2.

Mean-Variance Theory (MVT)

These slides were originally created by R Gaspar and have in this version been editied and complemented by M Hinnerich.

Revised on slide 62. For a more detailed derivation please see the notes "partial_derivative_of_sharpe_ratio.pdf

2. Mean-Variance Theory (MVT)

Section A

2.1 Mean-Variance Approach

2.2 The two-assets case

2.3 Two assets where one is the riskless asset

2.4 Two risky assets and the riskless asset

2.1 Mean-Variance Approach

- Learning Objectives
- Defining Mean-Variance Analysis
- Efficient Portfolios
- Investment Opportunity Set

Learning objectives

- define mean-variance efficiency,
- distinguish between the opportunity set and efficient frontier,

Mean variance analysis

MVT has its flaws, but it provides a good starting point for portfolio theory. It is still the most commonly used tool for portfolio construction.

MVT assumptions

1. Investors only care about the mean and variance of future returns.
    - Investors prefer higher means to lower means.
    - Investors prefer lower variances to higher variances.
2. We know the means, variances and covariances of future returns for the assets we can invest.

Investor Preferences

On Assumption 1:

- Clearly, in general, investors do not care only about mean and variance of future returns.
- Most investor worry about bad outcomes, i.e. also care about the left tail of return distributions. Investors may want to impose, for instance, safety restrictions.
- Most investor preferences cannot be represented just in terms of mean and variance of returns.
- However assumption 1 gets to be automatically satisfied if:
  - If investor have quadratic utility function (or approximately); OR
  - The mean and variance are sufficient statistics to the future return distribution.

MVT estimation of inputs

On Assumption 2:

- In a world with n risky assets. The MVT inputs are a vector of future expected returns.

$$\bar{R} = \begin{pmatrix} \bar{R}_1 \\ \bar{R}_2 \\ \vdots \\ \bar{R}_n \end{pmatrix}$$

and the future variance-covariance matrix (recall $V_{ij} = \sigma_{ij}$)

$$V = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_n^2 \end{pmatrix}$$

- They are not things we can observe in the market.
- Historical data only tell us about past realised returns, not future returns .

Investment Opportunity set

Definition (Investment Opportunity Set)

The set of all possible pairs of standard deviations and returns attainable from investing in a collection of assets is called the *investment opportunity set*.

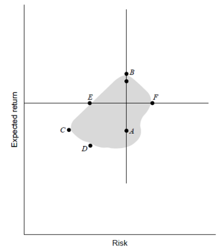

Figure 5.8 Risk and return possibilities for various assets and portfolios.

Mean-variance criteria

We say portfolio A dominates portfolio B if either

1) A has at least as much expected return and lower standard deviation: $E(r_A) \geq E(r_B)$ and $\sigma_A < \sigma_B$

2) A has higher expected return and a standard deviation which is smaller or equal. $E(r_A) > E(r_B)$ and $\sigma_A \leq \sigma_B$

Q: Which portfolios dominates P?

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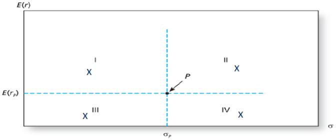


Mean Variance Efficient Portfolios

Definition (Efficiency)

A portfolio (A) is *efficient* provided either

No other portfolio B has at least as much expected return and lower standard deviation, and

$$E(r_A) \geq E(r_B) \quad \underline{and} \quad \sigma_A < \sigma_B$$

No other portfolio B has higher expected return and standard deviation which is smaller or equal.

$$E(r_A) > E(r_B) \quad \underline{and} \quad \sigma_A \leq \sigma_B$$

Efficient Frontier

Remark

An efficient asset is on the edge of the opportunity set. If it would not be on the edge, there would be a direction that gives better return or risk.

Definition (Efficient Frontier)

The subset of the investment opportunity set which is efficient is called the *efficient frontier*.

Investment Opportunity set

Q: What is i) the investment opportunity set?
ii) the global minimum variance?
iii) the efficient frontier?

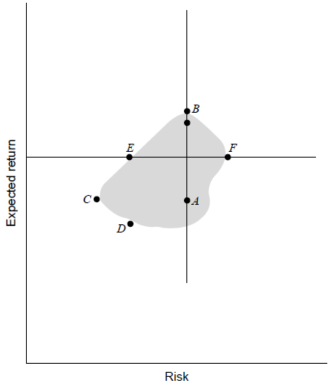


Figure 5.8 Risk and return possibilities for various assets and portfolios.

Efficiency and opportunities

- It is important to realize that efficiency is only defined relative to a set of investment opportunities.
- If we change the set of assets which the investor can put his money into then the set of efficient portfolios changes too.
- In general, if we allow an extra asset then portfolios that were previously efficient are no longer efficient.
- Similarly, if we throw away an asset both from the set of investment opportunities and from an efficient portfolio, then the portfolio containing the remaining assets may not be efficient.

2.2 The two-assets case

- Learning Objectives
- Graphs
- Geometry
- Questions

Learning objectives

- Derive and sketch the opportunity set and efficient frontier for two assets for a variety of correlations with and without short selling,
- find the minimum variance combination of two assets,
- state what sort of curve the opportunity set takes,
- discuss convexity in the context of efficiency.

The two assets case

- Let us consider there are only two assets C and S, both are risky.
- All possible combinations of C and S consist of investing fractions x_S and x_C such that

$$x_S + x_C = 1.$$

Ex. X_s=0.4 X_c=0.6

- Because we can always write $x_C = 1 - x_S$, the return of a portfolio can be described in terms of only the variable which is the fraction of investments put into C

$$R_P = x_c R_c + \underbrace{(1 - x_c)}_{x_s} R_s$$

- The portfolio return is just a weighted average of the returns of the individual assets.

The two assets case: mean and variance

- The expected portfolio return can be written as

$$\begin{aligned}\bar{R}_P = \mathbb{E}(R_P) &= x_C\mathbb{E}(R_C) + (1 - x_C)\mathbb{E}(R_S),\\ &= x_C(\bar{R}_C - \bar{R}_S) + \bar{R}_S\ .\end{aligned}$$

- The variance of any combination of C and S can be written as

$$\sigma_P^2 = \mathbb{E}\left[\left(R_p - \bar{R}_p\right)^2\right] = x_C^2\sigma_C^2 + (1 - x_C)^2\sigma_S^2 + 2x_C(1 - x_C)\sigma_{CS}$$

OBS: The expected return is linear in x_C whilst the variance is quadratic.

Q: What is the portfolio return and portfolio standard deviation if x_c=40% and correlation =0?

