + q_t B_{t+1} \text{ for } t = 1, 2. \end{aligned}$$

The real difference in this version of the household's problem has to do with the impact of choosing M_2 and B_2.

Second Step

The household's Lagrangian is now given by

$$\mathcal{L} = \max_{\{C_t, L_t, M_{t+1}, B_{t+1}\}_{t=1,2}} \min_{\{\lambda_t, \mu_t\}_{t=1,2}} \tag{12}$$

$$u(C_1) - v(L_1) + \beta\left[u(C_2) - v(L_2)\right] + \beta^2 V(M_3, B_3) \tag{13}$$
$$+ \sum_{t=1,2} \lambda_t \left\{M_t - P_t C_t\right\}$$
$$+ \sum_{t=1,2} \mu_t \left\{P_t Z_t L_t + M_t - P_t C_t + B_t + T_t - M_{t+1} - q_t B_{t+1}\right\}.$$

In this problem we are now assuming the discounting of future utils by multiplying the next period's payoff by $\beta < 1$, and the payoff two periods ahead by β^2.

Can readily push out modeling to arbitrary T.

Second Step

The first-order conditions for consumption and labor are still given by

$$\beta^{t-1} u'(C_t) = [\lambda_t + \mu_t] P_t \tag{14}$$

and

$$\beta^{t-1} v'(L_t) = \mu_t P_t Z_t. \tag{15}$$

Second Step

FOC for Money:

- To derive the first-order conditions for M_2 note that it shows up in the date $t = 1$ budget constraint as M_{t+1}, and it also shows up in the date $t = 2$ cash-in-advance and budget constraints as M_t.
- Doing the differentiation correctly leads to

$$-\mu_t + \lambda_{t+1} + \mu_{t+1} = 0.$$

- This says that the cost of increasing money holdings, M_{t+1}, is the shadow price of the first period budget constraint, μ_t, while the gain is the sum of the shadow price of the second period cash-in-advance (c.i.a.) constraint, λ_{t+1}, and the shadow price of the second period budget constraint, μ_{t+1}.
- Here λ_{t+1} is capturing the service yield of money.
- This condition now makes clear what stood behind the mystery term $V_1(M', B')$ in our static model of the previous chapter.

Second Step

FOC for Bonds:

- Once again this term shows up as B_{t+1} when $t = 1$ and B_t when $t = 2$.
- When we differentiate correctly, we get the following expression,

$$-\mu_t q_t + \mu_{t+1} = 0.$$

- This expression highlights the difference between buying bonds vs. holding money. With bonds we get a price break to the extent that $q_t < 1$, but we don't get the future service yield, λ_{t+1}, just the future benefit of having more wealth, μ_{t+1}.
- This condition now makes clear what stood behind the mystery term $V_2(M', B')$ in our static model of the previous chapter.
- Note that

$$V_1(M', B') - V_2(M', B') = \lambda_{t+1} \geq 0$$

Second Step

To get a bit more insight into the gain from having more money in period 2, use the f.o.c. for C_2

$$\beta u'(C_2) = (\lambda_2 + \mu_2) P_2.$$

Note that the cost of second period consumption is the combination of the shadow prices of the cash-in-advance (c.i.a.) and budget (b.c.) constraints. Using this expression we can rewrite the first-order condition for money as

$$\mu_1 = \frac{\beta u'(C_2)}{P_2}.$$

From this condition we can see more sharply that the household is trading off the benefit of being able to buy consumption tomorrow vs. the lower return on savings offered by money (again to the extent that $q_1 < 1$).

Second Step

Consumption vs. Labor:

- Starting from the first-order condition for L_1, we use the f.o.c. for M_2 to replace μ_1, and then the f.o.c. for C_2 to finally get

$$\begin{aligned} v'(L_1) &= \mu_1 P_1 Z_1 \\ &= [\mu_2 + \lambda_2] P_1 Z_1 \\ &= \left[\frac{\beta u'(C_2)}{P_2}\right] P_1 Z_1 \\ &= \frac{P_1}{P_2} Z_1 \beta u'(C_2). \end{aligned}$$

This is the optimal labor-consumption condition in our model.

- To the extent that money is a bad asset here, $P_1/P_2 << 1/q$, and this will discourage people from working.
- This is one of the key inefficiency wedges that the presence of money has created in our model.

