2.

Mean-Variance Theory (MVT)

Section B

These slides were originally created by R. Gaspar and have in this version been edited and complemented by M. Hinnerich

Mean-Variance Theory

2.5 The General Case

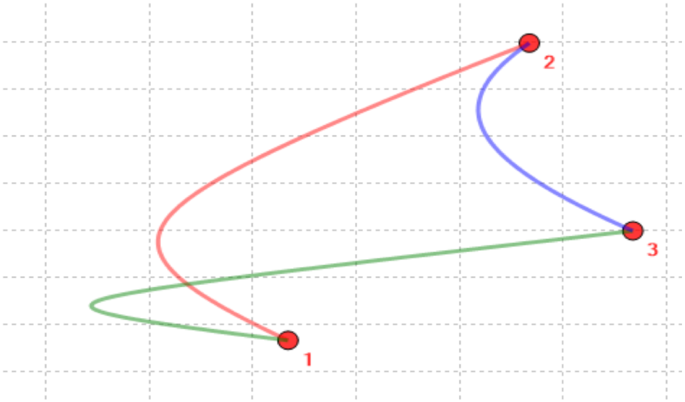
- Learning objectives
- Three or more risky assets
- Solving for the efficient frontier:
  - Tangent portfolios
  - Minimal variance portfolio
  - The envelop hyperbola
- Multi-asset Example
- Questions

Learning objectives

- understand the impact of three or more risky assets in the investment opportunity set and the efficient frontier
- find the efficient frontier under various market conditions
- find the tangent portfolio(s) in the multi-asset case,
- find the minimal variance portfolio in the multi-asset case,
- describe the geometry of the efficient frontier in weight space with and without a risk-free asset,
- solve problems which involve finding a portfolio prescribed expected return or standard deviation in the multi-asset case.

3 risky assets

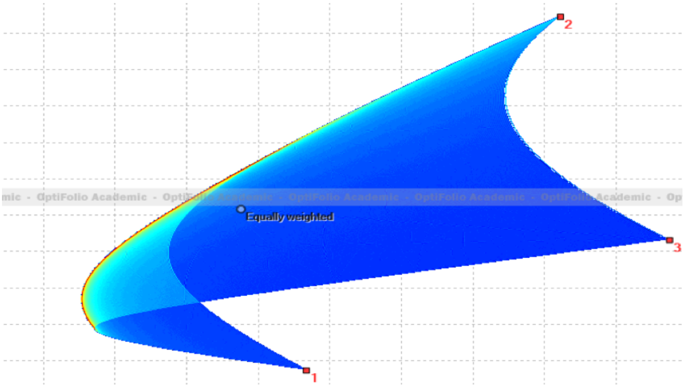
Just pairwise combinations ...



... and assuming shortselling is not allowed.

3 risky assets

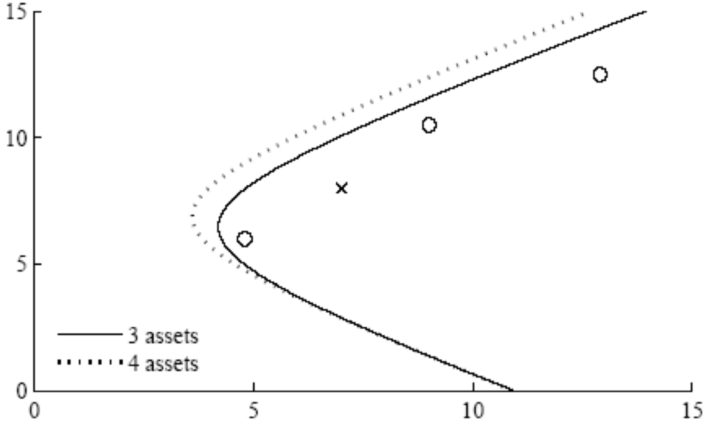
A more realistic picture...



... still assuming shortselling is not allowed.

From 3 to 4 assets ...

Let us just focus on the frontier of the investment opportunity set



Solving for the efficient frontier

We have the consider the same various scenarios:

- Scenario 1: Shortselling allowed + same R_f for lending and borrowing
- Scenario 2: Shortselling not allowed + same R_f
- Scenario 3: Shortselling allowed + no borrowing
- Scenario 4: different deposit and lending rate
- Scenario 5: no riskless asset

$$\Downarrow$$

From before we know it all reduces to be able :

A. determine tangent portfolios.
B. determine the minimum variance portfolio of risky assets.
C. derive the equation of the efficient frontier without the r iskless asset.

