# 1 Complements of Linear Algebra

**1.1.** Determine the eigenvalues of each of the following matrices and, if they are real, determine the corresponding eigenvectors together with the algebraic and geometric multiplicities.

a) $\begin{bmatrix} 2 & -7 \\ 3 & -8 \end{bmatrix}$ b) $\begin{bmatrix} 2 & 4 \\ -2 & 6 \end{bmatrix}$ c) $\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$

d) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ e) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ f) $\begin{bmatrix} 1 & -1 & -2 \\ 0 & 3 & 0 \\ -2 & 5 & 1 \end{bmatrix}$

g) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ h) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ i) $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

**Solution:** a) $\lambda_1 = -5,\ a.m. = 1$; eigenvectors $u = (c, c)$, with $c \neq 0,\ g.m. = 1$;
$\lambda_2 = -1,\ a.m. = 1$; eigenvectors $u = (7/3c, c)$, with $c \neq 0,\ g.m. = 1$;
b) The characteristic polynomial does not have real eigenvalues ($\lambda_1 = 4 + 2i,\ \lambda_2 = 4 - 2i$);
c) $\lambda_1 = 1 + \sqrt{2};\ a.m. = 1$; eigenvectors $u = ((1 + \sqrt{2})c, c)$, with $c \neq 0,\ g.m. = 1$;
$\lambda_2 = 1 - \sqrt{2};\ a.m. = 1$; eigenvectors $u = ((1 - \sqrt{2})c, c)$, with $c \neq 0,\ g.m. = 1$;
d) $\lambda = 0,\ a.m. = 2$; eigenvectors $u = (c, 0)$, with $c \neq 0,\ g.m. = 1$;
e) $\lambda_1 = 2,\ a.m. = 1$; eigenvectors $u = (c, 0, 0)$, with $c \neq 0,\ g.m. = 1$;
$\lambda_2 = 3,\ a.m. = 1$; eigenvectors $u = (0, c, 0)$, with $c \neq 0,\ g.m. = 1$;
$\lambda_3 = 4,\ a.m. = 1$; eigenvectors $u = (0, 0, c)$, with $c \neq 0,\ g.m. = 1$;
f) $\lambda_1 = 3,\ a.m. = 2$; eigenvectors $u = (-c, 0, c)$, with $c \neq 0,\ g.m. = 1$;
$\lambda_2 = -1,\ a.m. = 1$; eigenvectors $u = (c, 0, c)$, with $c \neq 0,\ g.m. = 1$;
g) $\lambda_1 = 0,\ a.m. = 2$; eigenvectors $u = (-c_1 - c_2, c_1, c_2)$, with $c_1^2 + c_2^2 \neq 0\ ,\ g.m. = 2$;
$\lambda_2 = 3,\ a.m. = 1$; eigenvectors $u = (c, c, c)$, with $c \neq 0,\ g.m. = 1$;
h) $\lambda = 1,\ a.m. = 3$; eigenvectors $u = (c, 0, 0)$, with $c \neq 0,\ g.m. = 1$;
i) $\lambda_1 = 2,\ a.m. = 2$; eigenvectors $u = (c_1, c_1, c_2)$, with $c_1^2 + c_2^2 \neq 0,\ g.m. = 2$;
$\lambda_2 = 0,\ a.m. = 1$; eigenvectors $u = (-c, c, 0)$, with $c \neq 0,\ g.m. = 1$.

**1.2.** Show that

a) Every eigenvalue of $A$ is also an eigenvalue of $A^T$.

b) If $\lambda$ is an eigenvalue of $A$ and $|A| \neq 0$ then $\frac{1}{\lambda}$ is an eigenvalue of $A^{-1}$.

c) If $\lambda$ is an eigenvalue of $A$ then $\lambda^k$ is an eigenvalue of $A^k$, $k \in \mathbb{N}$.

d) If $u$ and $v$ are eigenvectors associated to an eigenvalue $\lambda$ of $A$, then

   i) $tu$ is an eigenvector of $A$ associated to $\lambda$ provided that $t \in \mathbb{R} \setminus \{0\}$,

   ii) $u + v$ is an eigenvector of $A$ associated to $\lambda$ provided that $u + v \neq 0$.

**1.3.** Let $A = \begin{bmatrix} 1 & 4 \\ 6 & -1 \end{bmatrix}$

a) Determine the eigenvalues and eigenvectors of $A$.

b) Determine the eigenvalues and eigenvectors of $A^{100}$.

c) Is $A$ diagonalizable?

d) Compute the trace of $A^{100}$.

**Solution:** a) $\lambda_1 = -5$, eigenvectors $u = (-2/3c, c)$, with $c \neq 0$, ;
$\lambda_2 = 5$, eigenvectors $u = (c, c)$, with $c \neq 0$, ;
b) $\lambda = 5^{100}$, eigenvectors $u = (-2/3c_1, c_1) + (c_2, c_2)$, with $c_1, c_2$ not simultaneously zero;
c) Yes, $A$ has distinct eigenvalues;
d) $2 \times 5^{100}$.

**1.4.** Determine which of the following matrices are diagonalizable.

a) $\begin{bmatrix} 2 & 2 & 2 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ b) $\begin{bmatrix} 6 & 0 \\ 1 & 6 \end{bmatrix}$ c) $\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$

d) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ e) $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ f) $\begin{bmatrix} 1 & -1 & -2 \\ 0 & 3 & 0 \\ -2 & 5 & 1 \end{bmatrix}$

g) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ h) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ i) $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

**Solution:** a) Yes; b) No; c) Yes; d) No; e) Yes; f) No; g) Yes; h) No; i) Yes.

**1.5.** Two square $n$-by-$n$ matrices $A$ and $B$ are called *similar* if there is an invertible $n$-by-$n$ matrix $P$ such that $A = PBP^{-1}$. Suppose that $A$ and $B$ are similar. Show that

a) $|A| = |B|$.

b) $A$ and $B$ have the same characteristic polynomial.

c) $(\lambda, v)$ is an eigenpair of $B$ iff $(\lambda, Pv)$ is an eigenpair of $A$.

**1.6.** Consider a matrix $A$ and a vector $\boldsymbol{x}$ given by

$$A = \begin{bmatrix} a & a & 0 \\ a & a & 0 \\ 0 & 0 & b \end{bmatrix} \qquad \text{e} \qquad \boldsymbol{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad (a, b \in \mathbb{R})$$

a) Write down the characteristic polynomial $p(\lambda)$.

b) Determine the eigenvalues and eigenvectors of matrix $A$.

c) Compute $\boldsymbol{x}^T A\boldsymbol{x}$.

d) Consider that $a, b \geq 0$. Without performing any calculations show that there exists a a vector $\boldsymbol{x} \neq \mathbf{0}$ such that $\boldsymbol{x}^T A\boldsymbol{x} = 0$

e) Classify the quadratic form $\boldsymbol{x}^T A\boldsymbol{x}$ for all possible values of $a, b$.

**Solution:** a) $P(\lambda) = (b - \lambda)(-\lambda)(2a - \lambda)$; b) $\lambda_1 = b, \lambda_2 = 0, \lambda_3 = 2a$; c) $\mathbf{x}^T A\mathbf{x} = ax^2 + 2axy + ay^2 + bz^2$;
e) PSD if $(a \geq 0 \; e \; b \geq 0)$ ; NSD if $(a \leq 0 \; e \; b \leq 0)$;
Ind. in the remaining cases, i.e. $(a > 0 \; e \; b < 0)$ ou $(a < 0 \; e \; b > 0)$.

**1.7.** Classify the following quadratic forms:

a) $q(x, y) = x^2 + 2xy + y^2$

b) $q(x, y) = x^2 - 2xy + y^2$

c) $q(x, y) = x^2 - y^2$

d) $q(x, y, z) = x^2 + 4xy - 2xz + 7y^2 - 3z^2$

e) $q(x, y, z) = x^2 - 4xy + 4xz - z^2$

f) $q(x, y) = 6x^2 + 4xy + 3y^2$

g) $q(x, y) = x^2 + 4xy + y^2$

h) $q(x, y) = 2x^2 + 6xy + 4y^2$

i) $q(x, y, z) = 3y^2 + 4xz$

j) $q(x, y) = x^2 + 4xy + ay^2, \quad a \in \mathbb{R}$

**Solution:** a)PSD b)PSD c)Ind. d)Ind. e)Ind. f)PD g)Ind. h)Ind. i)Ind. j)Ind. if $a < 4$, PSD if $a = 4$ and PD if $a > 4$.