	Expected Return	Standard Deviation
c	14%	6%
s	8%	3%

Curve in expected return vs standard deviation space

- To represent all combinations of C and S in the space $(\sigma, \bar{R})$.
- We need to solve

$$\begin{cases} \bar{R}_P = x_C(\bar{R}_C - \bar{R}_S) + \bar{R}_S \\ \sigma_P^2 = x_C^2\sigma_C^2 + (1 - x_C)^2\sigma_S^2 + 2x_C(1 - x_C)\sigma_{CS} \end{cases}$$

- We can solve f or x_C in terms of the expected return in the 1st egn, provided the two assets have different expected returns, and we get x_C is a linear function on the expected return $\bar{R}_P$:

$$x_C = \frac{\bar{R}_P - \bar{R}_S}{\bar{R}_C - \bar{R}_S}.$$

- If one substitutes this back into the expression for variance, one gets the **investment opportunity curve**

$$\sigma_P^2 = \alpha\bar{R}_P^2 + \beta\bar{R}_P + \gamma$$

for some constants α, β and γ that depend only on the MVT inputs.

Example: Investment opportunity set (curve)

	Expected Return	Standard Deviation
C	14%	6%
S	8%	3%

=>

X_C	0	0.2	0.4	0.6	0.8	1.0
$\bar{R}_P$	8.0	9.2	10.4	11.6	12.8	14.0
σ_P	3.00	2.68	3.00	3.79	4.84	6.0

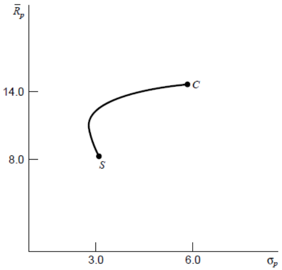


Figure 5.4 Relationship between expected return and standard deviation when $\rho = 0$. (no shortselling)

Q: What is the smallest possible variance a portfolio of S and C can have, i.e. **what is the minimum variance portfolio?**

Minimum variance portfolio

- The variance of all combinations of C and S is given by

$$\sigma_P^2 = x_C^2\sigma_C^2 + (1-x_C)^2\sigma_S^2 + 2x_C(1-x_C)\sigma_{CS}$$

- To find the minimum variance portfolio (MV) we can solve

$$\frac{\partial \sigma_P^2}{\partial x_c} = 0$$

to find the value of x_C^* that gives least variance,

$$x_C^{MV} = \frac{\sigma_S^2 - \sigma_{CS}}{\sigma_C^2 + \sigma_S^2 - 2\sigma_{CS}} = \frac{\sigma_S^2 - \sigma_C\sigma_S\rho_{CS}}{\sigma_C^2 + \sigma_S^2 - 2\sigma_C\sigma_S\rho_{CS}}.$$

- In the two risky assets case, the portfolio of minimum variance will always be efficient.

Special cases

- Having done a little work on the general case for two assets, we now study some special cases in order to develop some intuition.
- For illustration purposes we consider the following parameters

$$\begin{aligned} \sigma_S &= 15\%, \qquad \text{S is riskier than C} \\ \sigma_C &= 10\%, \\ \bar{R}_S &= 6\%, \\ \bar{R}_C &= 5\%. \end{aligned}$$

- What will change across cases is the correlation coefficient ρ_{CS}.

$\rho = -1$: perfect negative correlation

- Suppose we have $\rho_{CS} = -1$.
- We have

$$\begin{aligned}\sigma_P^2 =& x_C^2\sigma_C^2 + (1-x_C)^2\sigma_S^2 - 2x_C(1-x_C)\sigma_C\sigma_S, \\ =& (x_C\sigma_C - (1-x_C)\sigma_S)^2.\end{aligned}$$

  This implies

$$\sigma_P = | x_C\sigma_C - (1-x_C)\sigma_S | \ .$$

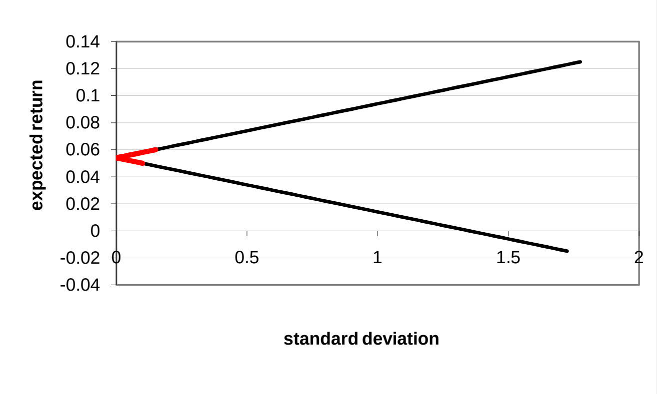
- In this case, there will be a point where the two pieces of risk cancel each other out and we obtain zero risk.

Question: What proportion of the investment in asset C will provide a portfolio with zero risk? What is the return of this portfolio?

$\rho = -1$: investment opportunity curve (no shortsell)

In the expected return against standard deviation space mark the portfolio
-100% invested into the asset S
-100% invested into the asset C
-the 0-risk portfolio in your previous answer

$\rho = -1$: investment opportunity curve in $(\sigma, \bar{R})$



- The red portion is the opportunity curve without shortselling.

Q: Which portfolios are efficient?

$\rho = 1$: two perfectly correlated assets

- If the assets are perfectly correlated, i.e. $\rho_{CS} = 1$
- The variance of any combination is

$$\begin{aligned}\sigma_P^2 =& x_C^2\sigma_C^2 + (1 - x_C)^2\sigma_S^2 + 2x_C(1 - x_C)\sigma_C\sigma_S,\\ =& (x_C\sigma_C + (1 - x_C)\sigma_S)^2,\end{aligned}$$

which implies

$$\sigma_P = | x_C\sigma_C + (1 - x_C)\sigma_S |$$

- The expected return remains as before:

$$\bar{R}_P = x_C\bar{R}_C + (1 - x_C)\bar{R}_S = x_C(\bar{R}_C - \bar{R}_S) + \bar{R}_S$$

$\rho = 1$: two perfectly correlated assets continued

In the expected return against standard deviation space mark the portfolio

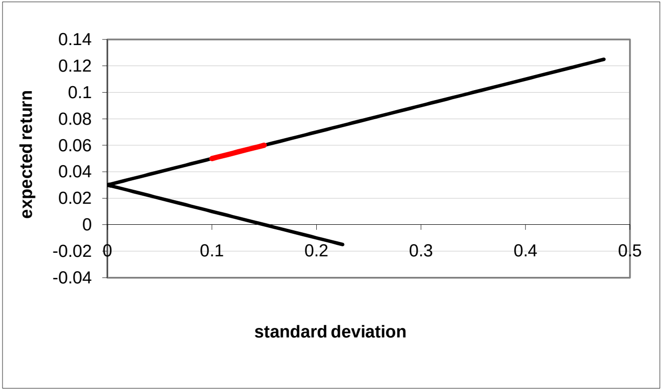
-100% invested into the asset S
-100% invested into the asset C
- 50% invested in C
-draw the opportunity set if no shortselling

Remark: If shortselling is not allowed, there is no risk reduction arising from diversification, because in this case both the volatility and expected returns are linear functions of x_C.

$\rho = 1$: two perfectly correlated assets continued

Q: How does the zero risk portfolio look like (what proportions) - if it exist?