A) Tangent portfolio

Recall 2 variables $\max_{x_A, x_B} \quad \theta = \frac{\bar{R}_p - R_f}{\sigma_p}$

$$\bar{R}_p = x_A \bar{R}_A + x_B \bar{R}_B$$
$$\sigma_p = \left(x_A^2 \sigma_A^2 + x_B^2 \sigma_B^2 + 2 x_A x_B \sigma_{AB}\right)^{\frac{1}{2}}$$
$$x_A + x_B = 1$$

- Shortselling allowed

If X is a vector of portfolio weights, $\bar{R}$ is the vector of the assets' expected returns and V is the variance-covariance matrix, we have for all risky portfolios P

$$\bar{R}_p = X'\bar{R}, \text{ and } \sigma_p = (X'VX)^{\frac{1}{2}}.$$

So we must maximize

$$\theta(X) = \frac{X'\bar{R} - R_f}{(X'VX)^{\frac{1}{2}}}$$

subject to $X'\mathbb{1} = 1$.

Tangent portfolio

Recall 2 variables

$$\begin{pmatrix} \bar{R}_A - R_f \\ \bar{R}_B - R_f \end{pmatrix} = \begin{pmatrix} \sigma_A^2 & \sigma_{AB} \\ \sigma_{AB} & \sigma_B^2 \end{pmatrix} \begin{pmatrix} z_A \\ z_B \end{pmatrix}$$

$$\left(\bar{R} - R_f \mathbb{1}\right) = VZ \quad \Leftrightarrow \quad Z = V^{-1}\left(\bar{R} - R_f \mathbb{1}\right)$$

$$x_A^T = \frac{z_A}{z_A + z_B} \quad \text{and} \quad x_B^T = \frac{z_B}{z_A + z_B},$$

- From 2 variables case we know:
- that solving the the FOC is equivalent to solving the system
- $$\left(\bar{R} - R_f \mathbb{1}\right) = VZ \quad \Leftrightarrow Z = V^{-1}\left(\bar{R} - R_f \mathbb{1}\right)$$

  where $Z = \lambda X$ with λ constant.

- So, from Z we can obtain the individual weights x_i of the tangent portfolio as

$$x_i = \frac{z_i}{\sum_{j=1}^{n} z_j} \qquad X = (Z'\mathbb{1})^{-1} Z \; .$$

Tangent portfolio

Note that if we set

$$\mathbb{1}' = (1, 1, \ldots, 1),$$

we can rewrite the algorithm as

- Set
$$\tilde{R} = \bar{R} - R_f \mathbb{1},$$
- then
$$Z = V^{-1}\tilde{R} \quad .$$
- and
$$X = (Z'\mathbb{1})^{-1} Z .$$

Tangent portfolio

- Shortselling not allowed

As before, we must maximize

$$\theta(X) = \frac{\bar{R}_p - R_f}{\sigma_p} = \frac{X'\bar{R} - R_f}{(X'VX)^{\frac{1}{2}}}$$

such that

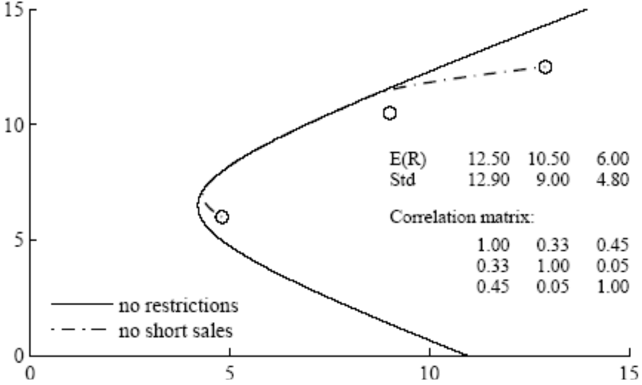
$$\sum x_i = X'\mathbb{1} = 1,$$
$$x_i \geq 0 \quad \text{for all} \quad i = 1, 2, \cdots, n \ .$$

$$\Downarrow$$

Additional n inequality restrictions.
We have to rely on numerical solutions.