Second Step

Consumption today vs. tomorrow:

- The first-order condition for C_1 is given by

$$\begin{aligned} u'(C_1) &= [\mu_1 + \lambda_1] P_1, \text{ and for } C_2 \\ \beta u'(C_2) &= [\mu_2 + \lambda_2] P_2. \end{aligned}$$

- The first-order condition for money says that $\mu_1 = \mu_2 + \lambda_2$.
- Making this substitution and rearranging we get that

$$\begin{aligned} \frac{u'(C_1)}{P_1} &= [\mu_2 + \lambda_2 + \lambda_1], \text{ and for } C_2 \\ \frac{\beta u'(C_2)}{P_2} &= [\mu_2 + \lambda_2]. \end{aligned}$$

So, for these two right-hand-side (r.h.s.) expressions to be equal, we need $\lambda_1 = 0$ or the shadow price of the c.i.a. constraint to be 0. Otherwise, we strictly prefer to spend our money in the first period.

Second Step

When will the shadow price of money be 0?

- Note that from our money and bond conditions

$$\begin{aligned} \mu_1 &= \mu_2 + \lambda_2 \text{ and} \\ \mu_1 q_1 &= \mu_2. \end{aligned}$$

- So, if $q_1 < 1$, these conditions say that $\lambda_2 > 0$.
- In other words, the household will adjust the composition of its savings between money and bonds to ensure that the cash-in-advance constraint binds enough to offset the extra interest return that they get from holding bonds.

Second Step

General Version of first-order conditions:

- So far, we only have modeled two periods.
- But if we extended the modeling out to an arbitrary period T, then we would get similar conditions.
- For consumption
$$\beta^{t-1} u'(C_t) = [\lambda_t + \mu_t] P_t$$
- For labor
$$\beta^{t-1} v'(L_t) = \mu_t P_t Z_t.$$
- For money
$$\mu_t = \mu_{t+1} + \lambda_{t+1}$$
- For bonds
$$\mu_t q_t = \mu_{t+1}$$
- When we solve for the steady state we will take these as our conditions.

Second Step

Closing the Model:

- Now we want to think about the aggregate equilibrium variables.
- The aggregate resource constraint implies that per capita output is equal to per capita consumption, so

$$Z_t L_t = Y_t = C_t.$$

- The per capita money supply evolves over time here because of net transfers. Let $\bar{M}_t$ denote the per capita money supply at the beginning of period t.
- If we denote the net growth rate of money in period t by τ_t, then

$$\begin{aligned} \bar{M}_{t+1} &= (1+\tau_t)\bar{M}_t, \text{ and} \\ T_t &= \tau_t \bar{M}_t. \end{aligned}$$

- The money market clearing condition requires that the amount of money with which the household leaves the asset market must equal the supply, or

$$M_{t+1} = \bar{M}_{t+1}.$$

Second Step

In closing the model, let's assume that cash-in-advance (c.i.a.) constraint binds. Then, this implies that

$$C_t = \frac{\bar{M}_t}{P_t}, \text{ or } P_t = \frac{\bar{M}_t}{Z_t L_t}. \tag{16}$$

This is a simple velocity type equation, familiar from very old-school macro models: $Mv = PY$ where v is the velocity of money.

In our equations we can replace the price level with this simple relationship.

Second Step

Next, we turn to our optimality conditions to finish closing the model.
The consumption and labor conditions imply that

$$\beta^{t-1} u'(Z_t L_t) = (\lambda_t + \mu_t) \frac{\bar{M}_t}{Z_t L_t}$$

and

$$\beta^{t-1} v'(L_t) = \mu_t Z_t \frac{\bar{M}_t}{Z_t L_t},$$

where we have made use of the fact that $C_t = Z_t L_t$ from our resource constraint and replaced P_t.

The money and bond conditions imply that

$$\begin{aligned} \mu_t &= \mu_{t+1} + \lambda_{t+1} \text{ and} \\ \mu_t q_t &= \mu_{t+1}. \end{aligned}$$

Second Step

We can use the money condition along with our conditions for consumption and labor to get that

$$\begin{aligned}
\beta^{t-1} v'(L_t) &= \left[\mu_{t+1} + \lambda_{t+1}\right] \frac{\bar{M}_t}{L_t} \\
&= \left[\beta^t u'(Z_{t+1} L_{t+1}) \frac{Z_{t+1} L_{t+1}}{\bar{M}_{t+1}}\right] \frac{\bar{M}_t}{L_t} \\
&= \frac{1}{(1+\tau_t)} \frac{Z_{t+1} L_{t+1}}{L_t} \beta^t u'(Z_{t+1} L_{t+1}). \qquad (17)
\end{aligned}$$

This is our key dynamic equation and it involves both L_t and L_{t+1}.

