**Lisbon University**
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Ms in Economics, Mathematical Finance and
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## Mathematical Economics – 1st Semester - 2023/2024

Exercises - Group I

1. (DeMorgan's laws) Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$ and let $A, B, C \subset \Omega$. Show that:
   (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   (b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
   (c) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$

2. Sketch the following sets:
   (a) $A_1 = \{x \in \mathbb{R} \colon x^2 > 1\}$ and $A_2 = \{(x, y) \in \mathbb{R}^2 \colon x^2 > 1\}$
   (b) $B = \{x \in \mathbb{R} \colon x^3 > 1 \text{ and } x^4 < 16\}$
   (c) $C = \{x \in \mathbf{Z} \colon x^2 < 2 \text{ and } x \text{ is even}\}$
   (d) $B \cup C$ and $(A_1)^c$
   (e) $A_1 \times B$
   (f) $D = \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 \leq 4\}$
   (g) $E = \{(x, y) \in \mathbb{R}^2 \colon x^2 \leq y \text{ and } 1 \geq y + x^2\}$

3. Consider the following real values maps defined by $f(x) = \sqrt{x - 1}$ and $g(x) = x^2 - 4$.
   (a) Compute $(f - g)(1)$, $(fg)(0)$ and $f(3)/g(2)$ if they exist.
   (b) Define the maps $f \circ g$ and $g \circ f$ (including their domains).
   (c) Do $f$ and $g$ commute?
   (d) Define the inverse of $f$ and $g$ (in a suitable domain).

4. Sketch the graph of the following real valued functions $f : D \to \mathbb{R}$, determine if are injective/surjective and compute its inverse (if it exists):
   (a) $f(x) = x^3 + 1$ with $x \in \mathbb{R}$
   (b) $f(x) = x^2 - x$ with $x \in \mathbb{R}$
   (c) $f(x) = \sqrt{x + 1}$ with $x > 0$.
   (d) $f(x) = \frac{x-1}{x+1}$ with $x > -1$
   (e) $f(x) = 2e^{-x}$ with $x \in \mathbb{R}$
   (f) $f(x) = \log(x^2 + 1)$ with $x > 0$

5. (a) Find a bijective function mapping $\mathbb{N}$ to $\mathbf{Z}$.
   (b) Find a bijective function mapping $\mathbb{R}$ to $]-10, 10[$.

6. Show that the following functions $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ satisfy the four axioms of a metric:

   (a) $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$
   (b) $d(x, y) = |x_1 - y_1| + |x_2 - y_2|$
   (c) $d(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$

   For each of the metrics above sketch the set

$$\{x \in \mathbb{R}^2 \colon d(x, 0) = 1\}.$$

7. Consider the open intervals $A_n = ]-1/n, 1/n[$ with $n \in \mathbb{N}$. Show that

$$\bigcap_{n \in \mathbb{N}} A_n = \{0\}.$$

   Conclude that the intersection of a countable set of open sets might be closed.

8. Consider the following sequence in $\mathbb{R}^3$:

$$u_n = \left( \left( \frac{n+3}{n} \right)^{2n} ; \frac{n^2(-1)^{2n+1}}{5n^2+1} ; \frac{4n^5 + 5n^3}{8n^5 + 7n^2} \right), \quad n \in \mathbb{N}$$

   Does $(u_n)_n$ converge?

9. The (maximal) domain of $f : D_f \to \mathbb{R}$ is the set

$$D_f = \{(x, y) \in \mathbb{R}^2 : y > (x-1)^2 \wedge (x, y) \neq (1, 5)\}.$$

   (a) Define the closure and the set of accumulation points of $D_f$.
   (b) Is $D_f$ an open set? Why?
   (c) Given one possible analytical expression for $f$.

10. Which is the set of accumulation points of $\left\{ \left(\cos(n\pi), \left(\frac{1}{2}\right)^n\right), n \in \mathbb{N} \right\} \subset \mathbb{R}^2$ ?

11. Consider the map $f : D_f \to \mathbb{R}$, $D_f \subset \mathbb{R}^2$, given by:

$$f(x, y) = \frac{\ln(4 - (x-1)^2 - y^2)}{\sqrt{y - x}}$$

   (a) Define analytically the domain of $f$, $D_f$ and represent it in the plane $(x, y)$
   (b) Define analytically $int(D_f)$, $cl(D_f)$ and $\partial(D_f)$.
   (c) Is $D_f$ compact?