No shortselling illustration

Impact of no shortselling on the frontier of the investment opportunity set



Tangent portfolios

If we know the solution to the unrestricted problem
(when shorselling is allowed)
$\Downarrow$
we already know some results about the solution to the restricted problem
(when shortselling is forbidden).

- if the unrestricted solution requires no shortselling positions that is also the solution to the restricted problem.
- a short position in the unrestricted tangent portfolio implies no investment in the restricted tangent portfolio.
- a long position in the unrestricted tangent portfolio does NOT imply long position in the restricted tangent portfolio.

Tangent portfolios

- Shortselling allowed, but restricted *a la* Lintner

- Lintner definition of portfolio:

$$\sum_{i=1}^{n} |x_i| = 1$$

Q: How can this be connected to shortselling restrictions?

- For this portfolio definition the problem becomes

$$\theta(X) = \frac{X'\bar{R} - R_f}{(X'VX)^{\frac{1}{2}}}$$

subject to $\sum |x_i| = 1$.

Tangent portfolios

- Lintner solution:
  1. Convince yourself that from the FOC we get the same vector Z as in the unrestricted problem (Appendix 6A):
  $$Z = V^{-1}\left(\bar{R} - R_f \mathbb{1}\right) \ .$$
  2. Lintner weights for the risky assets can, thus, easily be obtained by
  $$x_i = \frac{z_i}{\sum_{i=1}^{n} |z_i|}$$
  3. What is not invested in risky assets is assumed to be invested in the risk-free asset
  $$x_f = 1 - \sum_{i=1}^{n} x_i$$

Tangent portfolios

- Real-life Shortselling limits

We must maximize

$$\theta(X) = \frac{X'\bar{R} - R_f}{(X'VX)^{\frac{1}{2}}}$$

subject to

$$\sum x_i = 1,$$

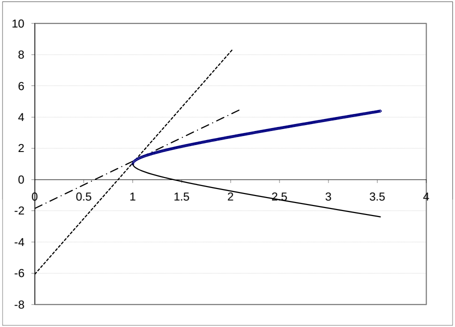
$$x_i \geq -c_i \ , \quad \text{for all} \quad i = 1, 2, \cdots, n$$

$$\sum_{x_i<0} x_i \leq -C$$

$$\left(\vdots \right)$$

for $c_1, c_2, \cdots, c_n$ and C positive constants.⇒ Numerical Solutions

B) Minimum variance portfolio



- as the risk-free rate gets lower and lower, the slope of the investment line gets steeper and steeper, and the tangent portfolio gets closer and closer to the tip.
- So we can find the weights in the minimal variance portfolio by letting the risk-free rate tend to minus infinity.

Minimum variance portfolio

- When there are no shortselling restrictions, we can write the tangent portfolio weights , as

$$X_T = \frac{Z}{Z'\mathbb{1}} = \frac{V^{-1}\left[\bar{R} - R_f \mathbb{1}\right]}{\mathbb{1}'V^{-1}\left[\bar{R} - R_f \mathbb{1}\right]} = \frac{V^{-1}\bar{R} - R_f V^{-1}\mathbb{1}}{\mathbb{1}'V^{-1}\bar{R} - R_f(\mathbb{1}'V^{-1}\mathbb{1})} = \frac{-R_f^{-1}V^{-1}\bar{R} + V^{-1}\mathbb{1}}{-R_f^{-1}(\mathbb{1}'V^{-1}\bar{R}) + (\mathbb{1}'V^{-1}\mathbb{1})}$$

Letting $R_f \to -\infty$ this converges to

$$X_{MV} = \frac{V^{-1}\mathbb{1}}{\mathbb{1}'V^{-1}\mathbb{1}} ,$$

so, we can find the minimal variance portfolio with n assets easily.