However, these are the only endogenous variables in the equation. So, solving this equation will essentially solve our model.

Second Step

Finally, we can determine the bond price off of the optimality conditions for labor and bonds. Note that

$$\mu_t q_t = \mu_{t+1} \ \Rightarrow \ \frac{v'(L_t)L_t}{\bar{M}_t} q_t = \frac{\beta v'(L_{t+1})L_{t+1}}{\bar{M}_{t+1}}$$

or

$$q_t = \frac{\beta}{(1+\tau_t)} \frac{v'(L_{t+1})L_{t+1}}{v'(L_t)L_t}. \tag{18}$$

The unusual aspect of this equation is that we are pinning down the interest rate through the intertemporal tradeoff of working more today vs. tomorrow rather than consuming more today vs. tomorrow.

Second Step

Steady State 1: We are going to shut down as much of the time variation as we can and solve for a steady-state equilibrium of our model.

- Assume that productivity is constant; i.e., $Z_t = Z$.
- Assume money supply to grows at a constant rate τ.
- Conjecture that labor, and hence consumption are constant.
- If $L_t = L$, then our key dynamic equation becomes

$$v'(L) = \frac{\beta}{(1+\tau)} Zu'(ZL).$$

- One can see from inspection that this equation is going to admit a unique solution in L. The left-hand-side (l.h.s.) is increasing in L because we assumed that $v'' > 0$. The right-hand-side (r.h.s.) is decreasing in L because u' is decreasing in C $(u'' < 0)$.

Second Step

Before actually solving our model, we have to take a stand on the explicit function forms of our preferences.

The standard assumption for consumption is that preferences are CRRA, or

$$u(C) = \begin{cases} \frac{C^{1-\alpha}-1}{1-\alpha} \text{ if } \alpha \neq 1 \\ \log(C) \text{ o.w.} \end{cases}$$

Standard values for α are 1 or 2.

The standard assumption with respect to labor is to also assume a power utility form, or

$$v(L) = \frac{L^{1+\gamma}}{1+\gamma}. \tag{19}$$

Second Step

Frisch elasticity of labor asks how labor will change if we change the wage while holding fixed the marginal value of wealth. From our first-order condition for labor, we get that

$$v'(L) = V'(W)PZ \quad \text{or } L^{\gamma} = V'(W)w.$$

where PZ is our stand-in for the nominal wage w.
Taking logs, holding fixed the marginal utility of wealth, and differentiating yields

$$\gamma \log(L) = \log\left[V'(W)\right] + \log(w) \implies \left|\frac{d\log(L)}{d\log(w)}\right|_{V'(W)} = \frac{1}{\gamma}.$$

The Frisch elasticity is therefore $1/\gamma$.

These's a lot of debate about this elasticity. Many micro studies estimate it to be quite low (0 to 0.5), while macro studies generally estimate a significantly higher value (2 to 4).

Second Step

Given our functional form assumption our steady state labor condition becomes:

$$L^{\gamma} = \frac{\beta}{(1+\tau)} Z (ZL)^{-\alpha}.$$

which implies a very simple analytic solution

$$L = \left[\beta \frac{Z^{1-\alpha}}{(1+\tau)} \right]^{1/(\gamma+\alpha)} .$$

We can see how productivity Z raises labor and money growth τ lowers it.

We can also see that the impact depends upon our preference parameters.

Second Step

Finally, using our condition for q_t and the fact that labor is constant in a steady state, we get the following expression for the steady-state bond price

$$q = \frac{\beta}{(1+\tau)},$$

where τ will be both the steady-state net growth rate of money and the inflation rate.

Second Step

Multipliers in the Steady State:
In the steady state we get that

$$\beta^{t-1} v'(L) \frac{L}{\bar{M}_1 (1+\tau)^{t-1}} = \mu_t,$$

and

$$\beta^{t-1} u'(ZL) \frac{ZL}{\bar{M}_1 (1+\tau)^{t-1}} = (\lambda_t + \mu_t).$$

- The first equation implies that μ_t is shrinking at the rate $\beta^{t-1}/(1+\tau)^{t-1}$.
- Given this, the second equation implies that λ_t is shrinking at the same rate.
- But the *relative values* of the multipliers are staying constant.
- For future reference, note that if we adjust our multipliers as follows

$$\mu_t (1+\tau)^{t-1}/\beta^{t-1}$$

the result will be constant. This is an important insight we will use later.