**1.8.**

Classify the following symmetric matrices (with respect to being PD, PSD, ND, NSD, Ind.)

a) $\begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix}$ b) $\begin{bmatrix} -5 & 1 \\ 1 & 5 \end{bmatrix}$ c) $\begin{bmatrix} -5 & 1 \\ 1 & -5 \end{bmatrix}$ d) $\begin{bmatrix} 1 & 2 \\ 2 & a \end{bmatrix}, a \in \mathbb{R}$ e) $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 7 & 0 \\ -1 & 0 & -3 \end{bmatrix}$

f) $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 7 & -5 \\ -1 & -5 & 4 \end{bmatrix}$ g) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -4 \end{bmatrix}$ h) $\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & a & 0 \\ 0 & a & 2 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}, a \in \mathbb{R}$ i) $\begin{bmatrix} -1 & 1 & 1 \\ 1 & -3 & 0 \\ 1 & 0 & -2 \end{bmatrix}$

j) $\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

**Solution:** a) PD b) Ind. c) ND d)Ind. if $a < 4$, PSD if $a = 4$ and ND if $a > 4$ e) Ind. f) PSD
g) Ind. h)Ind. if $a < -\sqrt{2} \vee a > \sqrt{2}$, PD if $-\sqrt{2} < a < \sqrt{2}$ and PSD if $a = \pm\sqrt{2}$ i) ND j) Ind.

# 2 Functions of several variables: Topology, limits and continuity

**2.1.** Determine the domain of the following functions and represent it graphically.

$a) f(x,y) = \dfrac{\sqrt{9 - x^2 - y^2}}{1 - \ln x}$ $b) f(x,y) = \dfrac{\sqrt{e - e^x}}{\ln(4 - x^2 - y^2)}$ $c) f(x,y) = \ln(x-y)\sqrt{(y-x)(x^2+y^2-1)}$

$d) f(x,y) = \dfrac{\sqrt[3]{x+y}}{\ln x^2 - \ln(3-x)^2}$ $e) f(x,y) = \ln(x-y)^2$

**Solution:** a) $\{(x,y) \in \mathbb{R}^2 : x > 0 \wedge x \neq e \wedge x^2 + y^2 \leq 9\}$ b) $\{(x,y) \in \mathbb{R}^2 : x \leq 1 \wedge x^2 + y^2 < 4 \wedge x^2 + y^2 \neq 3\}$ c) $\{(x,y) \in \mathbb{R}^2 : x - y > 0 \wedge x^2 + y^2 - 1 \leq 0\}$ d) $\{(x,y) \in \mathbb{R}^2 : x \neq 0 \wedge x \neq 3 \wedge x \neq 3/2\}$ e) $\{(x,y) \in \mathbb{R}^2 : x \neq y\}$

**2.2.** Determine the interior, exterior and boundary of the following subsets of $\mathbb{R}^2$. Classify them with respect to being open, closed and bounded.

$$A = \{(x,y) \in \mathbb{R}^2 : (x-1)^2 + (y+2)^2 < 4\} \qquad B = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \geq 9\}$$

$$C = \{(x,y) \in \mathbb{R}^2 : \frac{x^2}{9} + \frac{y^2}{16} \leq 1\} \qquad D = \{(x,y) \in \mathbb{R}^2 : \frac{(x-2)^2}{4} + \frac{(y+1)^2}{9} = 1\}$$

**Solution:** $\text{Int}(A) = A$, $\text{Bdy}(A) = \{(x,y) \in \mathbb{R}^2 : (x-1)^2 + (y+2)^2 = 4\}$, $ext(A) = \{(x,y) \in \mathbb{R}^2 : (x-1)^2 + (y+2)^2 > 4\}$, $A$ is open and bounded. $\text{Int}(B) = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 > 9\}$, $\text{Bdy}(B) = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 9\}$, $\text{Ext}(B) = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 9\}$, $B$ is closed and not bounded. $\text{Int}(C) = \{(x,y) \in \mathbb{R}^2 : \frac{x^2}{9} + \frac{y^2}{16} < 1\}$, $\text{Bdy}(C) = \{(x,y) \in \mathbb{R}^2 : \frac{x^2}{9} + \frac{y^2}{16} = 1\}$, $\text{Ext}(C) = \{(x,y) \in \mathbb{R}^2 : \frac{x^2}{9} + \frac{y^2}{16} > 1\}$, $C$ is closed and bounded. $\text{Int}(D) = \emptyset$, $fr(D) = D$, $\text{Ext}(D) = \{(x,y) \in \mathbb{R}^2 : \frac{(x-2)^2}{4} + \frac{(y+1)^2}{9} \neq 1\}$, $D$ is closed and bounded.

**2.3.**

Represent graphically and analitically the the domain $D_f$, as well as $\text{Int}(D_f)$, $\text{Bdy}(D_f)$ and $D'_f$. State in each case if $D_f$ open, closed, bounded and compact.

$$a) f(x,y) = \frac{|x| - 4}{\ln(4 - x^2 - y^2)} \qquad b) f(x,y) = \sqrt{x(1-x)} + \frac{\ln(x^2 - y)}{\sqrt{y + x}}$$

$$c) f(x,y) = \sqrt{y + x - 1}.\ln(4 - (x+1)^2 - (y-1)^2) \qquad d) f(x,y) = \sqrt{x - |y|} + \sqrt{2 - y^2 - x}$$

$$e) f(x,y) = x\sqrt{y^2 - 4} + \sqrt[4]{16 - x^2 - y^2}$$

**Solution:** a) $D_f = \{(x,y) \in \mathbb{R}^2 : 4 - x^2 - y^2 > 0 \wedge \ln(4 - x^2 - y^2) \neq 0\} = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 4 \wedge x^2 + y^2) \neq 3\}$, $\text{Int}(D_f) = D_f$, $\text{Bdy}(D_f) = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 4 \vee x^2 + y^2 = 3\}$, $D'_f = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\}$. $D_f$ is open, not closed, bounded, not compact.

b) $D_f = \{(x,y) \in \mathbb{R}^2 : x(1-x) \geq 0 \wedge x^2 - y > 0 \wedge y + x > 0\} = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1 \wedge -x < y < x^2\}$, Int$(D_f) = \{(x,y) \in \mathbb{R}^2 : 0 < x < 1 \wedge -x < y < x^2\}$, Bdy$(D_f) = \{(x,y) \in \mathbb{R}^2 : (y = x^2 \ \wedge \ 0 \leq x \leq 1) \vee (y = -x \ \wedge \ 0 \leq x \leq 1) \vee (x = 1 \ \wedge \ -x \leq y \leq x^2)\}$, $D'_f = \{(x,y) \in \mathbb{R}^2 : (0 \leq x \leq 1) \wedge (-x \leq y \leq x^2)\}$. $D_f$ is not open nor closed, bounded, not conmpact.

c) $D_f = \{(x,y) \in \mathbb{R}^2 : y + x - 1 \geq 0 \wedge 4 - (x+1)^2 - (y-1)^2 > 0\} = \{(x,y) \in \mathbb{R}^2 : y \geq 1 - x \wedge (x+1)^2 + (y-1)^2 < 4\}$, Int$(D_f) = \{(x,y) \in \mathbb{R}^2 : y > 1 - x \wedge (x+1)^2 + (y-1)^2 < 4\}$, Bdy$(D_f) = \{(x,y) \in \mathbb{R}^2 : y = 1 - x \wedge (x+1)^2 + (y-1)^2 \leq 4\} \cup \{(x,y) \in \mathbb{R}^2 : y \geq 1 - x \wedge (x+1)^2 + (y-1)^2 = 4\}$, $D'_f = \{(x,y) \in \mathbb{R}^2 : y \geq 1 - x \wedge (x+1)^2 + (y-1)^2 \leq 4\}$. $D_f$ is not open nor closed, bounded, not compact.

d)$D_f = \{(x,y) \in \mathbb{R}^2 : x - |y| \geq 0 \wedge 2 - y^2 - x \geq 0\}$, Int$(D_f) = \{(x,y) \in \mathbb{R}^2 : x - |y| > 0 \wedge 2 - y^2 - x > 0\}$, Bdy$(D_f) = \{(x,y) \in \mathbb{R}^2 : x - |y| = 0 \wedge 2 - y^2 - x \geq 0\} \cup \{(x,y) \in \mathbb{R}^2 : x - |y| \geq 0 \wedge 2 - y^2 - x = 0\}$, $D'_f = D_f$. $D_f$ is not open, is closed, bounded, compact.

e)$D_f = \{(x,y) \in \mathbb{R}^2 : y^2 - 4 \geq 0 \wedge 16 - x^2 - y^2 \geq 0\} = \{(x,y) \in \mathbb{R}^2 : (y \geq 2 \vee y \leq -2) \wedge x^2 + y^2 \leq 16\}$, Int$(D_f) = \{(x,y) \in \mathbb{R}^2 : (y > 2 \vee y < -2) \wedge x^2 + y^2 < 16\}$, Bdy$(D_f) = \{(x,y) \in \mathbb{R}^2 : (y = 2 \vee y = -2) \wedge x^2 + y^2 \leq 16\} \cup \{(x,y) \in \mathbb{R}^2 : (y \geq 2 \vee y \leq -2) \wedge x^2 + y^2 = 16\}$, $D'_f = D_f$. $D_f$ is not open, is closed, bounded, compact.