Minimum variance portfolio

- Alternatively, one could explicitly solve the optimization problem:

$$\min_{X} \quad \sigma_p^2 = X'VX$$
$$\text{s.t.} \quad X'\mathbb{1} = 1,$$

using the Lagrangean to get the same solution as on previous slide.

- In the case of no shortselling or real-life shortselling restrictions we would need to include additional short selling conditions and solve the problem numerically.

C) Efficient risky portfolio for fixed $\bar{R}_P$

Consider only the n risky assets.

Often we are given a predetermined level of expected return $\bar{R}_P$ and our task is to find, among all risky portfolios with that specific expected return, the only efficient one.

I.e, we need to solve the optimization problem:

$$\begin{aligned} \min_X \quad & \sigma_p^2 = X'VX \\ \text{s.t.} \quad & X'\mathbb{1} = 1 \\ & X'\bar{R} = \bar{R}_p^* \ , \end{aligned}$$

OBS: There is only two equality restrictions. So, the problem can be solved for instance using Lagrange.

The Envelope Hyperbola Result

Result (envelope)

When there are n risky assets, the efficient frontier is the upper-part of some enveloping Hyperbola.

To get the exact expression of an hyperbola it is enough to know two portfolios on that hyperbola and their return covariance.

The envelop Hyperbola

"Two tangents strategy" to find the outer hyperbola:

1. Choose two fictitious values for the return of the riskless asset, R_h and R_g
2. Find the two tangent portfolios, H and G associated with each of the fictitious riskless returns.
3. Determine $\bar{R}_H, \bar{R}_G, \sigma_H, \sigma_G, \sigma_{HG}$
4. Derive the expression for the hyperbola that represents all combinations of H and G. This is the envelop hyperbola!

$$\Downarrow$$

That hyperbola is nothing but our Envelop Hyperbola!

The envelop Hyperbola

For the case of unlimited shortselling be get:

$$\sigma_P^2 = \frac{A\bar{R}_P^2 - 2B\bar{R}_P + C}{AC - B^2}$$

where A, B, C are the scalars

$$A = \mathbb{1}'V^{-1}\mathbb{1} \qquad B = \mathbb{1}'V^{-1}\bar{R} \qquad C = \bar{R}'V^{-1}\bar{R}\ .$$

Using this simpler notation the minimum variance portfolio is

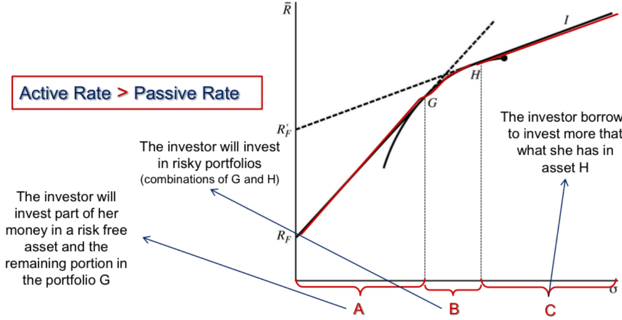
$$X_{MV} = \frac{1}{A}V^{-1}\mathbb{1}$$

OBS: *For a particular instance with $n \geq 3$ check that the hyperbola you get from the above expression is the same as the hyperbola you get from the previous slide "two tangents strategy".*

The two-fund theorem

Theorem

Two efficient funds (portfolios) can be established so that any efficient portfolio can be duplicated, in terms of mean and variance, as a combination of these two. In other words, all investors seeking efficient portfolios need only invest in combinations of these funds.



Example

We are given

Asset	$\bar{R}$	σ
A	15%	10%
B	10%	6%
C	20%	15%
R_f	3%	

and pairwise correlations $\rho_{AB} = 0.4, \rho_{BC} = 0.3$, and $\rho_{AC} = 0.5$.

Setup A

- there is a single risk-free rate R_f for both lending and borrowing,
- shortselling is allowed.