Second Step

Optimal Policy:

- Consider a social planner who could ignore the cash-in-advance constraint.
- The social planner's problem is

$$\max_{L_1} \; u(Z_1 L_1) - v(L_1) + \beta V,$$

  and V is the continuation payoff to the planner.
- The first-order condition for this problem is

$$u'(Z_1 L_1) Z_1 - v'(L_1) = 0. \tag{20}$$

- Next we compare this condition to the one that emerges from our model.

Second Step

Competive equilibrium:

- The FOCs are

$$\beta^{t-1} u'(C_t) = [\lambda_t + \mu_t]\, P_t \tag{21}$$

$$\beta^{t-1} v'(L_t) = \mu_t P_t Z_t. \tag{22}$$

$$\mu_t = \mu_{t+1} + \lambda_{t+1} \tag{23}$$

$$\mu_t q_t = \mu_{t+1} \tag{24}$$

- Using (21) and (22)

$$\frac{v'(L_t)}{u'(C_t)} = \frac{\mu_t Z_t}{\lambda_t + \mu_t} \tag{25}$$

- Using (23) and (24) in (25) get

$$\frac{v'(L_t)}{u'(C_t)} = \frac{\mu_t Z_t}{\mu_{t-1}} = q_{t-1} Z_t \tag{26}$$

- Rewriting (26)

$$v'(L_t) = q_{t-1} Z_t u'(Z_t L_t) \tag{27}$$

Second Step

- Condition (27) which we rewrite here again

$$v'(L_t) = q_{t-1} Z_t u'(Z_t L_t)$$

  and (20) which we rewrite here again

$$u'(Z_1 L_1) Z_1 - v'(L_1) = 0.$$

  are the same if $q = 1$.

- This means that the net nominal interest rate is being set to zero. The notion that the net nominal interest rate should be zero is often called the *Friedman rule*, after Milton Friedman, who first proposed it.

Second Step

- For $q = 1$, it must be the case that

$$\begin{aligned} q &= \frac{\beta}{(1+\tau)} = 1, \text{ or} \\ (1+\tau) &= \beta. \end{aligned}$$

- This condition implies that $\tau < 0$.
- But if the money supply is shrinking, prices will also be falling and we will have *deflation*.
- In fact, with L constant, it follows that in the optimum $P_{t+1} = \beta P_t$. So the discount rate sets the optimal level of deflation and the optimal rate at which money must shrink.

Second Step

What is the cost of inflation?

- Compute this cost in terms of consumption equivalents, i.e., the fraction of lifetime consumption, $\phi(\tau)$, that we would have to add or take away to make you just as well off.
- Denote by $L(\tau)$ the competitve allocation for a given value of inflation. Then lifetime utility is given by

$$\begin{aligned} U(\tau) &= \sum_{t=1}^{\infty} \beta^{t-1} \left[u(ZL(\tau)) - v(L(\tau)) \right] = \\ &= \frac{1}{1-\beta} \left[\frac{(ZL(\tau))^{1-\alpha} - 1}{1-\alpha} - \frac{L(\tau)^{1+\gamma}}{1+\gamma} \right]. \end{aligned}$$

- If we fix some particular τ_0 as our benchmark then $\phi(\tau)$ is defined as:

$$\frac{1}{1-\beta} \left[\frac{(\phi(\tau) ZL(\tau))^{1-\alpha} - 1}{1-\alpha} - \frac{L(\tau)^{1+\gamma}}{1+\gamma} \right] = U(\tau_0)$$

Second Step

What is the cost of inflation?