**2.4.** Determine the domain $D_f$ of the function defined by

$$f(x,y) = \frac{\ln(x^2 + y^2 - 4)}{|x| - 4}.$$

Represent $D_f$ graphically and show that it is open and unbounded. Check if Ext$(D_f)$ is also open and unbounded.

**Solution:** $D_f = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 - 4 > 0 \wedge |x| - 4 \neq 0\}$. $Ext(D_f) = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 4\}$.

**2.5.**

Determine the interior, the boundary and the limit points of the following subsets of $\mathbb{R}^2$.

$a)A = \{(0,0), (0,1), (1,0), (1,1)\}$

$b)B = \{(x,y) \in \mathbb{R}^2 : (x,y) = (\frac{1}{n}, \frac{1}{n}), n \in \mathbb{N}\}$

$c)C = \{(x,y) \in \mathbb{R}^2 : x \geq 0, \ y = (-1)^n \frac{1}{n}, n \in \mathbb{N}\}$.

**Solution:** a) $\text{Int}(A) = \emptyset, \text{Bdy}(A) = A, A' = \emptyset$. b) $\text{Int}(B) = \emptyset$, $\text{Bdy}(B) = B \cup \{(0,0)\}$, $B' = \{(0,0)\}$. c) $\text{Int}(C) = \emptyset$, $\text{Bdy}(C) = C \cup \{(x,y) \in \mathbb{R}^2 : x \geq 0,\ y = 0\}$, $C' = fr(C)$.

**2.6.**

Compute the limits of the following sequences, or show that they no not exist.

a) $\lim \left( \left( \frac{2n^2+3}{1+2n^2} \right)^{n^2}, \ln \left( \frac{2n}{2n+1} \right)^{n+\frac{1}{2}} \right)$ b) $\lim \left( n^3 + n - n^2 - 1), \sqrt{n}.\frac{\sqrt{n}+3}{(\sqrt{n}+1)^2} \right)$

c) $\lim \left( n \left( e^{\frac{1}{n}} - 1 \right), \sin \frac{n\pi}{2} \right)$

**Solution:** a) $(e, -\frac{1}{2})$; b) $(+\infty, 1)$; c) does not exist.

**2.7.** Compute the following limits:

a) $\lim \overline{x}_n$, com $\overline{x}_n = \left[ \frac{n}{2n+1}, \left(1 + \frac{2}{n}\right)^n, \ln \left(1 + \frac{1}{n}\right)^n \right]$

b) $\lim \overline{x}_n$ com $\overline{x}_n = \left[ \sqrt{n} - \sqrt{n-1},\ \left(\sqrt[n]{e} - 1\right).n,\ n. \ln \frac{n+2}{n}, \left(1 - \frac{n^2}{n^2+1}\right)^{\frac{1}{3}} \cdot \left(\frac{n^2+1}{2n^2}\right)^{\frac{1}{3}} \right]$

**Solution:** a) $(1/2, e^2, 1)$; b) $(0, 1, 2, 0)$.

**2.8.**

For each of the following functions, investigate the existence of limit at the point $(0,0)$.

a) $f(x,y) = \dfrac{x^2 - y^2}{x(x+y)}$ b) $f(x,y) = \dfrac{x^3 - y^3}{x^2 + y^2}$ c) $f(x,y) = \dfrac{x^2 + y}{\sqrt{x^2 + y^2}}$

d) $f(x,y) = xy\dfrac{x^2 - y^2}{x^2 + y^2}$ e) $f(x,y) = \dfrac{xy}{x^2 + y^2}$ f) $f(x,y) = \dfrac{x^2 y}{x^4 + y^2}$

g) $f(x,y) = \dfrac{x^2(x+y)}{x^2 + y^2}$ h) $f(x,y) = \dfrac{x^2 - y^2 + 2x^3}{x^2 + y^2}$, i) $f(x,y) = \begin{cases} \dfrac{y}{x}\sqrt{x^2 + y^2}, & \text{se } x > 0,\ y > 0 \\ 0, & \text{se } x < 0 \text{ ou } y < 0. \end{cases}$

**Solution:** a) does not exist  b) 0  c)does not exist  d) 0  e) does not exist  f) does not exist  g) 0  h) does not exist  i) does not exist.

**2.9.**

Determine, if possible, a continuous extention of each of the functions in the previous exercise to the point $(0, 0)$.

**Solution:**

When the limit does not exist it is not possible to define such extension. In the other cases simply define $f(0, 0) = \lim_{(x,y)\to(0,0)} f(x, y)$.

**2.10.** Investigate the existence of the following limits

$a)\ \lim_{(x,y)\to(2,1)} \frac{(x-2)(y-1)}{(x-2)^2+(y-1)^2}$ $\qquad b)\ \lim_{(x,y)\to(1,-3)} \frac{\sqrt{(x-1)(y+3)}+\sin(x-1)(y+3)}{\sqrt{(x-1)(y+3)}}$

$c)\ \lim_{(x,y)\to(2,1)} \frac{3(x-2)^2(y-1)}{(x-2)^2+(y-1)^2}$ $\qquad d)\ \lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x}$

$e)\ \lim_{(x,y)\to(1,0)} \frac{x^2-y^2-1}{x-1}$ $\qquad f)\ \lim_{(x,y)\to(0,0)} \frac{x^2y+x^2+y^2}{x^2+y^2}$

$g)\ \lim_{(x,y)\to(0,1)} \frac{x^2\sqrt{|y-1|}}{x^2+(y-1)^2}$ $\qquad h)\ \lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2}$

**Solution:** a) does not exist  b) 1  c) 0  d) does not exist  e) does not exist  f) 1  g) 0  h) 0.

**2.11.** Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = \dfrac{x^3 - y^3}{x - y}$.

a) Determine the domain $D_f$..

b) Show that $x^3 - y^3 = (x - y)p(x, y)$ , where $p(x, y)$ is a polynomial.

c) Can $f$ be extended by continuity to the line $y = x$?

**Solution:** a) $D_f = \mathbb{R}^2 \setminus \{(x,x) : x \in \mathbb{R}\}$. c) yes, just define $f(x,x) = 3x^2$.

**2.12.** Consider the function $f(x,y) = \frac{x^2 y}{x^2 - y^2}$.

a) Compute $\lim_{\substack{(x,y)\to(0,0),\\ y=x+x^2}} f(x,y)$. What can you conclude about $\lim_{(x,y)\to(0,0)} f(x,y)$ ?

b) Compute $\lim_{\substack{(x,y)\to(0,0),\\ y=mx}} f(x,y)$, $|m| \neq 1$. What can you conclude about $\lim_{(x,y)\to(0,0)} f(x,y)$ ?

**Solution:**

a) $-\frac{1}{2}$. If it exists $\lim_{(x,y)\to(0,0)} f(x,y)$, is $-\frac{1}{2}$. b) 0. It does not exist $(0 \neq -\frac{1}{2})$.

# 3 Differential Calculus in $\mathbb{R}^n$

**3.1.**

Determine the first order partial derivatives of the following functions, defining them in the largest possible domain.

a)$f(x,y,z) = 3xy + x^2 - zy + z^2$; b) $f(x,y) = \begin{cases} x^2 - yx, & y \neq x \\ x, & y = x. \end{cases}$

**Solution:** a) $f'_x = 3y + 2x$ ; $f'_y = 3x - z$ ; $f'_z = -y + 2z$ ; $\mathcal{D}f'_x = \mathcal{D}f'_y = \mathcal{D}f'_z = \mathbb{R}^3$
b) $f'_x = 2x - y$ if $x \neq y$ and $f'_x = 0$ if $x = y = 0$ ; $f'_y = -x$ if $x \neq y$ and $f'_y = 0$ if $x = y = 0$
$\mathcal{D}f'_x = \mathcal{D}f'_y = \mathbb{R}^2 \setminus \{(a,a) : a \neq 0\}$

**3.2.** Show that $f(x,y) = \dfrac{x - y + 1}{x + y}$ is a solution of the equation

$$\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} = \frac{2}{x+y}$$

in any of the sets defined by $x + y > 0$ or $x + y < 0$.

