Microeconomics

Chapter 3
Profit function

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Profit function

The previous chapter introduced the **profit function**:

$$\pi(\mathbf{p}) = \max_{\mathbf{y}} \mathbf{py}.$$

The **profit function** $\pi(\mathbf{p})$ gives us the maximum profits as a function of the prices: For each **p** it uses the feasible technologically **y** that maximizes profits.

This chapter discusses several (intuitive) insights into the profit function. These insights are often derived via **comparative statics** exercises.

Nondecreasing (nonincreasing) in output (input) prices

What happens to the profit function if we weakly increase all output prices and weakly decrease all input prices?

Nondecreasing in output prices, nonincreasing in input prices: If $p'_i \geq p_i$ for all outputs i and $p'_j \leq p_j$ for all inputs j, then $\pi(\mathbf{p}') \geq \pi(\mathbf{p})$.

Let $\mathbf{y}$ be the profit maximizing net output at price $\mathbf{p}$, so $\pi(\mathbf{p}) = \mathbf{py}$.

Let $\mathbf{y}'$ be the profit maximizing net output at price $\mathbf{p}'$, so $\pi(\mathbf{p}') = \mathbf{p}'\mathbf{y}'$.

By WAPM, we have $\mathbf{p}'\mathbf{y}' \geq \mathbf{p}'\mathbf{y}$.

Since $p'_i \geq p_i$ for all which $y_i \geq 0$ (outputs) and $p'_j \leq p_j$ for all which $y_j \leq 0$ (inputs) we also have $\mathbf{p}'\mathbf{y} \geq \mathbf{py}$.

> This is easier to follow if we clearly distinguish input and output prices. Denote output prices by $\mathbf{p}$, with $\mathbf{p}' \geq \mathbf{p}$ since $p'_i \geq p_i \; \forall \; i$. Denote input prices by $\mathbf{w}$, with $\mathbf{w}' \leq \mathbf{w}$ since $w'_j \leq w_j \; \forall \; j$. Then $\mathbf{p}'\mathbf{y} - \mathbf{w}'\mathbf{x} \geq \mathbf{py} - \mathbf{wx}$.

Putting these two inequalities together, we have that:

$$\pi(\mathbf{p}') = \mathbf{p}'\mathbf{y}' \geq \mathbf{p}'\mathbf{y} \geq \mathbf{py} = \pi(\mathbf{p}).$$

Homogeneous of degree one

What happens to the profit function if we multiply all prices by a factor $t > 0$? So we want to know how $\pi(t\mathbf{p})$ and $\pi(\mathbf{p})$ are related.

Profit function is homogeneous of degree one: If we multiply all prices by $t > 0$, the profit is multiplied by t:

$$\pi(t\mathbf{p}) = t\pi(\mathbf{p}).$$

This statement can be proved with the WAPM logic.

Exercise

Proof that the profit function is homogeneous of degree one.

Demand and supply functions from the profit function

Consider a firm with one input and one output, so the prices are (p, w).

If you were given the factor demand function $x(p, w)$ finding the profit function is easy: just substitute the factor demand function into π:

$$\begin{aligned}\pi(p, w) &= pf(x) - wx \\ &= pf(x(p, w)) - wx(p, w).\end{aligned}$$

It turns out that if you know the profit function, it is also easy to find the factor demand (and supply) function. This is what **Hotelling's lemma** shows us.

Hotelling's lemma

Hotelling's lemma shows that we can find the factor demand and supply function from the profit function as follows:

$$\frac{\partial \pi(p, w)}{\partial w} = -x(p, w).$$
$$\frac{\partial \pi(p, w)}{\partial p} = f(p, w).$$

In words, **Hotelling's lemma** is that:
(1) the derivative of the profit function towards the input price gives us the (negative) factor demand function, and
(2) the derivative of the profit function towards the output price gives us the factor supply function.

Proof Hotelling's lemma for the supply function

We know the profit function $\pi(p, w)$ is given by

$$\pi(p, w) = pf(x(p, w)) - wx(p, w).$$

Note that we write the profit function, after you substituted for $x(p, w)$, as $\pi(p, w)$: when you look at the function, only (p, w) is in it, hence $\pi(p, w)$.