- Solving

$$\frac{1}{1-\beta}\left[\frac{(\phi(\tau)ZL(\tau))^{1-\alpha}-1}{1-\alpha}-\frac{L(\tau)^{1+\gamma}}{1+\gamma}\right]=U(\tau_0)$$

for $\phi(\tau)$

$$(\phi(\tau)ZL(\tau))^{1-\alpha}=\left\{U(\tau_0)\left(1-\beta\right)+\frac{L(\tau)^{1+\gamma}}{1+\gamma}\right\}(1-\alpha)+1$$

Which leads to

$$\phi(\tau)^{1-\alpha}=\frac{\left\{U(\tau_0)(1-\beta)+\frac{L(\tau)^{1+\gamma}}{1+\gamma}\right\}(1-\alpha)+1}{(ZL(\tau))^{1-\alpha}}. \tag{28}$$

Once we compute $\phi(\tau)$ using this last expression, the consumption equivalent variation is just given by $\phi(\tau)-1$.

Second Step

What is the cost of inflation?

- Suppose we want to compare two levels of inflation: 10% and $(\beta - 1) * 100\%$. According to the model higher inflation is leads to lower welfare. But how much worse is it? What is the additional consumption we have to give to the household so that the household is indifferent between that two situations? To get this number we have to compute $L(\tau = 10\%)$, $L(\tau_0 = (\beta - 1) * 100\%)$ and replace these two values in (28).
- In general (depends on the parameters) the inflation costs are below 0.5% of consumption.

Second Step

Accounting for Growth:

- Assume that $Z_{t+1} = (1+g)Z_t$.
- Return to our fundamental equation now modified to take account of growth in Z and normalized $Z_0 = 1$:

$$\begin{aligned} v'(L_t) &= \frac{\beta}{(1+\tau_t)} \frac{Z_{t+1}L_{t+1}}{L_t} u'(Z_{t+1}L_{t+1}). \\ &= \frac{\beta}{(1+\tau_t)} \frac{(1+g)^{t+1}L_{t+1}}{L_t} u'((1+g)^{t+1}L_{t+1}). \end{aligned}$$

- Unless the growth terms cancel in the r.h.s. the solution will not be invariant to the level of Z. So L will be changing over time.
- In order to have a steady state with L constant we must assume that $u(C) = \log(C)$.

Second Step

Accounting for Growth:

- When we specialize our utility function for consumption in this fashion our fundamental equation becomes

$$\beta^{t-1} v'(L_t) = \frac{1}{(1+\tau_t)} \frac{Z_{t+1} L_{t+1}}{L_t} \beta^t \frac{1}{Z_{t+1} L_{t+1}}. \tag{29}$$

- and this leads to a very simple expression for the equilibrium level of labor

$$L^{1+\gamma} = \frac{\beta}{1+\tau}.$$

- The price level is given by cash in advance constraint

$$P_t = \frac{\bar{M}_t}{Z_t L_t}$$

$$P_t = \frac{M_1 (1+\tau)^{t-1}}{Z_1 (1+g)^{t-1} L} = \frac{M_1}{Z_1 L} \left(\frac{1+\tau}{1+g} \right)^{t-1}$$

So, the inflation rate $1+\pi = (1+\tau)/(1+g)$ depends on the growth rate

Second Step

Accounting for Growth:

- Since the labor condition is

$$\beta^{t-1} v'(L_t) = \mu_t Z_t \frac{\bar{M}_t}{Z_t L_t},$$

and

$$\beta^t v'(L_{t+1}) = \mu_{t+1} Z_{t+1} \frac{\bar{M}_{t+1}}{Z_{t+1} L_{t+1}}$$

$$\Longrightarrow \beta = q_t \frac{\bar{M}_{t+1}}{\bar{M}_t},$$

- it follows that the steady-state interest rate is still

$$q = \frac{\beta}{1+\tau}.$$

Second Step

Consumption Equivalence with Growth:

$$\begin{aligned} U(\tau) &= \sum_{t=1}^{\infty} \beta^{t-1} \left[u(ZL(\tau)) - v(L(\tau)) \right] = \\ & \sum_{t=1}^{\infty} \beta^{t-1} \left[\log \left(ZL(\tau)(1+g)^{t-1} \right) - \frac{L(\tau)^{1+\gamma}}{1+\gamma} \right]. \end{aligned}$$

- To see how adding growth changes our calculation of the consumption equivalent variation, let us focus on the consumption term in the payoff. This becomes

$$\sum_{t=1}^{\infty} \beta^{t-1} \log \left(ZL(\tau)(1+g)^{t-1} \right) = \frac{\log(ZL(\tau))}{1-\beta} + \sum_{t=1}^{\infty} \beta^{t-1} (t-1) \log(1+g)$$