12. Consider $A = \{(x, y) \in \mathbb{R}^2 : x^2 + (y+2)^2 < 4 \wedge xy \neq 0\}$.

(a) Is $A$ a compact set?

(b) Define analytically its topological border.

13. Define the topological interior, closure and accumulation points of the following sets:

    (a) $A = \{(1,1),(1,0)\}$

    (b) $B = \left\{(x,y) \in \mathbb{R}^2 : x \geq 0 \wedge y = \frac{(-1)^n}{n}, n \in \mathbb{N}\right\}$

14. Sketch the following sets and decide which are open, closed, bounded and compact.

    (a) $A = \{(x,y) \in \mathbb{R}^2 \colon x \geq 0\}$

    (b) $B = \{(x,y) \in \mathbb{R}^2 \colon 1 \leq x^2 + y^2 \leq 4\}$

    (c) $A \cap B$

    (d) $C = \{(x,y) \in \mathbb{R}^2 \colon x^2 + 2x > y\}$

    (e) $C \setminus A$

    (f) $D = \{(x,y,z) \in \mathbb{R}^3 \colon z > x^2 + y^2\}$

    (g) $E = \{(x,y,z) \in \mathbb{R}^3 \colon -1 \leq z \leq 1\}$

    (h) $F = \{(x,y,z) \in \mathbb{R}^3 \colon z^2 \geq x^2 + y^2 \quad \text{and} \quad z \leq 0\}$

    (i) $D \cap E$

    (j) $(\overline{D} \cup F) \cap E$

15. Show that:

    (a) Any finite union of compact sets is compact.

    (b) $A \subset \mathbb{R}^n$ is bounded if and only if $\overline{A}$ is compact.

16. Exhibit an example of a compact set of $\mathbb{R}^2$ without accumulation points.

17. In this exercise you will study the Cantor set $C$. Let $A_0 = [0,1]$ and define $A_1$ by cutting $A_0$ in three equal parts and then removing the middle part from $A_0$. i.e., $A_1 = [0,1/3] \cup [2/3,1]$. The set $A_1$ is a union of two disjoint intervals. Now for each interval of $A_1$ we proceed as before. Cut in three equals parts and remove from each its middle part. Call this new set $A_2$. Continue this process indefinitely to obtain a sequence of sets $A_n$, $n \geq 0$. The Cantor set is the intersection of all these sets, i.e.,

$$C = \bigcap_{n \geq 0} A_n$$

    (a) Obtain the analytic expression for $A_2$.

    (b) Show that each set $A_n$ is a union of $2^n$ disjoint closed intervals.

    (c) Is $A_n$ closed? Why?

    (d) Prove that $C$ is compact.

18. Show that the limit of a converging sequence is unique, i.e., a converging sequence cannot more than one limit.

19. For each of the following functions $f\colon D \to \mathbb{R}$ determine its continuity points and, using the Weierstrass theorem, decide if the function has a maximum or minimum.

    (a) $f(x) = x^2$ where $D = \{x \in \mathbb{R}\colon |x| \leq 1\}$
    (b) $f(x) = x^3 - x^2 + x - 1$ where $D = [-2, -1] \cup [1, 2]$
    (c) $f(x) = x\cos^2(1/x)$ where $D = \left\{\frac{(-1)^n}{2\pi n} : n \in \mathbb{N}\right\} \cup \{0\}$
    (d) $f(x, y) = xy$ where $D = [-1, 1]^2$
    (e) $f(x, y) = x\log(y)$ where $D = ]0, 1]^2$
    (f) $f(x, y) = e^{-x^2-y^2}$ where $D = \mathbb{R}^2$

20. Consider the function $f : [-1, 1] \to [-1, 1]$ defined by $f(x) = \sin(\lambda\, x)$ where $\lambda \in ]0, 1[$. Sketch the graph of $f$ and interpret the fixed point geometrically (also draw the bissectrix, i.e., $y = x$).