$\rho = 1$: investment opportunity curve in $(\sigma, \bar{R})$



- The red portion is the opportunity curve without shortselling.

  The opportunity set is describe by two straight lines reflecting at the zero volatility axis, since standard deviation is always positive.

$\rho = 0$: uncorrelated assets

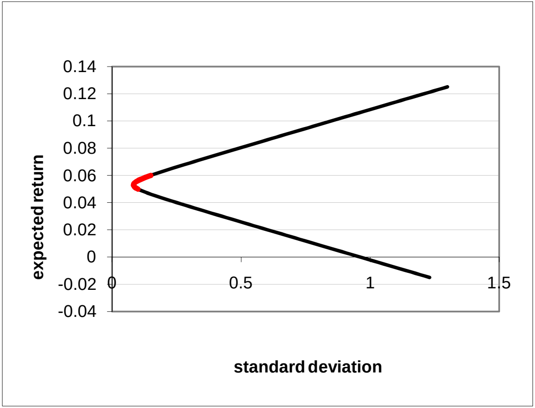
- We now consider the case where $\rho_{CS} = 0$.
- The last term in the variance formula disappear and simplifies to:

$$\sigma_P^2 = x_C^2 \sigma_C^2 + (1 - x_C)^2 \sigma_S^2.$$

- The opportunity set becomes a hyperbola.
- The minimum variance portfolio simplifies to

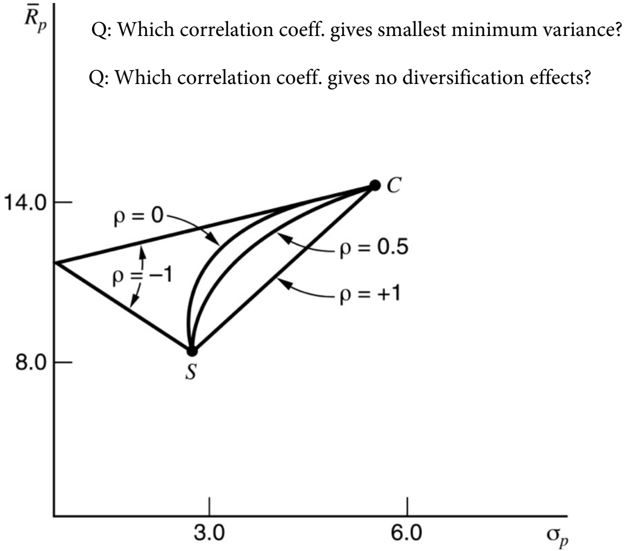
$$x_C^{MV} = \frac{\sigma_S^2}{\sigma_C^2 + \sigma_S^2}.$$

$\rho = 0$: investment opportunity curve in $(\sigma, \bar{R})$

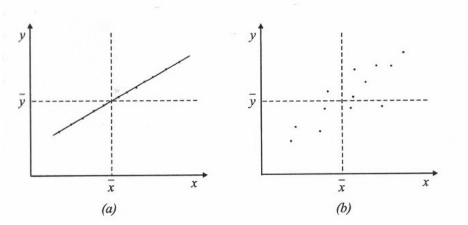


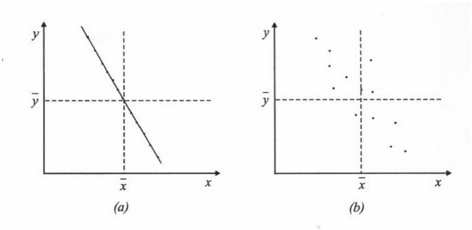
- The red portion is the opportunity curve without shortselling.

Real-life two-assets investment curves (no shortselling)



Real-life correlation examples





Q: Which picture has correlation -1?

Q: Which picture has positive correlation ?

Q: Which picture has 0 correlation?

Real-life correltaions_: no correlation

Q: How does no correlation btw x and y look like?

Understanding the curve

Consider two risky assets ($n = 2$).

- We saw that, in general, the investment opportunity set is a *hyperbola*.

$$\sigma_P^2 = \alpha \bar{R}_P^2 + \beta \bar{R}_P + \gamma$$

  for some constants α, β and γ that depend only on the MVT inputs.
- Recall

$$\begin{cases} \bar{R}_P = x_C(\bar{R}_C - \bar{R}_S) + \bar{R}_S \\ \sigma_P^2 = x_C^2\sigma_C^2 + (1 - x_C)^2\sigma_S^2 + 2x_C(1 - x_C)\sigma_{CS} \end{cases}$$

HW: Find out the mathematical expressions for α, β and γ.

Convexity

- Defn: A function is said to be convex if the chord between any two points lies above the graph.
- Defn: A function is said to be concave if the chord between two points lies below the graph.
- Hyperbolas are concave above the minimum variance portfolio
- Straight lines are trivially both convex and concave.
- Hence, the efficient frontier of two assets is concave.
- Note the area below the turning point is not concave and not efficient.

Theory questions

1. What are the assumptions of mean-variance portfolio theory?
2. What does it mean for an asset to be mean-variance efficient?
3. Define the opportunity set and efficient frontier in mean-variance analysis? How do they relate to each other?
4. If a portfolio is efficient and we add in a new asset to invest in will the original portfolio remain efficient?
5. If a portfolio is efficient and we discard one of its elements from the set of possible assets, will the part of the portfolio excluding this element always be efficient?
6. Derive expressions for the variance and expected returns of a portfolio of two assets in terms of the amount invested in the first, their expected returns, their variances and covariance.

2.3 Two assets where one is the riskless asset

- Recap
- Learning objectives
- The risk-free asset
- The investment opportunity set
- The efficient frontier

Recap

Up to now, we studied efficiency for two risky assets:

- The investment opportunity set is a hyperbola in $(\sigma, \bar{R})$ space.
- The efficient frontier is the upper part of the hyperbola in $(\sigma, \bar{R})$. space.
- Expected return of any combinations is linear in the portfolio weight of one of the assets. $\bar{R}_P = x_C(\bar{R}_C - \bar{R}_S) + \bar{R}_S$
- Variance of any combination is quadratic in the portfolio weight of one of the assets $\sigma_P^2 = x_C^2\sigma_C^2 + (1 - x_C)^2\sigma_S^2 + 2x_C(1 - x_C)\sigma_{CS}$
- Variance of any combination of the two assets is quadratic in its own expected return. $\sigma_P^2 = \alpha\bar{R}_P^2 + \beta\bar{R}_P + \gamma$

Learning objectives

- define a riskless asset,
- identify the opportunity set with a riskless asset F and one risky asset A,
- define and compute the market price of risk,
- prove that an efficient portfolio containing a riskless asset remains efficient after the riskless asset has been discarded,
- sketch the efficient set when there is a riskless asset,
- state and prove the Tobin separation theorem,
- discuss why tangency is required for optimality,
- show how the investment line with three assets meets the opportunity set with two assets.

Risk-free asset

Definition

An asset whose return is known in advance is said to be risk-free. An asset, F, is *risk-free* if and only if

- The variance of its returns is zero ($\sigma_f^2 = 0$)
- $\Rightarrow$ The standard deviation of returns is zero ($\sigma_f = 0$).