But if we are taking derivatives, it is useful to realize that p affects π directly and indirectly via x. Hence, it is useful to write the profit function as:

$$\pi(p, w) = \pi(p, w, x(p, w)).$$

Then, remember that partially differentiating the profit function, after you substituted for $x(p, w)$, is similar to taking the total derivative towards p,

$$\frac{\partial \pi(p, w, x(p, w))}{\partial p} = \frac{\partial \pi(p, w, x)}{\partial p} + \frac{\partial \pi(p, w, x)}{\partial x} \frac{\partial x(p, w)}{\partial p} = \frac{d\pi(p, w, x)}{dp}.$$

Proof Hotelling's lemma for the supply function

Lets label the two effects of the total derivative as "direct effect" (of p) and "indirect effect" (of p):

$$\frac{\partial \pi(p,w)}{\partial p} = \underbrace{\frac{\partial \pi(p,w,x)}{\partial p}}_{\text{direct effect}} + \underbrace{\frac{\partial \pi(p,w,x)}{\partial x}\frac{\partial x(p,w)}{\partial p}}_{\text{indirect effect}}.$$

Using that the profit function is $\pi(p,w) = pf(x(p,w)) - wx(p,w)$, we can now write the total derivative as:

$$\begin{aligned}\frac{\partial \pi(p,w)}{\partial p} &= f(x(p,w)) + p\frac{\partial f(x)}{\partial x}\frac{\partial x(p,w)}{\partial p} - w\frac{\partial x(p,w)}{\partial p} \\ &= \underbrace{f(x(p,w))}_{\text{direct effect}} + \underbrace{\left(p\frac{\partial f(x)}{\partial x} - w\right)\frac{\partial x(p,w)}{\partial p}}_{\text{indirect effect}}.\end{aligned}$$

Proof Hotelling's lemma for the supply function

The total derivative of the profit function is:

$$\frac{\partial \pi(p,w)}{\partial p} = \underbrace{f(x(p,w))}_{\text{direct effect}} + \underbrace{\left(p\frac{\partial f(x)}{\partial x} - w\right)\frac{\partial x(p,w)}{\partial p}}_{\text{indirect effect}}.$$

From the FOC of profit maximization we know that the indirect effect is zero at $x = x^* = x(p,w)$,

$$p\frac{\partial f(x)}{\partial x} - w = 0.$$

So that only the direct effect remains:

$$\frac{\partial \pi(p,w)}{\partial p} = f(x(p,w)).$$

Exercise

Proof Hotelling's Lemma for the factor demand function. That is, show that

$$\frac{\partial \pi(p, w)}{\partial w} = -x(p, w)$$

The envelope theorem

Hotelling's lemma follows from a more general result known as the envelope theorem.

Envelope theorem: if you want to know how an optimized function (e.g., $\pi(p, w)$) changes when an exogenous variable changes (e.g., p), only the *direct* effect of this exogenous variable needs to be considered, even if the exogenous variable also enters the optimized function *indirectly* as part of the solution to endogenous choice variables (e.g., $x(p)$).

The envelope theorem

Consider an arbitrary maximization problem

$$\max_x f(a, x),$$

where x is the endogenous choice variable and a is the exogenous variable. The solution to this maximization problem is a function $x(a)$: the optimal x as function of the exogenous variable a.

We can substitute $x(a)$ in $f(a, x)$ which will give us the maximum value of f, $f(a, x(a)) = f(a)$, as function of the exogenous variable a.

Note the analogy with Chapter 2: $f(a, x(a))$ is the profit function, $x(a)$ is the factor demand, and a is the exogenous price.

We are often interested in how the maximum value $f(a, x(a))$ changes when a changes. That is, we are interested in:

$$\frac{\partial f(a, x(a))}{\partial a}.$$

This is a similar question to Hotelling's lemma: how do profits change when prices change?

The envelope theorem

Remember that partially differentiating the objective function, after you substituted for $x(a)$, is similar to taking the total derivative,

$$\frac{\partial f(a, x(a))}{\partial a} = \underbrace{\frac{\partial f(a, x)}{\partial a}}_{\text{direct effect}} + \underbrace{\frac{\partial f(a, x)}{\partial x} \frac{\partial x(a)}{\partial a}}_{\text{indirect effect}} = \frac{df(a, x)}{da}.$$

The FOC of maximizing $f(a, x)$ is

$$\frac{\partial f(a, x)}{\partial x} = 0.$$

Hence, we know that the indirect effect is zero at $x = x^* = x(a)$, and only the direct effect remains:

$$\frac{\partial f(a, x(a))}{\partial a} = \frac{\partial f(a, x)}{\partial a}.$$

Intuitively, the partial derivative of $f(a, x(a))$ with respect to a is given by the partial derivative of f with respect to a "holding x fixed at the optimal choice $x(a)$". Indeed, since $x(a)$ is chosen optimally, f will not change when $x(a)$ changes slightly because of a change in a.

Exercise

Consider the function $f(a, x) = ln(x) - ax$.

1. Find the function $x(a)$ that maximizes the function $f(a, x)$.
2. Plug $x(a)$ into $f(a, x)$ to find $f(a, x(a))$. Explicitly show that:

$$\frac{\partial f(a, x(a))}{\partial a} = \frac{\partial f(a, x)}{\partial a}.$$

Homework exercises

Exercises: 3.3, 3.4 (only give the factor demand functions), and exercises on the slides