Second Step

- Then, note that

$$\frac{d}{d\beta}\sum_{t=1}^{\infty}\beta^{t-1}=\sum_{t=1}^{\infty}\frac{d}{d\beta}\beta^{t-1}=\sum_{t=1}^{\infty}(t-1)\beta^{t-2},$$

- while at the same time,

$$\frac{d}{d\beta}\sum_{t=1}^{\infty}\beta^{t-1}=\frac{d}{d\beta}\frac{1}{1-\beta}=\frac{1}{(1-\beta)^2}.$$

- Thus

$$\sum_{t=1}^{\infty}(t-1)\beta^{t-2}=\frac{1}{(1-\beta)^2}.$$

Second Step

Consumption Equivalence with Growth:

- We rewrite again the expression

$$\sum_{t=1}^{\infty} \beta^{t-1} \log\left(ZL(\tau)(1+g)^{t-1}\right) = \frac{\log(ZL(\tau))}{1-\beta} + \sum_{t=1}^{\infty} \beta^{t-1}(t-1)\log(1+g)$$

- Hence, it follows that our consumption payoff is

$$= \frac{\log(ZL(\tau))}{1-\beta} + \frac{\beta}{(1-\beta)^2}\log(1+g).$$

Second Step

Consumption Equivalence with Growth:

- It follows that our lifetime utility is given by

$$U(\tau) = \frac{1}{1-\beta}\left[\log(ZL(\tau)) - \frac{L(\tau)^{1+\gamma}}{1+\gamma}\right] + \log(1+g)\frac{\beta}{(1-\beta)^2}.$$

- This expression implies that the impact of money growth and growth in productivity comes in through two completely separate terms. Moreover, it means that our prior results on the cost of the money growth rate deviating from the optimal rate are unaffected by adding in productivity growth.

Second Step

Consumption Equivalence with Growth:

- So how important can growth be? Real risk-free interest rates around 1-2 percent generally, so this suggests that an annual value for $\beta = 0.98$.
- Plugging that in we get that

$$\frac{\beta}{(1-\beta)^2} = \frac{0.98}{.02^2} = \frac{0.98}{.0004} = 2450$$

  which looks big.
- For small g (for instance $g \approx 2\%$) we have

$$\log(1+g)\frac{\beta}{(1-\beta)^2} \approx g * 2450$$

  In other words growth is an important determinant of welfare.

Ext. 1: Varying Velocity

The velocity of money has been increasing over time because of the increased sophistication in the payments system.

- Want to extend our model to allow for this. So assume that the c.i.a. constraint is given by

$$M_t \geq \kappa P_t C_t, \quad \kappa \in (0, 1]$$

- $\kappa = 1$ is our old model.
- The amount $(1 - \kappa) P_t C_t$ is settled up later in the asset market.
- When the c.i.a. binds, we now get that

$$P = \frac{M}{\kappa Z L}.$$

So velocity is given by

$$v = \frac{PZL}{M} = \frac{M}{\kappa Z L} \frac{ZL}{M} = \frac{1}{\kappa}.$$

Ext. 1: Varying Velocity

Here is our Lagrangian, adjusted for this change,

$$
\begin{aligned}
\mathcal{L} = & \max_{\{C_t, L_t, M_{t+1}, B_{t+1}\}_{t=1,2}} \min_{\{\lambda_t, \mu_t\}_{t=1,2}} \\
& u(C_1) - v(L_1) + \beta \left[u(C_2) - v(L_2) \right] + \beta^2 V(M_3, B_3) \\
& + \sum_{t=1,2} \lambda_t \left\{ M_t - \kappa P_t C_t \right\} \\
& + \sum_{t=1,2} \mu_t \left\{ P_t Z_t L_t + M_t - P_t C_t + B_t + T_t - M_{t+1} - q_t B_{t+1} \right\}.
\end{aligned}
$$

Only change is w.r.t. the c.i.a. constraint. This changes the f.o.c. for consumption, which becomes

$$
\beta^{t-1} u'(C_t) = \left[\mu_t + \kappa \lambda_t \right] P_t .
$$

Ext. 1: Varying Velocity

We still get the following first-order conditions (f.o.c.s) for labor, money and bonds:

$$\begin{aligned} \beta^{t-1} v'(L_t) &= \mu_t Z_t P_t \\ \mu_t &= \mu_{t+1} + \lambda_{t+1}, \\ \mu_t q_t &= \mu_{t+1}, \end{aligned}$$