Result: (based on no-arbitrage assumption)

All risk-free assets have the same return.

If there would be two risk-free assets with different returns, then everyone would sell the risk-free asset with lower return and buy the one with higher return until the returns agreed.

The opportunity set with a risk-free asset

Consider one risk-free asset F with return R_f and one risky asset (or portfolio) A with expected return $\bar{R}_A$ and volatility σ_A.

- If our portfolio p is $1-x$ units of the risk-free asset, F and x units of some risky asset A,
- its expected return is

$$\bar{R}_p = (1-x)R_f + x\bar{R}_A \ ,$$

- its variance is given by

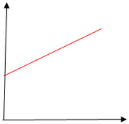
$$\begin{aligned} \mathrm{Var} R_p &= \mathrm{Var}(xR_A), \\ \sigma_p^2 &= x^2\sigma_A^2. \end{aligned}$$

- So, taking square roots.

$$\sigma_p = |x|\sigma_A.$$

The investment opportunity line (Capital allocation line)

- If we restrict to $x \geq 0$ (positive investment risky asset), we have from the previous slide
$$x = \frac{\sigma_p}{\sigma_A}.$$
i.e., the investment fraction in A is the ratio of the portfolio's standard deviation to the risky asset's standard deviation.
- This implies
$$\bar{R}_p = \left(1 - \frac{\sigma_p}{\sigma_A}\right) R_f + \frac{\sigma_p}{\sigma_A} \bar{R}_A,$$
- Hence all combinations of the risk-free asset F with the risky asset A are represented by a straight line
$$\bar{R}_p = R_f + \frac{\bar{R}_A - R_f}{\sigma_A} \sigma_p.$$

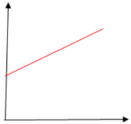


Q: What would mean an $x < 0$?
How can that be represented graphically?

Interpreting the line

- All the efficient combinations of a riskless asset and a risky asset are

$$\bar{R}_p = R_f + \frac{\bar{R}_A - R_f}{\sigma_A}\sigma_p .$$



- which is a straight line in the space $(\sigma, \bar{R})$.
- Its y-cross (intercept) is R_f.
- Its slope

$$\theta_A = \frac{\bar{R}_A - R_f}{\sigma_A} ,$$

The slope is the market price of risk of asset A, and it represents the excess expected return per unit of risk of the risky asset A.

- θ_A is also know as the Sharpe ratio of asset A.

Example of market price of risk

If we have

$$R_f = 3\%,$$
$$\sigma_A = 12\%,$$
$$\bar{R}_A = 12\%$$

The market price of risk for A is

$$\theta_A = \frac{12\% - 3\%}{12\%} = \frac{3}{4} = 0.75$$

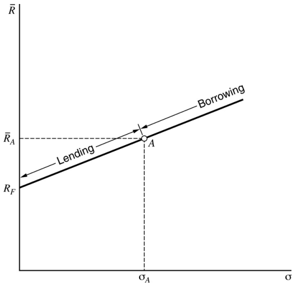
Q: Consider another risky asset B has $\bar{R}_B = 11\%$, and $\sigma_B = 6\%$. What is the market price of risk for B?

In which asset would you rather invest? Why?

Interpreting the capital allocation line

We can interpret the line as follows:

- Between the two points F and A, i.e. desired risk levels between $[0, \sigma_A]$, we are dividing our portfolio into the risk-free asset and the risky-asset.
- Above the risky point A, we are short-selling the risk-free asset (borrowing), and putting more money into the risky asset, to be able to reach volatilities above σ_A.



Q: What does this line say? If I you are an investment advisor-what is your advice?

Different active and passive riskless rates

In real life lending rates (passive rates) and borrowing rates (active rates) are not the same:

- Riskless borrowing is only possible if you are a government with an impeccable credit rating. $\Rightarrow$ Only then can one talk about buying or shorting the a risk-less asset at a constant rate R_f.
- In general, since shorting the risk-less asset is the same as borrowing, the lender is taking the risk that you will not pay back the money (credit risk). The lender may
  - not give a loan for investment in risky assets.
  - give a loan, but demand a risk-premium, making the active rate higher than the passive rate $R_f^a > R_f^p$.

Q: *How would this reality change the previously derived investment* capital allocation *line?*

Capital allocation line when lending/borrowing rates differ

If borrowing rate (9%) is higher than the risk free (lending) rate (7%)

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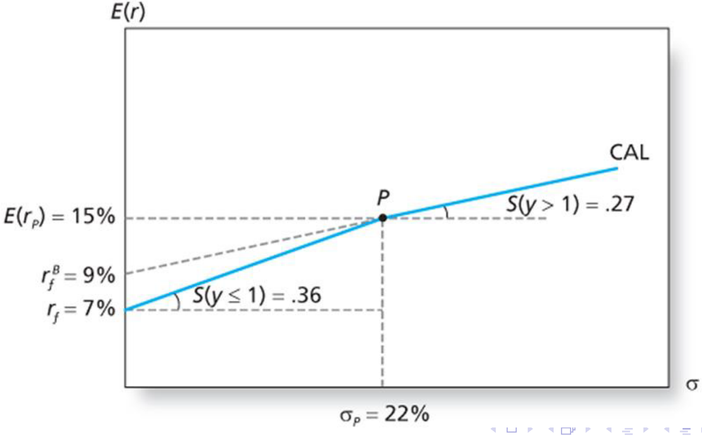


2.4 Two risky assets and the riskless asset

- Recap
- Learning Objectives
- Efficiency Results
- Tangent Portfolios
- Examples
- Borrowing restrictions
- Questions

Recap

Up to now we have considered ...

- Combinations of two risky assets (with and without shortselling allowed)
- Combinations of the riskless asset with one risky asset. (perhpas with the lending and borrowing rates differ)

Now it is time to ...

- Consider two risky-assets and one riskless asset (already 3 assets).
- For reasons that will become clear later, this is a reference situation .
- Even the general case (with n risky assets), in the end, reduce to this one.

Learning objectives

- Be able to use graphs and formulas to find the minimum variance portfolio, tangent portfolio and efficient frontiers.
- Sketch and derive the efficient frontier with two risky assets and one riskless asset with:
  - equal lending and borrowing rates => Scenario 1
  - equal lending and borrowing rates, but no shortselling => Scenario 2
  - one riskless asset, but no borrowing => Scenario 3
  - different borrowing and lending rates => Scenario 4
- Be able to find efficient portfolios for a specified level of expected returns, under different market scenarios.
- Understand how different shortselling restrictions impact the investment opportunity sets and efficient frontiers.