**3.3.** Consider the function

$$f(x,y) = \begin{cases} \dfrac{e^{x-y} - (x-y+1)}{x-y}, & x \neq y \\ 0, & x = y \end{cases}$$

a) Discuss the continuity of $f(x, y)$ at $(1, 1)$.

b) Check that $f'_x(a, a) + f'_y(a, a) = 0,\ \forall a \in \mathbb{R}$.

**Solution:**

a) $f$ is continuous at $(1, 1)$.

**3.4.** Given the function

$$f(x, y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2} & , x^2 + y^2 \neq 0 \\ 0 & , x = y = 0 \end{cases},$$

compute the directional derivatives at $(0, 0)$, whenever they exist.

**Solution:** $\partial_{\overrightarrow{v}} f(0, 0)$ exists for $\overrightarrow{v} = (\alpha, \alpha)$ and $\overrightarrow{v} = (\alpha, -\alpha)$, $\alpha \in \mathbb{R} \setminus \{0\}$. In that case, $\partial_{\overrightarrow{v}} f(0, 0) = 0$.

**3.5.** Consider the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} \dfrac{xy^2}{x^2 + y^4} & , x \neq 0 \\ 0 & , x = 0 \end{cases},$$

a) Show that $f$ admits a directional derivative at $(0, 0)$ along any direction and compute it.

b) Show that $f$ is not continuous at $(0, 0)$.

c) Without performing any calculations, state the value of $\dfrac{\partial f}{\partial x}(0, 0)$ and of $\dfrac{\partial f}{\partial y}(0, 0)$.

**Solution:** a) $\partial_{(\alpha,\beta)} f(0, 0) = \begin{cases} \dfrac{\beta^2}{\alpha} & \text{se } \alpha \neq 0 \\ 0 & \text{se } \alpha = 0 \end{cases}$, $(\alpha, \beta) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. c) $\dfrac{\partial f}{\partial x}(0, 0) = \dfrac{\partial f}{\partial y}(0, 0) = 0$.

**3.6.**

Study the differentiability of the following functions at the proposed points and obtain the expression of the first order differentials (in case they are differentiable).

a) $f(x,y) = x^2 + y^2$, at point (0,0);

b) $f(x,y) = \begin{cases} x+y, & x \neq y \\ x+1, & x = y \end{cases}$, at (1,1);

c) $f(x,y) = \begin{cases} xy - 2y + 3x, & x \neq y \\ x^2y^2 + 3x - 2y, & x = y \end{cases}$, at (0,0);

d) $y = (x^2+1, x)$, at $x = 1$;

**Solution:**
a) $f$ is differentiable at $(0,0)$; $Df(0,0)(\mathbf{h}) = 0$;
b) $f$ is not differentiable at em $(1,1)$;
c) $f$ is differentiable at $(0,0)$; $Df(0,0)(\mathbf{h}) = 3h_1 - 2h_2$;
d) $y$ is differentiable at $x = 1$; $Df(1)(\mathbf{h}) = (2h, h)$.

**3.7.** Write down the expressions of the first order differentials of each given function, at the proposed points:

a) $f(x,y) = y^x$, at a generic point $(a,b)$, with $b > 0$;

b) $f(x_1,x_2,x_3) = \dfrac{x_1 - x_2 + x_3}{\sqrt{x_3 - 1}}$, at (1,-3,2).

Note: Admit that the functions are differentiable.

**Solution:** a) $Df(a,b)(\mathbf{h}) = b^a \log b.h_1 + ab^{a-1}.h_2$ b) $Df(1,-3,2)(\mathbf{h}) = h_1 - h_2 - 2h_3$.

**3.8.** Show that the following functions are continuous but not differentiable at the given points:

a) $f(x,y) = \begin{cases} \dfrac{-3x(y-2)^2 + x^3}{x^2 + (y-2)^2}, & \text{if } (x,y) \neq (0,2) \\ 0, & \text{if } (x,y) = (0,2), \end{cases}$, at $(0,2)$.

b) $g(x,y) = \begin{cases} \dfrac{2xy}{\sqrt{x^2+y^2}}, & \text{if } x^2+y^2 \neq 0 \\ 0, & \text{if } x = y = 0, \end{cases}$ at $(0,0)$

c) $h(x,y) = \sqrt{|x|}\cos y$, at $(0,0)$.

**3.9.** Use the chain rule to compute

a) $\dfrac{df}{dt}$, where $f = x^2y^3$, knowing that $x = te^t$ e $y = t^2 + 1$;

b) $\dfrac{df}{dt}$, where $f = u^2 + v^3$, knowing that $u = \dfrac{x}{y}$, $v = (x + 2y)^3$ e $x = \dfrac{1}{t}$, $y = tg\ t$;

c) $\dfrac{dz}{dt}$, knowing that $z = \dfrac{2xy}{x^2 + y^2}$ e $x = \cos t$, $y = \sin t$.

d) $\nabla f(1, 1)$, where $f(x, y) = \sin(2u - v^3 + w)$, knowing that $u = e^{x^2-y}$, $v = xy^2$ e $w = x^3y^2$;

e) $\dfrac{\partial f}{\partial y}(0, 1, 1)$, where $f(x, y, z) = (u^2 - 3v)^5$, knowing that $u = e^{\frac{xy}{z}}$ e $v = \ln(y^2z^3)$;

f) $\nabla f(1, 2, 3)$, where $f(x, y, z) = g(u, v, w)$, with $u = 5x + 3z$, $v = 8x + 2y$, $w = -y + z$ and knowing that $\nabla g(14, 12, 1) = (4, 5, 6)$.

**Solution:**

a) $\frac{df}{dt} = 2te^{2t}(t + 1)(t^2 + 1)^3 + 6t^3e^{2t}(t^2 + 1)^2$;
b) $\frac{df}{dt} = -2\frac{1}{t^3}\frac{1}{tg^2\ t} - 2\frac{1}{t^2}\frac{\sec^2 t}{tg^3\ t} + 9(-\frac{1}{t^2} + 2\sec^2 t)(\frac{1}{t} + 2tg\ t)^8$; c) $\frac{dz}{dt} = 2 - 4\sin^2 t$; d) $\nabla f(1, 1) = (4\cos 2, -6\cos 2)$; e) $\frac{\partial f}{\partial y}(0, 1, 1) = -30$; f) $\nabla f(1, 2, 3) = (60, 4, 18)$.

**3.10.** If a function $f(u, v, w)$ is differentiable at $u = x - y$, $v = y - z$ and $w = z - x$, show that setting $F(x, y, z) = f(x - y, y - z, z - x)$ we have

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} = 0.$$

**3.11.** Consider the function

$$g(x, y) = \begin{cases} \dfrac{(x-1)^2y^2}{(x-1)^2 + y^2}, & (x, y) \neq (1, 0) \\ 0, & (x, y) = (1, 0) \end{cases}$$

a) Determine the partial derivatives $g'_x(x, y)$ and $g'_y(x, y)$, as well as their domain of definition.

b) Show that $g'_x(x, y)$ and $g'_y(x, y)$ are continuous over $\mathbb{R}^2$.

c) Study the differentiability of $f$ at $(1, 0)$.

d) Discuss the continuity of $f$ at $(1, 0)$.

**Solution:**

a) $g'_x(x,y) = \begin{cases} \dfrac{2(x-1)y^4}{((x-1)^2+y^2)^2}, & (x,y) \neq (1,0) \\ 0, & (x,y) = (1,0) \end{cases}$ $\quad g'_y(x,y) = \begin{cases} \dfrac{2(x-1)^4 y}{((x-1)^2+y^2)^2}, & (x,y) \neq (1,0) \\ 0, & (x,y) = (1,0) \end{cases}$

Therefore, $D_{g'_x} = D_{g'_y} = \mathbb{R}^2$. c) $g$ is differentiable at $(1,0)$. d) $g$ continuous at (1,0).

**3.12.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined by

$$f(x,y) = \begin{cases} (x^2+y^2)\sin \dfrac{1}{\sqrt{x^2+y^2}}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

a) Compute $\dfrac{\partial f}{\partial x}(0,0)$ and $\dfrac{\partial f}{\partial y}(0,0)$.

b) Determine $\dfrac{\partial f}{\partial y}(x,y)$ and show that it is discontinuous at $(0,0)$.

c) Check that$f$ is differentiable at $(0,0)$.

d) Compute $\partial_{(\frac{3}{5},\frac{4}{5})} f(0,0)$.

e)Discuss the continuity of $f$ at $(0,0)$.