Ext. 1: Varying Velocity

Unfortunately, things don't reduce quite as neatly once we conjecture that L is constant and use the fact that M grows at a constant rate. To keep things simple assume that Z is constant and $\bar{M}_0 = 1$, and hence we can rewrite these conditions as

$$
\begin{aligned}
\beta^{t-1} u'(ZL) &= [\mu_t + \kappa\lambda_t] \frac{M}{\kappa ZL} \Longleftrightarrow \\
\beta^{t-1} u'(ZL) &= [\mu_t + \kappa\lambda_t] \left(\frac{(1+\tau)^t}{\kappa ZL} \right) \\
\beta^{t-1} v'(L_t) &= \mu_t Z \left(\frac{(1+\tau)^t}{\kappa ZL} \right), \\
\mu_t &= \mu_{t+1} + \lambda_{t+1}.
\end{aligned}
$$

Note that here too our multipliers won't be constant over time.

Ext. 1: Varying Velocity

Make a "change in variables," to rewrite our equations in terms of variables that are stationary.

Construct new multipliers

$$
\begin{aligned}
\tilde{\mu}_t &= \mu_t \left((1+\tau) \frac{1}{\beta} \right)^t \beta, \\
\tilde{\lambda}_t &= \lambda_t \left((1+\tau) \frac{1}{\beta} \right)^t \beta.
\end{aligned}
$$

Using a change in variables along these lines will turn out to be a much more robust method of analyzing our models.

Ext. 1: Varying Velocity

- Then, I'm going to rewrite my equations in terms of these new variables and cancel anything out that I can to get

$$u'(ZL) = \left[\tilde{\mu}_t + \kappa\tilde{\lambda}_t\right]\left(\frac{1}{\kappa ZL}\right) \tag{30}$$

$$v'(L) = \tilde{\mu}_t Z\left(\frac{1}{\kappa ZL}\right), \tag{31}$$

$$\left(\frac{1+\tau}{\beta}\right)\tilde{\mu}_t = \tilde{\mu}_{t+1} + \tilde{\lambda}_{t+1}. \tag{32}$$

  Now, this is a nice equation system and we can guess that there is a constant solution where $\tilde{\mu}_t = \tilde{\mu}$ and $\tilde{\lambda}_t = \tilde{\lambda}$.

- System is too complicated to solve directly unfortunately.
- Could use a nonlinear equation solver to solve 3 equations in 3 unknowns.

Ext. 1: Varying Velocity

- Here is a brute force method that one could use for this system of equations because of their block structure.
- First, consider a grid on labor of $\mathbf{L} = [L_0, L_1, ..., L_N]$ values where the highest level of labor is the efficient level and the lowest is so low that we don't think our answer will be below this value.
- Then use a for loop to sequentially compute the following:
  1. Use (31) to solve for $\tilde{\mu}_i$ given L_i.
  2. Use (30) to solve for $\tilde{\lambda}_i$ given $\tilde{\mu}_i$ and L_i.
  3. Given this, construct the deviations

$$ERR(i) = \left(\frac{1+\tau}{\beta}\right)\tilde{\mu}_i - \tilde{\mu}_i - \tilde{\lambda}_i.$$

  4. Plot $ERR(i)$ and see where the zeros are.
- These are our solutions. If the plot is nice, there is only one solution.
- Once we know where the solution is, make our grid finer in this region.

Ext. 3: Costly Money Holding

It turns out interest rates can be negative in the real world, but not in our model. So we need to extend our model to include costs to holding money.

- We will assume that these costs come in the form of labor costs associated with each unit of real balances the individual holds.
- As a result, this version of our model will allow for negative interest rates.
- We will use this version of our model to undertake our first calibration exercise and to revisit the costs of inflation

Ext. 3: Costly Money Holding

The household's problem can be written as

$$\max_{\{C_t, L_t, H_t, M_{t+1}, B_{t+1}\}_{t=1,2}} u(C_1) - v(L_1 + H_1) + \beta\left[u(C_2) - v(L_2 + H_2)\right]$$
$$+\beta^2 V(M_3, B_3) \text{ subject to}$$

$$\begin{aligned} M_t &\geq P_t C_t, \\ Z_t H_t &= \phi M_{t+1}/P_t, \\ P_t Z_t L_t + [M_t - P_t C_t] + B_t + T_t &\geq M_{t+1} + q_t B_{t+1} \text{ for } t = 1, 2. \end{aligned}$$

