Lisbon University
Lisbon School of Economics and Management
Ms in Economics, Mathematical Finance and
Monetary and Financial Economics

<u>Mathematical Economics</u> – 1st Semester - 2023/2024

Exercises - Group II

1. Complete the following sentences:

   (a) If $f(x,y) = \frac{3}{(x-1)^2+y^2}$ and $g : \mathbb{R}\backslash\{3\} \to \mathbb{R}$ is the map defined by $g(x) = 2 + \frac{5}{x-3}$, then
   $$\lim_{(x,y)\to(0,0)} [g \circ f(x,y)] = \ldots\ldots\ldots$$

   (b) The gradient vector of $f : \mathbb{R}^2 \to \mathbb{R}$ is given by $(2x\cos y, 1 - x^2 \sin y)$. If $f(x,y)$ does not have constant terms in both components, then $f(1,\pi) = \ldots\ldots$

   (c) With respect to the map $f : \mathbb{R}^2 \to \mathbb{R}$, we know that $\nabla f(3,2) = (0,0)$ and
   $$H_f(3,2) = \begin{pmatrix} \ldots\ldots & 0 \\ 0 & \ldots\ldots \end{pmatrix}.$$
   Then, $f(3,2)$ is a local maximum of f.

   (d) The point $x = 5$ is a saddle-point of the map $f(x) = \ldots\ldots\ldots\ldots\ldots\ldots$, $x \in \mathbb{R}$.

2. Classify the critical points of the following functions,

   (a) $f(x,y,z) = x^2 + 2y^2 + 3z^2 + 2xy + 2xz$ on $\mathbb{R}^3$

   (b) $f(x,y,z,w) = 20y + 48z + 6w + 8xy - 4x^2 - 12z^2 - w^2 - 4y^3$ on $\mathbb{R}^4$

   (c) $f(x,y,z) = z\log(x^2+y^2+z^2)$ on $\mathbb{R}^3\backslash\{(0,0,0)\}$

3. Consider the map f defined in $\mathbb{R}^2$ as
   $$f(x,y) = x^3 + y^3 + x^3y^3.$$

   (a) Compute $f(x,0)$, $x \in \mathbb{R}$.

   (b) Identify and classify the critical points of f.

4. Find and classify the critical points of $f(x,y) = e^{x^2-ay^2}$ as function of the parameter $a \in \mathbb{R}\backslash\{0\}$.

5. Consider the map $f(x,y) = (y-\alpha)xe^x$, where $\alpha \in \mathbb{R}\backslash\{0\}$.

   (a) If $(0,1)$ is a critical point of f, find α and classify the critical point.

   (b) Show that f is not limited.

6. Consider the map $f(x,y) = x^2e^{y^3-3y}$.

   (a) Find the maximal domain of f.

   (b) Find and classify all critical points of f.

   (c) Show that f attains its global minimum at points of the form $(0,b), b \in \mathbb{R}$.

   (d) Show that f is not limited.

   (e) Show that f has a maximum and a minimum on

$$B = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 5\}.$$

7. Consider the following function defined on $\mathbb{R}^2$:

$$f(x,y) = (1+y)^3x^2 + y^2$$

   Show that f has a unique critical point (x^*, y^*) which is a local minimizer. Is (x^*, y^*) a global minimizer ?

8. Determine the maximum and minimum distance to the origin of the points in the ellipse defined by $5x^2 + 6xy + 5y^2 = 8$.

9. Determine the point in the ellipse $x^2 + 2xy + 2y^2 = 2$ with smallest x coordinate.

10. Find and classify all critical points of

$$f(x,y) = 8y^2 - 4x^2(y-1).$$

   Among the critical points, is there some global one?

11. Find the extrema of $f(x,y) = x$ when restricted to the set $y^2 + x^4 - x^3 = 0$.

12. Find the extrema of $f(x,y) = x - 2y + 2z$ when restricted to

$$M = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

13. Determine the global extrema of f over the set M, where:

   (a) $f(x,y,z) = x - 2y + 2z$ and $M = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$

   (b) $f(x,y,z) = x^2 + 2xy + y^2$ and $M = \{(x,y,z) \in \mathbb{R}^3 : (x-3)^2 + y^2 = 2\}$

14. Determine the global extrema of f over the set M, where:

   (a) $f(x, y, z) = x - 2y + 2z$ and $M = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$
   (b) $f(x, y, z) = x^2 + 2xy + y^2$ and $M = \{(x, y, z) \in \mathbb{R}^3 : (x - 3)^2 + y^2 \leq 2\}$

15. Determine if the following functions are (strictly) convex/concave:

   (a) $f(x, y) = 2x - y - x^2 + 2xy - y^2$ on $\mathbb{R}^2$.
   (b) $f(x, y) = x^a y^b$ on $\mathbb{R}^2_+$ and $a + b \leq 1$ with $a, b \geq 0$.

16. Find the largest domain $D \subset \mathbb{R}^2$ on which the following function is concave,

$$f(x, y) = x^2 - y^2 - xy - x^3.$$

17. Use Lagrange convexity's Theorem to solve the following optimization problems:

   (a)

$$\begin{aligned} \text{maximize } & 2x + y \\ \text{subject to } & x^2 + y^2 = 1 \end{aligned}$$

   (b)

$$\begin{aligned} \text{minimize } & x^2 y^2 \\ \text{subject to } & (1/x)^2 + (1/y)^2 = 1 \end{aligned}$$

   (c)

$$\begin{aligned} \text{maximize } & x + 4y + z \\ \text{subject to } & x + 2y + 3z = 0 \\ & x^2 + y^2 + z^2 = 42 \end{aligned}$$

   (d)

$$\begin{aligned} \text{minimize } & x + 4z \\ \text{subject to } & x - y + z = 2 \\ & x^2 + y^2 = 1 \end{aligned}$$

18. Find the local optimal points of f on D where:

   (a)

$$\begin{aligned} f(x, y) &= \log(xy) \\ D &= \{(x, y) \in \mathbb{R}^2 \colon (1/x)^2 + (1/y)^2 = 1\} \end{aligned}$$

   (b)

$$\begin{aligned} f(x, y) &= x + y \\ D &= \{(x, y) \in \mathbb{R}^2 \colon xy = 16\} \end{aligned}$$

(c)

$$\begin{aligned} f(x, y, z) &= x^2 - z^2 \\ D &= \{(x, y, z) \in \mathbb{R}^3 \colon 2x + z = a,\ x - y = b\} \end{aligned}$$

19. Use Kuhn-Tucker Theorem to solve the following optimization problems:

(a)

$$\begin{aligned} \text{maximize } & x^2 + 2y \\ \text{subject to } & x^2 + y^2 \leq 5 \\ & y \geq 0 \end{aligned}$$

(b)

$$\begin{aligned} \text{maximize } & \frac{1}{2}x - y \\ \text{subject to } & x + e^{-x} + z^2 \leq y \\ & x \geq 0 \end{aligned}$$

(c)

$$\begin{aligned} \text{minimize } & 2x^2 + 3y^2 \\ \text{subject to } & x + 2y \leq 11 \\ & x \geq 0 \\ & y \geq 0 \end{aligned}$$

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