**Solution:**

a) $f'_x(0,0) = f'_y(0,0) = 0$;

b)$f'_y(x,y) = \begin{cases} 2y \sin \frac{1}{\sqrt{x^2+y^2}} - y \frac{1}{\sqrt{x^2+y^2}} \cos \frac{1}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$ ; d) 0; e) $f$ is continuous at $(0,0)$.

**3.13.** Use the function

$$g(x,y) = \begin{cases} \dfrac{\sin x}{y}, & y \neq 0 \\ 0, & y = 0 \end{cases}$$

and the point $(0,0)$ to show that a function with finite partial derivatives at a given point is not necessarily continuous at that point. Is the given function differentiable at $(0,0)$? Why?

**Solution:**

The function is not continuous at $(0,0)$, and so it is also not differentiable.

**3.14.** Considerer the function $f:\mathbb{R}^2\to\mathbb{R}$ defined by

$$f(x,y)=\begin{cases}\dfrac{xy}{|x|+|y|}, & \text{if } (x,y)\neq(0,0)\\ 0, & \text{if } (x,y)=(0,0).\end{cases}$$

a) Show that $f$ is continuous at $(0,0)$.

b) Determine $\dfrac{\partial f}{\partial x}(0,0)$ and $\dfrac{\partial f}{\partial y}(0,0)$.

c) Show that $f$ is not differentiable at (0,0). Without performing any calculations, what can you conclude about the continuity of $\dfrac{\partial f}{\partial x}$ e $\dfrac{\partial f}{\partial y}$ at (0,0)?

**Solution:**
b) $f'_x(0,0)=f'_y(0,0)=0$ c) Since $f$ is not differentiable at $(0,0)$, at least one of the functions $f'_x$ or $f'_y$ is not continuous at $(0,0)$.

**3.15.** Let $f:\mathbb{R}^2\to\mathbb{R}$ be the function defined by

$$f(x,y)=\begin{cases}\dfrac{x(x-y)}{x+y} & \text{if } x+y\neq 0\\ 0 & \text{if } x+y=0\end{cases}$$

a) Study the continuity of $f$ at $(0,0)$

b) Compute the partial derivative $\dfrac{\partial f}{\partial x}$ and discuss its continuity at $(0,0)$.

c) Study the differentiability of $f$ at $(0,0)$.

d) Show that $\left(\dfrac{\partial f}{\partial x}(0,0),\dfrac{\partial f}{\partial y}(0,0)\right)\cdot\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)\neq\delta_{\vec{(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2})}}f(0,0)$. Comment on the result.

**Solution:**

a) $f$ is not continuous at (0,0).
b) $f'_x(x,y)=\frac{x^2+2xy-y^2}{(x+y)^2}$ if $x+y\neq 0$ and $f'_x(x,y)=1$ if $(x,y)=(0,0)$ (it does not exist $f'_x(a,-a), a\neq 0$); $f'_x(x,y)$ is not continuous at $(0,0)$.
c) $f$ is not differentiable at $(0,0)$.
d) The two values onlym had to be equal if $f$ was differentiable at $(0,0)$.

**3.16.** Compute $\dfrac{\partial^2 f}{\partial x^2}$ and $\dfrac{\partial^4 f}{\partial x^2 \partial z \partial y}$, for $f(x, y, z) = z^2x^2y + xye^z$.

**Solution:** $\dfrac{\partial^2 f}{\partial x^2} = 2yz^2$ , $\dfrac{\partial^4 f}{\partial x^2 \partial z \partial y} = 4z$.

**3.17.** Compute $f''_{x^2}$, $f''_{xy}$ and $f'''_{xyx}$ for each of the following functions, indicating the corresponding domain of definition:

$$a)\ f(x, y) = x \sin(x + y); \qquad b)\ f(x, y) = \begin{cases} y \sin x, & y \neq 0 \\ 2, & y = 0 \end{cases}$$

**Solution:**

a) $f''_{x^2} = 2\cos(x + y) - x\sin(x + y)$, $f''_{xy} = \cos(x + y) - x\sin(x + y)$ and $f'''_{xyx} = -2\sin(x + y) - x\cos(x + y)$.
b) $f''_{x^2} = -y \sin x$, $f''_{xy} = \cos x$ and $f'''_{xyx} = -\sin x$;

**3.18.** Compute the differential of order 2, 3 and 4 of the function $f(x, y) = \sqrt{xy}$ at $(1, 1)$.

**Solution:**
$D^2 f(1, 1)(\mathbf{h}^2) = -\frac{1}{4}h_1^2 + \frac{1}{2}h_1h_2 - \frac{1}{4}h_2^2$,
$D^3 f(1, 1)(\mathbf{h}^3) = \frac{3}{8}h_1^3 - \frac{3}{8}h_1^2h_2 - \frac{3}{8}h_1h_2^2 + \frac{3}{8}h_2^3$,
$D^4 f(1, 1)(\mathbf{h}^4) = -\frac{15}{16}h_1^4 + 4\frac{3}{16}h_1^3h_2 + 6\frac{1}{16}h_1^2h_2^2 + 4\frac{3}{16}h_1h_2^3 - \frac{15}{16}h_2^4$.

**3.19.** Determine the differential of order $n$ of the function $f(x, y) = \sin(x + y)$, at the point (0,0).

**Solution:** $D^n f(0, 0)(\mathbf{h}) = \sin(n\pi/2) \displaystyle\sum_{i=0}^{n} \tbinom{n}{i} h_1^i h_2^{n-i}$

**3.20.** Show that $f(x, y) = log(e^x + e^y)$ satisfies the (differential) equation

$$\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 0$$

everywhere in $\mathbb{R}^2$.

**3.21.** Let $f \in C^2(\mathbb{R}^2)$ be a real function such that $\dfrac{\partial f}{\partial u}(0,0) = \dfrac{\partial f}{\partial v}(0,0) = 1$. Also, let $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by

$$g(x, y) = f(y\text{sen } x, y^2).$$

Show that the Hessian matrix of $g$ at $(0,0)$ is given by $\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$.

**3.22.** Consider $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by

$$f(x, y) = xy^2 + g(u, v, w), \text{ with } u = \text{sen } y^2, \ v = \ln x \text{ and } w = ye^x.$$

Assuming that $g$ is of class $C^2(\mathbb{R}^3)$, compute $\dfrac{\partial^2 f}{\partial y \partial x}(1, 0)$.

**Solution:** $\frac{\partial^2 f}{\partial y \partial x}(1, 0) = e\left(\frac{\partial^2 g}{\partial w \partial v}(0, 0, 0) + \frac{\partial g}{\partial w}(0, 0, 0)\right)$.

**3.23.** Show that the following functions are homogeneous or positively homogeneous. Determine in each case the degree of homogeneity and verify Euler's identity.

a) $f(x, y) = \log \dfrac{(x + y)^2}{xy}$ b) $f(x, y, z) = \dfrac{\sqrt{x^2 + y^2}}{z^2}$ c) $f(x, y) = \begin{cases} (x + y) \sin\left(\dfrac{xy}{x^2 + y^2}\right), & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

**Solution:**

a) $f$ is homogeneous with degree 0;
b) $f$ is positively homogeneous with degree $-1$;
c) $f$ is homogeneous with degree 1.

**3.24.** Study the function $g(x, y, z) = x^2 + x^\alpha y^{\beta-3} - z^{3\alpha} y^\beta$ e de $h(x, y) = \dfrac{x^3 y^\alpha + x^{\beta-1}}{y^{3-\beta}}$, with respect to its homogeneity in terms of the parameters $\alpha, \beta \in \mathbb{R}$,

a) Using the definition.

b) Using Euler's identity.

**Solution:**

$g$ is homogeneous with degree 2 for $\alpha = -\frac{3}{2}$ and $\beta = \frac{13}{2}$;
$h$ is homogeneous with degree $\alpha + \beta$ for $\beta = \alpha + 4, \alpha \in \mathbb{R}$.

**3.25.** Assuming that $g(u, v)$ is differentiable $\left(\frac{x}{y}, \frac{z}{x}\right)$, with $x, y \neq 0$, show that

$$f(x, y, z) = x^2.g\left(\frac{x}{y}, \frac{z}{x}\right),$$

satisfies the identity $x\, f'_x + y\, f'_y + z\, f'_z = 2.f$. Interpret this results in terms of homogeneity.

**Solution:** $f$ is positively homogeneous with degree 2.

**3.26.** Let $f(\mathbf{x}) : \mathbb{R}^n \backslash \{\mathbf{0}\} \to \mathbb{R}$ be an homogeneous, non constant function with degree 0. Show that $\lim_{\mathbf{x}\to\mathbf{0}} f(\mathbf{x})$ does not exist.