21. Complete the sentence:
    The continuous map $f(x, y) = x^2 + y^2$ has a maximum but not a minimum when restricted to the set

    $$M = \left\{(x, y) \in \mathbb{R}^2 : \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\right\}$$

    .....................'s Theorem cannot be applied because $M$ is not .........................

22. Show that $f : ]0, 1/4[ \to ]0, 1/4[$ defined by $f(x) = x^2$ is a Lipschitz contraction. Can you conclude that $f$ has a unique fixed point?

23. Determine whether the following functions $f : D \to \mathbb{R}$ are Lipschitz contractions. For each function determine its fixed points.

    (a) $f(x) = \frac{1}{4}x(1 - x^2)$ with $D = [-1, 1]$
    (b) $f(x) = \arctan(x/2)$ with $D = \mathbb{R}$
    (c) $f(x) = \frac{1}{4}\sin(x^3)$ with $D = [-1, 1]$
    (d) $f(x) = \sqrt{1 + x}$ with $D = [0, +\infty[$
    (e) $f(x, y) = \left(\frac{x}{2} + 5, \frac{y}{3} - 1\right)$ with $D = \mathbb{R}^2$

24. Use the Banach fixed point theorem to show that the sequence $x_{n+1} = \sqrt{1 + x_n}$, $n \in \mathbb{N}$ and $x_0 = 0$ converges to the golden ratio

    $$\frac{1 + \sqrt{5}}{2} = \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}.$$

25. Decide which of the following sets are convex:

(a) $A = \{(x, y) \in \mathbb{R}^2 \colon y \geq x^2\}$

(b) $B = \{(x, y) \in \mathbb{R}^2 \colon y < x^2\}$

(c) $C = \{(x, y) \in \mathbb{R}^2 \colon \frac{x^2}{4} + y^2 \leq 1\}$

(d) $A \cap C$

(e) $C \setminus B$

(f) $D = \{(x, y, z) \in \mathbb{R}^3 \colon x + y + z = 1,\ x, y, z \geq 0\}$

26. Consider the function $f(x) = \frac{1}{2}(x + 1)$ defined on the interval $]0, 1[$. Show that $f(]0, 1[) \subset ]0, 1[$. Does $f$ has a fixed point in $]0, 1[$? Can you apply the Brouwer fixed point theorem to $f$ ?

27. Consider the following matrix

$$A = \begin{pmatrix} 0 & 1/2 & 1 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}$$

and define the function $f(v) = Av$ where $v \in \Delta^2 = \{(x, y, z) \in \mathbb{R}^3 \colon x + y + z = 1,\ x, y, z \geq 0\}$.

(a) Show that $f(\Delta^2) \subset \Delta^2$.

(b) Can you apply the Brouwer fixed point theorem to $f$? Can you show that $f$ has a fixed point?

(c) If yes, compute the fixed points of $f$ explicitly.

28. Find a continuous function $f \colon [0, 1[ \to [0, 1[$ with no fixed points. Can you apply the Brouwer fixed point theorem to $f$?

29. Find a continuous function $f \colon [0, 1] \to [0, 1]$ with an infinite number of fixed points.

30. Consider the function $f \colon [0, 1] \to [0, 1]$ defined by

$$f(x) = \begin{cases} 2x, & x \leq 1/2 \\ 2 - 2x, & x > 1/2 \end{cases}$$

(a) Can you apply the Brouwer fixed point to $f$?

(b) Compute the fixed points of $f$ and of $f \circ f$ (hint: draw the graph of the functions).

(c) How many fixed points has $f^n = f \circ \cdots \circ f$? Here, $f^n$ denotes the composition of $f$ with itself $n$ times ($f^2 = f \circ f$, $f^3 = f \circ f \circ f$, etc.)