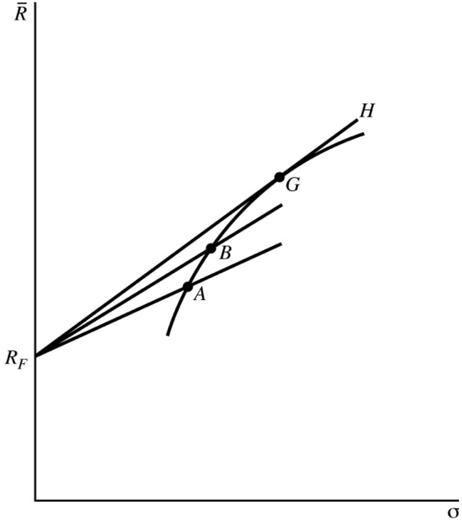
Scenario 1: two risky assets and one R_f

Scenario1

Consider two risky assets A and B with some $\bar{R}_A$, $\bar{R}_B$, σ_A, σ_B and σ_{AB}, and a riskless asset F with return R_f, that can be used for both lending and borrowing.

- Combinations of two risky assets give us an hyperbola.
- Combining a riskless asset with any risky portfolio, give us straight lines passing through F and the risky portfolio.

Scenario 1: Illustration



Q: Which of the risky portfolios A, B and G is optimal?

Hint: What is the slope of the capital allocation line for A, B and G respectively?

Scenario 1: Illustration

- From the previous picture, we see the capital allocation line (the efficent frontier of all assets) must be the straight line going through the riskless asset F and the risky portfolio G.
- Combining F with any other combination of just risky assets is not efficient (since they would have smaller sharpe ratio.)
- G can be characterised as the combination of risky assets with the highest slope. It is also the portfolio tangent to the hyperbola – so it is common to denote it by T (instead of G) to highlight that fact.

Tasks:

1. What can you say about the investment opportunity set?
2. Find the so-called tangent portfolio T. Compute its expected return, $\bar{R}_T$ and volatility σ_T.
3. Write down the efficient frontier equation and interpret it.

Scenario 1: the efficient set is a straight line

- We showed earlier that if we combined the tangent-portfolio of risky assets T (on the hyperbola), with the risk-free asset F we got a straight line (Capital allocation line).
- So any efficient portfolio P can be created by the investing in the risk-free asset and in tangency portfolio T:

$$P = x_f F + (1 - x_f) T \,,$$

- I.e. the entire line through the points $(0, R_f)$ and $(\sigma_T, \bar{R}_T)$ including P can be interpreted as combinations of T and F. The entire line is efficient.

Theorem (Investment line /Capital allocation line)

If there is a risk-free asset, all efficient portfolios lie on a straight line in $(\sigma, \bar{R})$ space.

Proof

Let P be an efficient portfolio. The line CAL going through point P and $(0, R_f)$ is also in the investment opportunity set.

Let P* be another efficient portfolio. The line CAL* going through point P* and $(0, R_f)$ is also in the investment opportunity set.

For two straight lines through R_f must hold that either:

1) one line will be below the other one for all $\sigma_P > 0$, or
2) the lines are the same,

In the first case, this means none of the portfolios on the lower line is efficient. I.e. not both P and P* can be efficient. (This cannot be since it violates the assumtions that both P and P* are efficient.)

So, the two lines must be the same (i.e. P and P* lie on the same line).

Scenario 1: Investors

- All efficient portfolios can be obtained as a mixture of a single portfolio of risky assets T and the risk-free asset F.
- So all mean-variance investors with the same investment situation, will hold the same portfolio of risky assets T, but may hold differing proportions of $T$and F . Ex 50%T and 50%F or 60%T and, 40%F
- The crucial point here is that the investors have to have the same views (expectations) in terms of the MVT inputs (expected returns, variances and covariances).
- But, they don't have to have the same risk preferences. As we will see later it is the risk preference that determine where on the line an investor choose to be, i.e. how much T and how much F to invest into.

Scenario 1: Tobin separation theorem

Theorem (Tobin separation theorem)

Two mean-variance investors facing the same investment situation will hold the same portfolio (T) *of risky assets (uniqueness).*

- Note that up to now we have not proven that the tangent portfolio T is unique – i.e. that it is the only combination of just risky assets that is efficient.
- However, T will be generally unique (think graphically).

HW: One exception occurs when it is possible to make a risk-free asset from a combination of risky assets. Think about this situation.
When can it happen?
What can you conclude about the efficient frontier in that case?

Scenario 1: investment opportunity set

1.
   - It is the cone with vertice at the riskless asset F and with limiting lines tangent to the hyperbola (the hyperbola comes from the investment opportunity set of just risky investments). It is an open set.
   - Sketch the investment opportunity set:
   -

Scenario 1: efficient frontier shape and portfolio T

- So, if we have a risk-free asset, F, with return R_f, and two risky assets.
- And additionally, we know portfolio E is efficient.
- Then, from before, we know we can write E in the form

$$E = x_f F + (1 - x_f) T,$$

for some portfolio of risky assets, T.
- Thus, we just need to know how to find the tangent portfolio T.

How do we find T?

Scenario 1: finding T

2. We know T is the portfolio with the highest Sharpe ratio.
So, it must be the portfolio P that solves the following optimisation problem

$$\max_{x_A, x_B} \quad \theta = \frac{\bar{R}_p - R_f}{\sigma_p}$$

s.t.

$$\bar{R}_p = x_A \bar{R}_A + x_B \bar{R}_B$$
$$\sigma_p = \left(x_A^2 \sigma_A^2 + x_B^2 \sigma_B^2 + 2 x_A x_B \sigma_{AB}\right)^{\frac{1}{2}}$$
$$x_A + x_B = 1$$

Scenario 1: finding T

2. Including the restrictions:
   - We can substitute the expressions for $\bar{R}_P$ and σ_P in the objective function.
   - Also, we have $R_f = (x_A + x_B)R_f$, because $x_A + x_B = 1$.

   So, the original problem is equivalent to the following unrestricted problem

$$\max_{x_A, x_B} \qquad \theta = \frac{x_A \bar{R}_A + x_B \bar{R}_B - R_f(x_A + x_B)}{\left(x_A^2 \sigma_A^2 + x_B^2 \sigma_B^2 + 2 x_A x_B \sigma_{AB}\right)^{\frac{1}{2}}}$$

And to get the optimal, we need to solve the FOC:

$$\begin{cases} \frac{\partial \theta}{\partial x_A} = 0 \\ \frac{\partial \theta}{\partial x_B} = 0 \end{cases}$$

Scenario 1: finding T

2

- Note the objective function is symmetric in A and B. So we can solve for A (and it will be similar for B)

$$\frac{\partial\theta}{\partial x_A} = \frac{\left(\bar{R}_A - R_f\right)\sigma_p - \frac{1}{2}\sigma_p^{-1}\left(2x_A\sigma_A^2 + 2x_B\sigma_{AB}\right)\left(\bar{R}_p - R_f(x_A + x_B)\right)}{\sigma^2{}_p}$$

where the blue terms depend on x_A and x_B.