Example

To get the covariance matrix, we multiply each element of the correlation matrix by the standard deviations for each of the corresponding assets:

$$\begin{pmatrix} 1 \times (10\%)^2 & 0.4 \times 10\% \times 6\% & 0.5 \times 10\% \times 15\% \\ 0.4 \times 10\% \times 6\% & 1 \times (6\%)^2 & 0.3 \times 6\% \times 15\% \\ 0.5 \times 10\% \times 15\% & 0.3 \times 6\% \times 15\% & 1 \times (15\%)^2 \end{pmatrix}$$

$$= \begin{pmatrix} 0.01 & 0.0024 & 0.0075 \\ 0.0024 & 0.0036 & 0.0027 \\ 0.0075 & 0.0027 & 0.0225 \end{pmatrix}$$

Example

We have

$$\bar{R} = \begin{pmatrix} 15\% \\ 10\% \\ 20\% \end{pmatrix}, \qquad R_f = 3$$

this implies that

$$\tilde{R} = \begin{pmatrix} 15\% - 3\% \\ 10\% - 3\% \\ 20\% - 3\% \end{pmatrix} = \begin{pmatrix} 12\% \\ 7\% \\ 17\% \end{pmatrix}.$$

The equation to solve is $\quad V\, Z = \tilde{R}$.

$$\begin{pmatrix} 0.01 & 0.0024 & 0.0075 \\ 0.0024 & 0.0036 & 0.0027 \\ 0.0075 & 0.0027 & 0.0225 \end{pmatrix} \begin{pmatrix} z_A \\ z_B \\ z_C \end{pmatrix} = \begin{pmatrix} 0.12 \\ 0.07 \\ 0.17 \end{pmatrix}.$$

Solving...

You can solve this using Gaussian elimination or as before

You can solve it using:

$$Z = V^{-1}\tilde{R} \quad .$$

In excel you get the 3 by 3 inverse "=MINVERSE(a1:c3)"

In excel to multiply a 3 row vector (a1, b1, c1) with a 3 column vector you can use "=MMULT(a1:c1;d1:d3)"

The inverse is:

$$\begin{pmatrix} 146.77 & -67.20 & -40.86 \\ -67.20 & 336.02 & -17.92 \\ -40.86 & -17.92 & 60.22 \end{pmatrix}$$

Example

The solution is

$$\begin{pmatrix} z_A \\ z_B \\ z_C \end{pmatrix} = \begin{pmatrix} 5.962 \\ 12.410 \\ 4.079 \end{pmatrix}$$

We need the weights x_i to add up to one.

Since we know the weights x_i are proportional to z_i, and $\sum z_i = 22.45$, we just need to compute

$$X_T = \begin{pmatrix} x_A^T \\ x_B^T \\ x_C^T \end{pmatrix} = \begin{pmatrix} \frac{z_A}{\sum z_i} \\ \frac{z_B}{\sum z_i} \\ \frac{z_C}{\sum z_i} \end{pmatrix} = \begin{pmatrix} 0.2656 \\ 0.5528 \\ 0.1817 \end{pmatrix}$$

Example

Now we can calculate the standard deviation $\sigma_T = \sqrt{X_T' V X_T} = 6.722\%$ and the expected return is $\bar{R}_T = X_T' \bar{R} = 13.145\%$.

The efficient line goes through F and T, i.e in the space $(\sigma, \bar{R})$ passes the points

$$(0, 3\%) \text{ and } (6.722\%, 13.145\%).$$

The slope of the line is $\dfrac{0.13145 - 0.03}{0.06722} = 1.509$.

The efficient frontier has equation

$$\bar{R}_p = 0.03 + 1.509\ \sigma_p,$$

and we are done!

All efficient portfolios can be seen as combinations of F and T.

Example - variation

What if shortselling is not allowed?

$\Downarrow$

Setup B

- there is a single risk-free rate R_f for both deposit and lending,
- shortselling not allowed.

OBS: Given the data in our Example, this is trivial!
Why?

Example - another variation

What if lending is possible at R_f but not borrowing ?

$$\Downarrow$$

Setup C

- riskless rate R_f only available for lending.
- shortselling allowed.

- The tangent portfolio is the same, but for volatility levels higher than $\sigma_T = 6.722\%$ it is not efficient to invest in the riskless asset.
- The efficient portfolios for higher volatilities lie on the hyperbola (just risky assets).