31. Consider a pure exchange economy with 2 commodities and 2 consumers. Denote by $p_1$ and $p_2$ the relative price of commodity 1 and 2, respectively. The 1st consumer has an initial amount of 1 unit of commodity 1 and 2 units of commodity 2. The 2nd consumer has an initial amount of 2 units of commodity 1 and 1 unit of commodity 2. Consumers engage in exchanging with the following demand functions

$$x_{1,1}(p) = \frac{\alpha(p_1 + 2p_2)}{p_1}, \quad x_{1,2}(p) = \frac{(1 - \alpha)(p_1 + 2p_2)}{p_2}$$

for the 1st consumer and

$$x_{2,1}(p) = \frac{\alpha(2p_1 + p_2)}{p_1}, \quad x_{2,2}(p) = \frac{(1-\alpha)(2p_1 + p_2)}{p_2}$$

for the 2nd consumer, where $\alpha \in [0, 1]$. Recall that $x_{i,j}(p)$ is the demand of commodity $j$ of consumer $i$. Check that the Walras's law is satisfied. Determine an equilibrium price for this economy.

32. Sketch the following hyperplanes and half-spaces:

    (a) $H((1, 1), -1)$

    (b) $H^+((2, -1), 1)$

    (c) $H((1, 0, -1), 1)$

    (d) $H^-((0, 1, 0), 2)$

33. (a) Let $A$ be the set of points in $\mathbb{R}^n$ whose first coordinate is equal to $a \in \mathbb{R}$. Find $p \in \mathbb{R}^n$ and $c \in \mathbb{R}$ such that $A = H(p, c)$.

    (b) Find a hyperplane $H(p, c)$ that separates $\Delta^2 = \{(x, y, z) \in \mathbb{R}^3_+ \colon x + y + z = 1\}$ and $A = \{(x, y, z) \in \mathbb{R}^3 \colon x = -2\}$.

    (c) Find the hyperplane $H(p, c)$ that contains the points $(1, 0, 0)$, $(1, 1, 0)$ and $(0, 1, 1)$.

34. Explain, using the hyperplane separation theorem, if it is possible to prove the existence of a hyperplane separating $A$ and $B$:

    (a) $A = \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 < 1\}$ and $B = \{(x, y) \in \mathbb{R}^2 \colon x + y = 2\}$

    (b) $A = \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 < 3\}$ and $B = \{(x, y) \in \mathbb{R}^2_+ \colon x + y = 1\}$

    (c) $A = C \cap D$ with $C = \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 = 2\}$ and $D = [-6, 6] \times [-5, 5]$, and $B = [-1, 1] \times [-1, 1]$

35. Decide if the following correspondences $F : [0, 1] \rightrightarrows [0, 1]$ are upper hemicontinuous and/or have the closed graph property:

    (a)
    $$F(x) = \begin{cases} [1/4, 3/4], & x < 1/2 \\ [1/2, 3/4], & x \geq 1/2 \end{cases}$$

    (b)
    $$F(x) = \begin{cases} [x^2, (x+1)/2], & x < 1 \\ \{0, 1\}, & x = 1 \end{cases}$$

    (c)
    $$F(x) = \begin{cases} ]x, 1-x[, & x < 1/2 \\ \{1/4, 3/4\}, & x \geq 1/2 \end{cases}$$

    (d)
    $$F(x) = ]x/2, (x+1)/2[$$

36. Explain if the following correspondences satisfy the hypothesis of the Kakutani fixed point theorem. Compute the fixed points if they exist.

(a) $F\colon [0,2] \rightrightarrows [0,2]$ defined by

$$F(x) = \begin{cases} \{1\}, & 0 \le x < 1 \\ [1,2], & x = 1 \\ \{2\}, & 1 < x \le 2 \end{cases}$$

(b) $F\colon [0,20] \rightrightarrows [0,20]$ defined by

$$F(x) = \begin{cases} \{10 - x\}, & 0 \le x < 7 \\ [3,20], & x = 7 \\ \{x - 4\}, & 7 < x \le 20 \end{cases}$$

(c) $F\colon [-6,20] \rightrightarrows [-6,20]$ defined by

$$F(x) = \begin{cases} \{x + 1\}, & -6 \le x < 7 \\ [-6,10], & x = 7 \\ \{(x+5)/2\}, & 7 < x \le 20 \end{cases}$$

(d) $F\colon [-6,20] \rightrightarrows [-6,20]$ defined by

$$F(x) = \begin{cases} \{x + 1\}, & -6 \le x < 7 \\ [-6,6] \cup [8,10], & x = 7 \\ \{(x+5)/2\}, & 7 < x \le 20 \end{cases}$$