- By setting $\frac{\partial\theta}{\partial x_A} = 0$, and recalling $x_A + x_B = 1$, we get

$$\frac{\left(\bar{R}_A - R_f\right)\sigma \quad -\frac{1}{2}\,\sigma_p^{-1}\left(2x_A\sigma_A^2 + 2x_B\sigma_{AB}\right)\left(\bar{R}_p - R_f\right)}{\sigma^2{}_p} = 0$$

$$\left(\bar{R}_A - R_f\right) - \left(x_A\sigma_A^2 + x_B\sigma_{AB}\right)\frac{\bar{R}_p - R_f}{\sigma_p^2} = 0$$

$$\left(\bar{R}_A - R_f\right) - \frac{\bar{R}_p - R_f}{\sigma_p^2}\left(x_A\sigma_A^2 + x_B\sigma_{AB}\right) = 0$$

Scenario 1: finding T

2

- For any concrete portfolio P (including the optimal/tangent portfolio) the ratio $(\bar{R}_p - R_f)/\sigma_p^2$ is a constant.
- So we can define $\lambda = (\bar{R}_p - R_f)/\sigma_p^2$ and do a change of variable

$$z_i = \lambda x_i \ , \qquad \text{for } i = A, B \ ,$$

  where the z values are proportional to x, but do not add up to 1.
- Using the variable z, $\dfrac{\partial \theta}{\partial x_A} = 0$ becomes

$$\left(\bar{R}_A - R_f\right) - \left(z_A \sigma_A^2 + z_B \sigma_{AB}\right) = 0$$
$$\bar{R}_A - R_f = z_A \sigma_A^2 + z_B \sigma_{AB}$$

- By symmetry, solving the FOC is equivalent to

Two eqns and 2 unknowns: we can solve this!

$$\begin{cases} \bar{R}_A - R_f = z_A \sigma_A^2 + z_B \sigma_{AB} & (1) \\ \bar{R}_B - R_f = z_A \sigma_{AB} + z_B \sigma_B^2 & (2) \end{cases}$$

Scenario 1: finding T

2

- In vector notation is

$$\begin{pmatrix} \bar{R}_A - R_f \\ \bar{R}_B - R_f \end{pmatrix} = \begin{pmatrix} \sigma_A^2 & \sigma_{AB} \\ \sigma_{AB} & \sigma_B^2 \end{pmatrix} \begin{pmatrix} z_A \\ z_B \end{pmatrix}$$

- Denoting $\mathbb{1} = (1,\ 1)'$, we can write the solution as

$$\left(\bar{R} - R_f \mathbb{1}\right) = VZ \quad \Leftrightarrow \quad Z = V^{-1}\left(\bar{R} - R_f \mathbb{1}\right)$$

- Since $Z = \lambda X$, the tangent portfolio weights follow from

$$x_A^T = \frac{z_A}{z_A + z_B} \qquad \text{and} \qquad x_B^T = \frac{z_B}{z_A + z_B},$$

  and we finally reached the composition of the tangent portfolio T,

$$X_T = \begin{pmatrix} x_A^T \\ x_B^T \end{pmatrix}$$

Scenario 1: efficient frontier

3

- The tangent portfolio T, expected return and variance are give by

$$\bar{R}_T = x_A^T \bar{R}_A + x_B^T \bar{R}_B$$

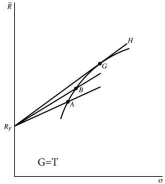
vector notation-> $\bar{R}_T = X_T' \bar{R}$

$$\sigma_T^2 = (x_A^T)^2 \sigma_A^2 + (x_B^T)^2 \sigma_B^2 + 2 x_A^T x_B^T \sigma_{AB}$$

vector notation-> $\sigma_T^2 = X_T' V X_T$

- The efficient frontier contains only combinations of the riskless asset F with the tangent portfolio T.
- Its equation in the $(\sigma, \bar{R})$ space is

$$\bar{R}_p = R_f + \frac{\bar{R}_T - R_f}{\sigma_T} \sigma_p$$



Remark: Compare to the picture on the earlier slide: illustration of scenario 1

Scenario 2: two risky assets and one R_f, no shortselling

What if shortselling is not allowed?

Scenario2

Consider two risky assets A and B with some $\bar{R}_A$, $\bar{R}_B$, σ_A, σ_B and σ_{AB}, and a riskless asset F with return R_f, that can be used to both lending and borrowing. Suppose you are not allowed no shortselling the risky assets A and B.

1. What can you say about the investment opportunity set.
2. What can you say about the tangent portfolio T.
3. What can you say about the efficient frontier.

Scenario 2: investment opportunity set

1. Opportunity set:

- Recall, that the possible combinations of the two risky assets are just a small portion of the hyperbola.
- Including the riskless asset with a fixed R_f to lending and borrowing allows to consider all combinations of the riskless asset with all feasible portfolios.
- The investment opportunity set will be a cone tangent from above and from below to portion of the hyperbola that do not require shortselling.
- It is and open set.

HW: Sketch this in the $(\sigma, \bar{R})$ space.

Scenario 2: tangent portfolio T

2. To find the tangent portfolio T we have the usual problem

$$\max_{x_A, x_B} \quad \theta = \frac{\bar{R}_p - R_f}{\sigma_p}$$

s.t.

$$\bar{R}_p = x_A \bar{R}_A + x_B \bar{R}_B$$
$$\sigma_p = \left(x_A^2 \sigma_A^2 + x_B^2 \sigma_B^2 + 2 x_A x_B \sigma_{AB}\right)^{\frac{1}{2}}$$
$$x_A + x_B = 1$$
$$x_A \geq 0$$
$$x_B \geq 0$$

with two inequality (because we just have two risky assets) additional restrictions.

Scenario 2: tangent portfolio T

The unrestricted solution

- If none of the two "no shortselling restrictions" is binding $\Rightarrow$ the solution to the restricted problem is the solution to the unrestricted problem.

Or, the trivial solution

- If the optimal solution of the unrestricted problem requires shortselling of the risky asset A, then the restrict solution implies zero investment in that asset $x_A = 0$. This holds in general, i.e. also for more than two risky assets, i.e. for $n \geq 2$.
- And the tangent portfolio implies full investment in the other asset B, i.e. trivially $x_B = 1$. This part of the solution only hold for $n = 2$.

OBS: *In the two risky assets case we can deal with shortselling restrictions.*

Scenario 2: efficient frontier

3. Given the tangent portfolio T, the efficient frontier contains only combinations of the riskless asset F with the tangent portfolio T. Its equation in the $(\sigma, \bar{R})$ space is

$$\bar{R}_p = R_f + \frac{\bar{R}_T - R_f}{\sigma_T}\sigma_p$$

Scenario 1+2: up to now ...

We have shown that

- If there is a risk-free asset with a unique rate R_f for both lending and borrowing, the efficient set is a straight line.
- The investment opportunity set is a open cone, with vertice at the riskless asset F and tangent, from above and below, to the hyperbola (or part of the hyperbola) which is the efficient set for the two risky assets.
- This efficient frontier is the straight line passing through the risk-free asset and tangent from above to the hyperbola (or part of the hyperbola).
- The efficient set of portfolios is linear combinations of the risk-free asset and the portfolio where the straight line is tangent from above to the hyperbola.
- That portfolio is called the tangent portfolio. and is the portfolio that maximizes the Sharpe ratio.