**3.27.** Consider the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by

$$f(x, y) = \ln\left(\frac{xy}{x + y}\right).$$

Write down Taylor's formula with degree 2, around (1,1).

**Solution:** $\ln \dfrac{(1+h)(1+k)}{2+h+k} \;=\; -\ln 2 + \frac{1}{2}h + \frac{1}{2}k + \frac{1}{2}\left(-\frac{3}{4}h^2 + \frac{1}{2}hk - \frac{3}{4}k^2\right) + r_3(h, k).$

# 4 Optimization Problems

**4.1.** Determine and classify the critical points of the following functions from $\mathbb{R}^2$ to $\mathbb{R}$.

| | | | |
|---|---|---|---|
| $a)x^2 + y^2$ | $b)x^2 - y^2$ | $c)x^3 + y^3$ | $d)x^3 - y^3$ |
| $e)x^4 + y^4$ | $f)x^4 - y^4$ | $g)3xy - x^3 - y^3$ | $h)x \ln x + y \ln y$ |
| $i)x^3 + ye^y$ | $j)2x^3 + xy^2 + 5x^2 + y^2$ | $k)x^4 + y^4 - 4xy + 1$ | $l)x^2y^2$ |

**Solution:** a) $(0,0)$ is a minimum point; b) c) d) $(0,0)$ is a saddle point ; e) $(0,0)$is a minimum point; f) $(0,0)$ is a saddle point; g) $(0,0)$ is a saddle point and $(1,1)$ is a maximum point; h) $(1/e,1/e)$ is a minimum point; i) $(0,-1)$ is a saddle point; j) $(0,0)$ is a minimum point, $(-5/3,0)$ éis a maximum point, $(-1,2)$ e $(-1,-2)$ are saddle points; k) $(0,0)$ is a saddle point, $(1,1)$ e $(-1,-1)$ are minimum points; l) $(0,b)$ e $(a,0)$ $\forall a,b\in\mathbb{R}$, are minimum points;

**4.2.** Determine and classify the critical points of the following functions, in terms of the parameter $a\in\mathbb{R}\setminus\{0\}$

$$a)f(x,y)=e^{x^2-ay^2} \qquad b)f(x,y)=ax^2-y^2$$

$$c)f(x,y)=x^3-ax^2-3y^2 \qquad d)f(x,y)=\frac{16}{5}x^5+ay^2-x$$

**Solution:** a) Critical point: $(0,0)$. if $a<0$, minimum point; if $a>0$, saddle point. b) Critical point: $(0,0)$. If $a>0$, $(0,0)$ is a saddle point; if $a<0$, $(0,0)$ is a maximum point. c) Critical points: $(0,0)$ and $(\frac{2a}{3},0)$. If $a>0$, $(0,0)$ is a maximum point and $(\frac{2a}{3},0)$ is a saddle point; if $a<0$, $(0,0)$ is a saddle point and $(\frac{2a}{3},0)$ is a maximum point. d) Critical points: $(-\frac{1}{2},0)$ and $(\frac{1}{2},0)$. if $a<0$, $(-\frac{1}{2},0)$ is a maximum point and $(\frac{1}{2},0)$ is a saddle point; if $a>0$, $(-\frac{1}{2},0)$ is a saddle point and $(\frac{1}{2},0)$ is a minimum point.

**4.3.** Consider the function $f(x,y)=(y-\alpha)\,xe^x$.

a) Knowing that $(0,1)$ is a critical point, determine $\alpha$ and classify this critical point.

b) Show that $f$ is unbounded.

**Solution:** a) $\alpha=1$. The critical point is a saddle point.

**4.4.** Let function $f:\mathbb{R}^2\longrightarrow\mathbb{R}$ be defined by $f(x,y)=4\alpha(y-2)^2+(\beta^2-1)(2x-2)^2$, where $\alpha\neq 0$, $\beta\neq 1$, $\beta\neq -1$. Show thate $(1,2)$ is the only critical point and classify it in terms of all possible values of $\alpha$ and $\beta$.

**Solution:** If $|\beta|<1$ and $\alpha<0$ then $(1,2)$ is a local maximum; if $|\beta|>1$ and $\alpha>0$ then $(1,2)$ is a local minimum; in all other cases it is a saddle point.

**4.5.** Let $f:\mathbb{R}^2\longrightarrow\mathbb{R}$ be defined by $f(x,y)=x^2e^{y^3-3y}$.

a) Determine all critical points of function $f$.

b) Show that $f$ attains its global minimum at points of the form $(0, b)$.

c) Justify that

 $(i)$ $f$ is unbounded over $\mathbb{R}^2$;

 $(ii)$ $f$ has a maximum and minimum over $B = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 9\}$.

**Solution:** a) Critical points: $(0, b)$ with $b \in \mathbb{R}$.

**4.6.** Determine the global extrema of $f$ over the set $M$, where

$$
\begin{aligned}
&a) f(x,y,z) = x - 2y + 2z, && M = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}\\
&b) f(x,y) = 4x^2 + y^2, && M = \{(x,y) \in \mathbb{R}^2 : 2x^2 + y^2 = 1\}\\
&c) f(x,y) = xy, && M = \left\{(x,y) \in \mathbb{R}^2 : \tfrac{x^2}{8} + \tfrac{y^2}{2} = 1\right\}\\
&d) f(x,y,z) = x^2 + 2y - 2z, && M = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 8\}\\
&e) f(x,y) = x^2 + 2xy + y^2, && M = \{(x,y) \in \mathbb{R}^2 : (x-3)^2 + y^2 = 2\}\\
&f) f(x,y,z) = 2x + 2y^2 + z^2, && M = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 2\}\\
&g) f(x,y,z) = e^{-x^2-y^2}, && M = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}\\
&h) f(x,y) = 4xy - 2x^2 - 2y^2, && M = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}\\
&i) f(x,y) = x^2 + 2xy + y^2, && M = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 8\}
\end{aligned}
$$

**Solution:** a) max. = 3, min.=-3; b) max.= 2, min. = 1; c) max. = 2, min. = $-2$; d) max. =10, min. = $-8$; e) max. = 25, min. = 1; f) max. = $9/2$, min. = $-2\sqrt{2}$. g) max. = 1, min. = $1/e$; h) max. = 0, min. = $-4$; i) max. = 16, min.= 0

**4.7.** Determine the global extrema of $f$ over the set $A$, where

$a) f(x,y,z) = x - 2y + 2z$, $A = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$
(note: compare with a) from previous exercise)

$b) f(x,y) = 4x^2 + y^2$, $A = \{(x,y) \in \mathbb{R}^2 : 2x^2 + y^2 \leq 1\}$
(note: compare with b) from previous exercise)

$c) f(x,y) = x^2 + 2xy + y^2$, $A = \{(x,y) \in \mathbb{R}^2 : (x-3)^2 + y^2 \leq 2\}$
(note: compare with e) from previous exercise)

**Solution:** a) max. = 3, min. = −3; b) max. = 2, min. = é 0; c) max. = 25, min. = 1.

**4.8.** Determine the maximum and minimum distance to the origin of the points in the ellipse $5x^2 + 6xy + 5y^2 = 8$.

**Solution:** The maximum distance is 2 and the minimum distance is 1.

**4.9.** Solve the optimization problem $\min(x + 4y + 3z)$ subject to the condition $x^2 + 2y^2 + \frac{1}{3}z^2 = b$, $(b > 0)$.

**Solution:** The minimum value is $-6\sqrt{b}$, attained at $(-\frac{\sqrt{b}}{6}, -\frac{\sqrt{b}}{3}, -\frac{3\sqrt{b}}{2})$.

**4.10.** Determine the point in the ellipse $x^2 + 2xy + 2y^2 = 2$ with smallest x coordinate.

**Solution:** $(-2, 1)$.

**4.11.** determine teh global extrema of $f(x,y) = e^{x^2+y^2+z^2}$ over the set $\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 = 1,\ z = 2 - y\}$.