Plug in the equality constraint for H_t to get

$$v\left(L_t + \frac{\phi M_{t+1}}{Z_t P_t}\right)$$

and we can drop the constraint since it is already being enforced

Ext. 3: Costly Money Holding

The Lagrangian for the individual's problem is

$$
\begin{aligned}
\mathcal{L} \;=\; & \max_{\{C_t, L_t, M_{t+1}, B_{t+1}\}_{t=1,2}} \; \min_{\{\lambda_t, \mu_t\}_{t=1,2}} \\
& \sum_{t=1,2} \beta^{t-1} \left[u(C_t) - v\left(L_t + \frac{\phi M_{t+1}}{Z_t P_t} \right) \right] + \beta^2 V(M_3, B_3) \\
& + \sum_{t=1,2} \lambda_t \left\{ M_t - P_t C_t \right\} \\
& + \sum_{t=1,2} \mu_t \left\{ P_t Z_t L_t + M_t - P_t C_t + B_t + T_t - M_{t+1} - q_t B_{t+1} \right\}.
\end{aligned}
$$

Ext. 3: Costly Money Holding

Two first-order conditions are unchanged

$$\beta^{t-1} u'(C_t) = [\mu_t + \lambda_t] P_t,$$

$$\mu_t q_t = \mu_{t+1},$$

The labor condition is new and given by

$$\beta^{t-1} v' \left(L_t + \frac{\phi M_{t+1}}{Z_t P_t} \right) = \mu_t Z_t P_t,$$

while that for money has a new cost term

$$\beta^{t-1} v' \left(L_t + \frac{\phi M_{t+1}}{Z_t P_t} \right) \frac{\phi}{Z_t P_t} + \mu_t = \mu_{t+1} + \lambda_{t+1},$$

Note that we can have $q_t > 1 \implies \mu_t < \mu_{t+1}$ without violating $\lambda_{t+1} \geq 0$. This means negative interest rates are possible now.

Ext. 3: Costly Money Holding

Following the standard path we find:

$$\begin{aligned}
\beta^t \frac{u'(C_{t+1})}{P_{t+1}} &= \mu_{t+1} + \lambda_{t+1} \\
&= \beta^{t-1} v' \left(L_t + \frac{\phi M_{t+1}}{Z_t P_t} \right) \frac{\phi}{Z_t P_t} + \mu_t \\
&= \beta^{t-1} v' \left(L_t + \frac{\phi M_{t+1}}{Z_t P_t} \right) \frac{\phi + 1}{Z_t P_t}.
\end{aligned}$$

The first step was to substitute for $\mu_{t+1} + \lambda_{t+1}$ using our money condition, and the second step was to substitute for μ_t using our labor condition.

Ext. 3: Costly Money Holding

Assuming constant growth rates, log preferences for consumption, and the c.i.a constraint binds, we get that

$$\frac{\beta^t}{\bar{M}_{t+1}} = \beta^{t-1} v' \left(L + \frac{\phi \bar{M}_{t+1}}{\frac{\bar{M}_t}{L_t}} \right) \frac{\phi + 1}{\frac{\bar{M}_t}{L}}$$

One more step of cleaning things up and we get our final expression

$$\frac{\beta}{1+\tau} = v' \left(L \left[1 + \phi(1+\tau) \right] \right) L(\phi + 1).$$

$$\implies L^{1+\gamma} = \frac{\beta}{1+\tau} \frac{1}{\left[1 + \phi(1+\tau) \right]^{\gamma} (\phi + 1)}$$

which is a neat analytic expression.

Ext. 3: Costly Money Holding

If we take log's and differentiate with respect to L and $1+\tau$ we get that

$$(1+\gamma)d\log L = -d\log(1+\tau) - \gamma d\log([1+\phi(1+\tau)])$$

$$\begin{aligned}\implies d\log L &= \frac{-1}{1+\gamma}\left[d\log(1+\tau) + \gamma\frac{\phi}{1+\phi(1+\tau)}d\log(1+\tau)\right] \\ &= \frac{-1}{1+\gamma}\left[1+\gamma\frac{\phi}{1+\phi(1+\tau)}\right]d\log(1+\tau)\end{aligned}$$

So, we can see that labor will become more sensitive to τ the bigger is ϕ.