Theorem: Efficiency of T after discarding the riskless asset

If we throw away the risk-free asset, it turns out that T is efficient amongst the set of risky assets:

Theorem (Efficiency of T even w/o the riskless asset)

If E is efficient then the portfolio T consisting of the risky assets in E is efficient relative to investing solely in risky assets.

- The intuitive reason this holds is that R_f has no variance or covariance.
- So no diversification effects can come into play.

OBS: If we added or subtracted a RISKY asset, a similar result would NOT hold.

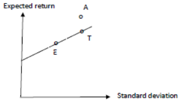
Proof by contra-positive

- We use proof by contra-positive.
- We show that if T is not efficient then E is not efficient.
- This is logically equivalent to the statement that if E is efficient then T is efficient.
- Note, in the proof F is riskless so $\mathbb{E}(R_f) = R_f$.

Logic of proof: If T is not efficient, then either there exists

1. a portfolio of risky assets A with higher return and the same or lower variance,
2. or a portfolio B with the same return and lower variance.

Proof by contra-positive (cont.)



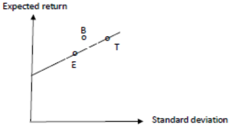
- If T is not efficient, and there exists a portfolio of risky assets A with higher return and the same or lower variance,

$$\begin{aligned}\mathbb{E}(x_f R_f + (1-x_f)R_A) &= x_f R_f + (1-x_f)\bar{R}_A, \\ &> x_f R_f + (1-x_f)\bar{R}_T,\end{aligned}$$

$$\begin{aligned}\mathrm{Var}(x_f R_f + (1-x_f)R_A) &= \mathrm{Var}((1-x_f)R_A), \\ &= (1-x_f)^2\sigma_A^2, \\ &\leq (1-x_f)^2\sigma_T^2.\end{aligned}$$

$\Rightarrow x_f F + (1-x_f)T$ is not efficient i.e. E is not efficinet.

Proof by contra-positive (cont.)



2. If T is not efficient, and there exist a portfolio B with the same expected return and lower variance,

$$\mathbb{E}(x_f R_f + (1 - x_f) R_B) = x_f R_f + (1 - x_f)\bar{R}_B, \\ = x_f R_f + (1 - x_f)\bar{R}_T,$$

$$\mathrm{Var}(x_f R_f + (1 - x_f) R_B) = \mathrm{Var}((1 - x_f) R_B), \\ = (1 - x_f)^2 \sigma_B^2), \\ < (1 - x_f)^2 \sigma_T^2.$$

$\Rightarrow x_f F + (1 - x_f) T$ is not efficient i.e. E is not efficient.

- So if T is not efficient then neither is $x_f F + (1 - x_f) T$.
- Turning this round, $x_f F + (1 - x_f) T$ efficient $\Rightarrow$ T is efficient. ■

Scenario 3: two risky assets and lending at R_f, but no borrowing

What if borrowing for investment in risky assets is not allowed?

Scenario 3

Consider two risky assets A and B with some $\bar{R}_A$, $\bar{R}_B$, σ_A, σ_B and σ_{AB}, and a riskless asset F that can be used only for lending/deposit at rate R_f.

1. What can you say about the investment opportunity set.
2. What can you say about the tangent portfolio T.
3. What can you say about the efficient frontier.

Scenario 3: investment opportunity set

1. Opportunity set:

- We saw the tangency portfolio T corresponds to having all money in the risky assets.
- If we say no borrowing then the investment capital allocation line must terminate at T.
- What we just said about T is actually true for any feasible combination of the two-risky assets.
- So the investment opportunity set is the cone with vertice in the riskless asset up to the hyperbola (to portion of the hyperbola is shortselling is forbidden) and the hyperbola itself (or portion of the hyperbola).

Scenario 3: tangent portfolio T

2. The tangent portfolio T can be determined using the methods describe in Scenario 1 or Scenario 2, depending on whether shortselling is allowed or not.

HW: Explain why this is so.

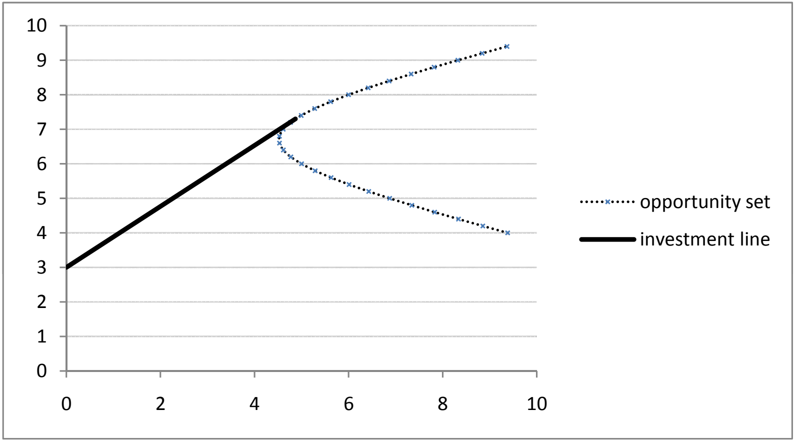
Scenario 3: efficient frontier

3. The efficient frontier when borrowing is not allowed comes in two pieces:
   - The investment line between the riskless asset F and the tangent portfolio T.
   - For volatility levels higher than σ_T, it is described by the upper part of the hyperbola that results from combining the two risky assets.

$$\begin{cases} \text{investment line} & \text{for } \sigma_p < \sigma_T \\ \text{hyperbola} & \text{for } \sigma_p \geq \sigma_T \end{cases}$$

OBS: Note that we already know how to determine both branches of the efficient frontier.

Scenario 3: efficient frontier



Question: Fill in the efficient frontier!

Scenario 4: Different borrowing and lending rates

What if borrowing is possible but only at a rate higher than the lending rate?

Scenario 4

Consider two risky assets A and B with some $\bar{R}_A$, $\bar{R}_B$, σ_A, σ_B and σ_{AB}, and a riskless asset F that can be used for lending at rate R_f^p or borrowing at rate R_f^a, with $R_f^a > R_f^p$.

1. What can you say about the investment opportunity set.
2. What can you say about the tangent portfolios.
3. What can you say about the efficient frontier.

Scenario 4: $R_f^p < R_f^a \Rightarrow$ two tangent portfolios T, T'

- When we buy the risk-less bond we are lending at the risk-less rate to an essentially risk-less counterparty: the government.
- When we borrow, we are charged a risk premium.
- So the borrowing rate R_f^a (active rate) should be higher than the lending rate R_f^p (passive rate).
- To obtain the efficient frontier we have to compute two tangent portfolios – one using the passive rate (T) and another using the active rate (T').