**Solution:** The maximum value is $e^{10}$, attained at $(0,-1,3)$; the minimum is $e^2$, attained at $(0,1,1)$.

# 5 Multiple Integrals

**5.1.** Compute the following integrals.

a) $\int_0^1 \int_{-1}^1 \int_1^2 x\, dxdydz$  b) $\int_{-1}^1 \int_0^1 ye^{xy} dxdy$  c) $\int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} e^{-x-y-z} dxdydz$

d) $\int_0^5 \int_0^{+\infty} (x^2e^{-2yx} + 3ye^{-y^2})dydx$  e) $\int_1^2 \int_1^2 (1 + x + \frac{y}{2})dxdy$  f) $\int_0^1 \int_0^1 \int_0^{\frac{\pi}{2}} y \cos x\, dxdydz.$

**Solution:** a) 3  b) $e - \frac{1}{e} - 2$  c) 1  d) $\frac{55}{4}$  e) $\frac{13}{4}$  f) $\frac{1}{2}$

**5.2.** Compute $\iint_A f(x, y)dxdy$, where

$a) f(x, y) = \frac{2x}{y^6}$,  $A = \{(x, y) \in \mathbb{R}^2 : 1 \le y \le 3,\ 0 \le x \le y^4\}$

$b) f(x, y) = e^{\frac{y}{x}}$,  $A = \{(x, y) \in \mathbb{R}^2 : 1 \le x \le 2,\ x \le y \le x^3\}$

$c) f(x, y) = x^2y^5$,  $A = \{(x, y) \in \mathbb{R}^2 : 0 \le x \le 1,\ x^2 \le y \le 1\}$

$d) f(x, y) = ye^x + x^2y$,  $A = \{(x, y) \in \mathbb{R}^2 : 0 \le y \le 1,\ 0 \le x \le y^2\}$

$e) f(x, y) = xe^{-xy}$,  $A = \{(x, y) \in \mathbb{R}^2 : 1 \le x \le 2,\ \frac{1}{x} \le y < +\infty\}$

$f) f(x, y) = x^3 + 4y$,  $A$ is the region bounded by the lines $y = x^2$ and $y = 2x$.

**Solution:** a) $\frac{26}{3}$  b) $\frac{e^4}{2} - 2e$  c) $\frac{2}{45}$  d) $\frac{e}{2} - \frac{23}{24}$  e) $\frac{1}{e}$  f) $\frac{32}{3}$

**5.3.** Let $A = \{(x, y) \in \mathbb{R}^2 : y \le 1 - x,\ y \le 1 + x,\ y \ge 0\}$, and compute

a) $\iint_A (x - 1)y\, dxdy$  b) $\iint_A (y - 2y^2)e^{xy}\, dxdy$  c) $\iint_A (x + y)\, dxdy.$

**Solution:** a) $-\frac{1}{3}$  b) 0  c) $\frac{1}{3}$

**5.4.** Compute $\int_{-1}^1 \int_{-1}^1 f(x, y)dxdy$, with $f(x, y) = \begin{cases} xy & x \ge 0 \text{ e } y \ge 0 \\ 1 - x - y, & \text{other } (x, y) \end{cases}$.

**Solution:** $\frac{17}{4}$

**5.5.** Using a double integral compute the area of set $A$, where

a) $A = \{(x, y) \in \mathbb{R}^2 : -1 \leq x \leq 1 \ \wedge \ x^2 \leq y \leq 1\}$;

b) $A = \{(x, y) \in \mathbb{R}^2 : 1 - x \leq y \leq 1 \ \wedge \ x \leq 1\}$;

c) $A = \{(x, y) \in \mathbb{R}^2 : x^2 \leq y \leq 2 - x^2\}$;

d) $A = \{(x, y) \in \mathbb{R}^2 : x + y + 2 \geq 0 \ \wedge \ x + y^2 \leq 0\}$;

e) $A = \{(x, y) \in \mathbb{R}^2 : y \leq x^2 \ \wedge \ y - x \geq 0 \ \wedge \ 2y - x \leq 3 \ \wedge \ x \geq 0\}$;

f) $A = \{(x, y) \in \mathbb{R}^2 : x \geq 0 \ \wedge \ x^2 + y^2 \leq 1\}$;

g) $A = \{(x, y) \in \mathbb{R}^2 : y^2 + x \leq 2 \ \wedge \ x - y \geq 0\}$;

h) $A = \left\{(x, y) \in \mathbb{R}^2 : y^2 \leq x \ \wedge \ x \leq \frac{y^2}{4} + 3\right\}$;

**Solution:** a) $\frac{4}{3}$ b) $\frac{1}{2}$ c) $\frac{8}{3}$ d) $\frac{9}{2}$ e) $\frac{35}{48}$ f) $\frac{\pi}{2}$ g) $\frac{9}{2}$ h) 8

**5.6.** Compute $\int_0^4 \int_{2x}^8 \sin(y^2) dy dx$ (suggestion: Draw the integration region and invert the integration order)

**Solution:** $\frac{1 - \cos 64}{4}$

**5.7.** Compute $\int \int_A g(x, y) dx dy$, where $A = [0, 2] \times [0, 4]$ e $g(x, y) = \begin{cases} x - y, & x \leq y \leq 2x \\ 0, & \text{otherwise} \end{cases}$.

**Solution:** $-\frac{4}{3}$

# 6 Differential equations

**6.1.** Determine the general solution of the following differential equations with separable variables.

$a) y' = xy - x$ | $b)\ dx\, e^y = dy(x+1) - dx$ | $c) \frac{dy}{dx} + \frac{1+y^3}{xy^2(1+x^2)} = 0$

$d) \sqrt{1-x^2}dy - \sqrt{1-y^2}dx = 0$ | $e) e^{x^4} yy' = x^3\left(9+y^4\right)$ | $f) e^y(4+x^2)y' = x\left(2+e^y\right)$

$g) e^{3x}dy + \left(4+y^2\right)dx = 0$ | $h) 4xe^y dx + \left(x^4+4\right)dy = 0$

**Solution:** a) $y(x) = 1 + e^{\frac{1}{2}x^2}C$ b) $\ln\left(e^{y(x)}+1\right) - y(x) + \ln(x+1) = C$ c) $\frac{1}{3}\ln|1+y^3| + \ln|x| - \frac{1}{2}\ln(1+x^2) = C$ d) $\arcsin(y(x)) - \arcsin x = C$ e) $\frac{1}{6}\arctan\left(\frac{1}{3}y^2(x)\right) + \frac{1}{4}e^{-x^4} = C$ f) $\ln\left(2+e^{y(x)}\right) - \frac{1}{2}\ln\left(4+x^2\right) = C$ g) $\frac{1}{2}\arctan\left(\frac{1}{2}y(x)\right) - \frac{1}{3}e^{-3x} = C$ h) $-e^{-y(x)} + \arctan\frac{1}{2}x^2 = C$

**6.2.** Solve the following initial value problems.

a) $y' + 4y = 0, \quad y(0) = 6$ | b) $\dfrac{dy}{dt} + y\sin t = 0, \quad y(\pi/3) = 3/2$

c) $(1+x^2)y' + y = 0, \quad y(1) = 1$ | d) $2y' + 4xy = 4x, \quad y(0) = -2$

e) $y' + y\sin x = \sin x\cos x, \quad y(\frac{\pi}{2}) = 0$

**Solution:** a) $y(x) = 6e^{-4x}$ b) $y(t) = \frac{3}{2}e^{\cos t - \frac{1}{2}}$ c) $y(x) = e^{\frac{\pi}{4} - \arctan(x)}$ d) $y(x) = 1 - 3e^{-x^2}$ e) $y(x) = \cos x + 1 - e^{\cos x}$.

**6.3.** show that any homogeneous first order linear differential equation can be written as a

**6.4.** Determine de general solution of the following differential equations.

$a) y'' - 7y' + 12y = 0$ | $b) y'' + 4y = 0$ | $c) y'' - 4y' + 4y = 0$

$d) y'' + 2y' + 10y = 0$ | $e) y'' + y' - 6y = 8$ | $f) y'' + 3y' + 2y = e^{5x}$

$g) y'' - y = \sin x$ | $h) y'' - y = e^{-x}$ | $i) y'' - 6y = 36(x-1)$

$j) y'' - 9y = 9x^2$ | $k) y'' + 3y' + 2y = \sin x$ | $l) y'' + 3y' + 2y = e^{-x}$

$m) y'' - 4y' + 4y = 6e^{2x}$

**Solution:** a) $y(x) = C_1e^{3x} + C_2e^{4x}$ b) $y(x) = C_1 \cos 2x + C_2 \sin 2x$ c) $y(x) = (C_1 + C_2x)e^{2x}$ d) $y(x) = (C_1 \cos 3x + C_2 \sin 3x)e^{-x}$ e) $y(x) = C_1e^{-3x} + C_2e^{2x} - \frac{4}{3}$ f) $y(x) = C_1e^{-2x} + C_2e^{-x} + \frac{1}{42}e^{5x}$ g) $y(x) = C_1e^{-x} + C_2e^{x} - \frac{1}{2}\sin x$ h) $y(x) = C_1e^{x} + C_2e^{-x} - \frac{1}{2}xe^{-x}$ i) $y(x) = C_1e^{\sqrt{6}x} + C_2e^{-\sqrt{6}x} - 6x + 6$ j) $y(x) = C_1e^{3x} + C_2e^{-3x} - x^2 - \frac{2}{9}$ k) $y(x) = C_1e^{-2x} + C_2e^{-x} - \frac{3}{10}\cos x + \frac{1}{10}\sin x$ l) $y(x) = C_1e^{-2x} + C_2e^{-x} + xe^{-x}$ m) $y(x) = C_1e^{2x} + C_2xe^{2x} + 3x^2e^{2x}$.

**6.5.** Solve the following initial value problems.

*a)* $\begin{cases} y'' + y' - 2y = 0 \\ y(0) = -1, y'(0) = 1 \end{cases}$ *b)* $\begin{cases} y'' + 2y' + 5y = 0 \\ y(0) = 0, y'(0) = 1 \end{cases}$ *c)* $\begin{cases} y'' + 2y' + y = x^2 \\ y\,(0) = 0,\ y'\,(0) = 1 \end{cases}$

*d)* $\begin{cases} y'' + 4y = 4x + 1 \\ y(\frac{\pi}{2}) = 0,\ y'(\frac{\pi}{2}) = 0 \end{cases}$ *e)* $\begin{cases} 9y'' + y = 0 \\ y\left(\frac{3}{2}\pi\right) = 2,\ y'\left(\frac{3}{2}\pi\right) = 0 \end{cases}$ *f)* $\begin{cases} 2y'' - 4y' + 2y = 0 \\ y(0) = -1, y'(0) = 1 \end{cases}$

*g)* $\begin{cases} y'' - 2y' + 10y = 10x^2 \\ y(0) = 0, y'(0) = 3 \end{cases}$

**Solution:** a) $y\,(x) = -\frac{2}{3}e^{-2x} - \frac{1}{3}e^{x}$ b) $y\,(x) = \frac{1}{2}\sin(2x)e^{-x}$ c) $y(x) = -(6 + x)e^{-x} + x^2 - 4x + 6$ d) $y(x) = \frac{2\pi+1}{4}\cos 2x + \frac{1}{2}\sin 2x + x + \frac{1}{4}$ e) $y\,(x) = 2\sin(\frac{x}{3})$ f) $y\,(x) = (-1 + 2x)e^{x}$ g) $y(x) = e^{x}\left(\frac{3}{25}\cos 3x + \frac{62}{75}\sin 3x\right) + x^2 + \frac{2}{5}x - \frac{3}{25}$.

**6.6.** Solve the following boundary value problems.

*a)* $\begin{cases} y'' - 6y' + 9y = e^{3x} \\ y\,(0) = 0,\ y\,(1) = 0 \end{cases}$ *b)* $\begin{cases} y'' + 4y = 0 \\ y\,(0) = 0,\ y\,(\pi) = 0 \end{cases}$ *c)* $\begin{cases} y'' - 10y' + 25y = 50 \\ y\,(0) = 0,\ y\,(2) = 2. \end{cases}$

*d)* $\begin{cases} y'' - 8y' + 16y = 0 \\ y\,(0) = 1,\ y\,(1) = e^4. \end{cases}$

**Solution:** a) $y(x) = (-\frac{1}{2}x + \frac{1}{2}x^2)e^{3x}$ b) $y(x) = C\sin 2x$ c) $y(x) = (-2 + x)e^{5x} + 2$ d) $y(x) = e^{4x}$.

**6.7.** Knowing that $y = e^{2x}$ is a solution of the differential equation

$$y'' - \alpha y' + 10y = 0, \quad \alpha \in \mathbb{R},$$

determine $\alpha$ and the general equation of this equation.

**Solution:** $\alpha = 7$; $y(x) = C_1e^{2x} + C_2e^{5x}$.

**6.8.** Knowing that $y\,(x) = xe^{2x}$ is a solution of the differential equation $2y'' - \alpha y' + 8y = 0$, with $\alpha \in \mathbb{R}$, solve the boundary value problem $\left\{ \begin{array}{l} 2y'' - \alpha y' + 8y = 16 \\ y\,(0) = 1;\ y\,(1) = 2 \end{array} \right.$.

**Solution:** $y(x) = -e^{2x} + xe^{2x} + 2$.

**6.9.** solve the following problem with periodic conditions.

$$\left\{ \begin{array}{l} y'' + 4y = 4, \\ y\,(0) = y\left(\frac{\pi}{2}\right), y'\,(0) = y'\left(\frac{\pi}{2}\right) \end{array} \right. .$$

**Solution:** $y(x) = 1$.

**6.10.** Determine the general solution of the following differential equations.

$a)y' + y^2 \sin x = 0$ $\quad b)yy' + x = 0$ $\quad c)y'' - 2y' = 0$

$d)y'y - x(2y^2+1)e^{x^2} = 0$ $\quad e)\frac{dy}{dx}\cos y = -x\frac{\sin y}{1+x^2}$ $\quad f)y' + 6yx^5 - x^5 = 0$

**Solution:** a) $y^{-1}(x) = -\cos x + C$ b) $y^2(x) = -x^2 + C$ c) $y(x) = C_1 + C_2e^{2x}$ d) $\frac{1}{4}\ln(2y^2+1) - \frac{1}{2}e^{x^2} = C$ e) $\ln|\sin y| + \frac{1}{2}\ln(1+x^2) = C$ f) $y(x) = \frac{1}{6} + Ce^{-x^6}$.

**6.11.** Determine the values of $a$ and $b$ for which $e^{2x}$ and $e^{-2x}$ are solutions to the differential equation $y'' + ay' + by = 0$. For those vales of $a$ and $b$, compute the general solution to the differential equation.

**Solution:** $a = 0$ e $b = -4$; $y_h(x) = Ae^{2x} + Be^{-2x}, A, B \in \mathbb{R}$.

**6.12.** [Malthus populational growth]

(a) Compute the time evolution of a population level $y(t)$ knowing that: i) at each time $t$, the growth rate of the population , $\frac{dy/y}{dt}$, is equal to $r$ (with $r > 0$); ii) at time $t = 0$ there are $y_0$ (millions of individuals).

(b) Estimate the value of the Portuguese population in the year 2020, knowing that in the year 2013 (by hypothesis $t = 0$) there is record of $y = 10.457$ (million individuals) and that $r = -0.00476$.

(c) Comment on the Malthusian hypothesis, by studying $\lim_{t\to\infty} y(t)$.

**Solution:**

a) $y(t) = y_0 e^{rt}$; b) $y(8) \approx 10.0663$; c) $+\infty$ if $r > 0$, $y_0$ if $r = 0$ and 0 if $r < 0$.

**6.13.** [Populational growth according to Verhulst]

(a) Compute the time evolution of a population level $y(t)$ knowing that: i) at each time $t$, the rate of growth of the population, $\frac{dy/y}{dt}$, is equal to $r$ minus $ay$ ($r$: natural growth rate; $a$: migratory or death rate; with $r > 0$ e $a > 0$); ii) at time $t = 0$ there is a record of $y_0$ million individuals.

(b) b)Estimate the value of the Portuguese population in the year 2020, knowing that in the year 2013 (by hypothesis $t = 0$) there is a record of $y = 10.457$ million individuals and that $r = -0.00229$, $a = 0.00033$.

c) Study $\lim_{t\to\infty} y(t)$.

**6.14.** [Domar's growth model]

Consider an economical model where i) the aggregated demand $y_d$, varies on time according to the equation $\frac{dy_d}{dt} = \frac{dI}{dt}\frac{1}{s}$, where $I = I(t)$ is the investment and $s$ the marginal propensity to saving ($1/s$ is Keynesian multiplier); ii) the productive capacity ($y_c = \rho K$ - note: the productive capacity depends only on the stock of capital) follows the differential equation $\frac{dy_c}{dt} = \rho\frac{dK}{dt}$, where $K = K(t)$ is the *stock* of capital of the economy (naturally, $\frac{dK}{dt} = I(t)$). Obtain the time trajectory of the investment, $I(t)$, satisfying the equilibrium of Domar's model - the variation of the aggregated demand = variation of the productive capacity.

**6.15.** Consider the demand and supply functions of a given commodity, $Q_d = a - bP$; $Q_s = -c + dP$.

(a) Determine the time evolution of the price level $P(t)$ knowing that a each time $t$, the variation rate of $P(t)$ is proportional to the excess demand, i.e. $\frac{dP}{dt} = \alpha(Q_d - Q_s)$, and that $P(0) = P_0 \neq P_e = (a + c)/(b + d)$ (equilibrium price).

(b) Verify under which conditions we have $\lim_{t\to\infty} P(t) = P_e$.