1. We *know hold to find T*.
2. We know how to find T'.

Scenario 4: efficient frontier

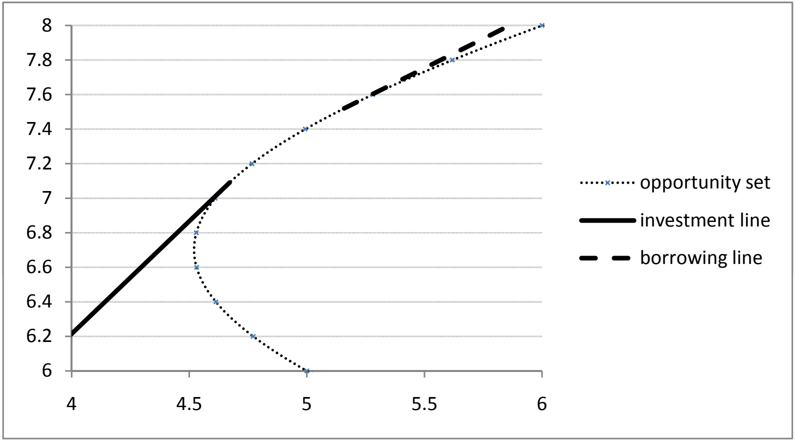
3. The efficient frontier will be in three pieces:

- the straight line to the tangent portfolio with riskless asset with the low lending late ⇒ investment line
- the upper part of the hyperbola between tangency portfolio (T) with the low lending rate and the tangent portfolio with high
- borrowing rate (T'),

  the investment line beyond the tangent portfolio for the high borrowing rate ⇒ borrowing line

$$\begin{cases} \bar{R}_p = R_f^p + \frac{\bar{R}_T - R_f^p}{\sigma_T}\sigma_p & \sigma_p < \sigma_T \\ hyperbola & \sigma_T \leq \sigma_p \leq \sigma_{T'}\ ,\ \bar{R}_p \geq \bar{R}_T \\ \bar{R}_p = R_f^a + \frac{\bar{R}_{T'} - R_f^a}{\sigma_{T'}}\sigma_p & \sigma_p > \sigma_{T'} \end{cases}$$

Efficient frontier with different borrowing and lending rates



Question: Fill in the efficient frontier!

Example: bond and stock funds

Suppose we can invest in a bond fund, B, and an index fund on the stock market, S. We want to find the efficient frontier.

Bonds offer lower returns but also lower volatility than stocks. The two funds are correlated since both are affected by the overall economy.

$$
\begin{aligned}
\bar{R}_S &= 10.3\%, \\
\sigma_S &= 12.2\%, \\
\rho_{S,B} &= 0.34, \\
\bar{R}_B &= 6.2\%, \\
\sigma_B &= 5.5\%
\end{aligned}
$$

Example: finding the minimal variance portfolio

Using the formula for the minimal variance portfolio, we compute

$$x_B = \frac{0.122^2 - 0.122 \times 0.055 \times 0.34}{0.12^2 + 0.055^2 - 2 \times 0.122 \times 0.055 \times 0.34}$$
$$= 0.944.$$

- To get *MV*, one invest 94.4% in bonds and only 5.6% in stocks.
- Substituting, we get that the minimal volatility of 5.46% which is slightly lower than the volatility of just bonds.

Example: adding the riskless investment

Suppose we add in a risk-less investment.

We let $R_f = 5\%$.

To find the tangent portfolio T we must solve:

$$\begin{pmatrix} 0.103 - 0.05 \\ 0.062 - 0.05 \end{pmatrix} = \begin{pmatrix} 0.122^2 & 0.122 \times 0.055 \times 0.34 \\ 0.122 \times 0.055 \times 0.34 & 0.055^2 \end{pmatrix} \begin{pmatrix} z_S \\ z_B \end{pmatrix}$$

$$(\cdots)$$

$$X_T = \begin{pmatrix} x_S^T \\ x_B^T \end{pmatrix} = \begin{pmatrix} 0.697 \\ 0.303 \end{pmatrix}$$

The tangent portfolio has

$$\bar{R}_T = 9.06\%,$$
$$\sigma_T = 9.21\%$$

Example: tangent line and portfolio

The tangent line has slope:

$$\frac{9.06\% - 5\%}{9.21\%} = 0.44,$$

and so its equation is

$$\bar{R}_p = 0.05 + 0.44\ \sigma_p\ .$$

- if instead we start from the above efficient frontier, we can recover the portfolio T composition. We know

$$\bar{R}_T = x_S \bar{R}_S + (1 - x_S)\bar{R}_B,$$

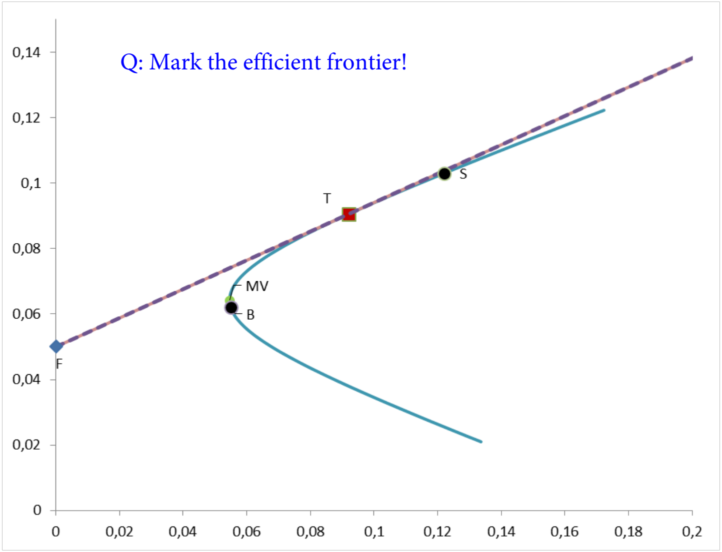
if $\bar{R}_T = 9.06\%$,

$$9.06\% = x_S(10.3\%) + (1 - x_S)6.2\%,$$

which implies

$$x_S = 0.697, \text{ and } x_B = 0.303.$$ Same as before :-)

Example: opportunity set (risky assets) and efficient frontier



Theory questions

1. What does the Tobin separation theorem say?
2. Show that if a portfolio consisting of x units of A and $(1 - x)$ units of the riskless assets is efficient then A is itself efficient amongst solely the risky assets
3. We have two risky assets and one riskless asset, sketch the efficient frontier with and without the riskless asset on the same graph.
4. Describe the shape of the efficient frontier in the space of weights when we have a riskless asset.
5. Show that if there is a risk-free asset, all efficient portfolios lie on a straight line in the standard (deviation, expected return) space.

Theory questions

6. How would borrowing and lending rates vary for most investors?
7. What is the shape of the efficient frontier with two risky assets and one riskless asset, if no borrowing is possible?
8. What is the shape of the efficient frontier with two risky assets and one riskless asset, with different borrowing and lending rates?
9. Describe how to find the weights in all efficient portfolios with two risky assets and different borrowing and lending rates.