Example

To get the hyperbola equation we can use

$$\sigma_P^2 = \frac{A\bar{R}^2 - 2B\bar{R} + C}{AC - B^2}\sigma_P$$

and for our case we have

$$A = \mathbb{1}'V^{-1}\mathbb{1} = 291.039$$
$$B = \mathbb{1}'V^{-1}\bar{R} = 31.1828$$
$$C = \bar{R}'V^{-1}\bar{R} = 3.8866$$

And we can conclude our efficient frontier is

$$\begin{cases} \bar{R}_p = 0.03 + 1.509\ \sigma_p & \sigma_p < 6.722\% \\ \sigma_p^2 = 1.8327\bar{R}_p^2 - 0.3927\bar{R}_p + 0.0245 & \sigma_p \geq 6.722\%\ ,\ \bar{R}_p \geq 13.245\% \end{cases}$$

Example - yet another variation

What if the active riskless rate differs from the passive riskless rate?

$\Downarrow$

Setup D

- active riskless rate R_f^a differs from the passive riskless rate R_f^p,
- shortselling allowed.

Let us keep $R_f^p = 3\%$ and set $R_f^a = 7\%$.

- The tangent portfolio T was found maximizing the slope $\frac{\bar{R}_P - 3\%}{\sigma_P}$;
- We now need to find the second tangent portfolio T' and, thus, maximize $\frac{\bar{R}_P - 7\%}{\sigma_P}$

Example

Solving for T'

$$Z = V^{-1}\left[\bar{R} - R_f \mathbb{1}\right] = \begin{pmatrix} 4.4140 \\ 2.3746 \\ 4.0215 \end{pmatrix}$$

Since we know the weights x_i are proportional to z_i, and $\sum z_i = 10.81$, we just need to compute

$$X_{T'} = \begin{pmatrix} 0.4083 \\ 0.2197 \\ 0.3720 \end{pmatrix}$$

and for our second tangent portfolio we have $\sigma_{T'} = \sqrt{X'_{T'} V X_{T'}} = 9\%$, $\bar{R}_{T'} = X'_{T'} \bar{R} = 15.76\%$.

The straight line passing through $(0, R_f^a)$ and $(\sigma_{T'}, \bar{R}_{T'})$ is:

$$\bar{R}_p = 0.07 + 0.9732\ \sigma_p$$

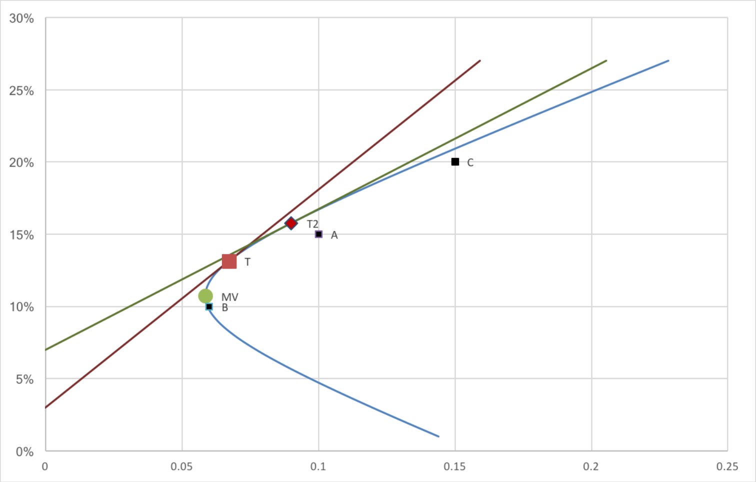
Example

The efficient frontier comes in three pieces

$$\begin{cases} \bar{R}_p = 0.03 + 1.509\ \sigma_p & \sigma_p < 6.722\% \\ \sigma_p^2 = 1.8327\bar{R}_p^2 - 0.3927\bar{R}_p + 0.0245 & 6.722\% \leq \sigma_p \leq 9\%\ , \\ & \bar{R}_p \geq 13.245\% \\ \bar{R}_p = 0.07 + 0.9732\ \sigma_p & \sigma_p > 9\% \end{cases}$$

HW: Determine the efficient portfolios with $\bar{R}_p = 10\%$, 15% *or* 20%*?*

Example



HW: Check that although close, asset B does not belong to the hyperbola. Even it would belong, it would not be efficient. Why?

Theory questions

1. What data is required to compute tangent portfolios?
2. Give the algorithm for finding the tangent portfolio.
3. Give the algorithm for finding the minimal variance portfolio.
4. How do the risky assets investment opportunities set looks like in $\left(\sigma, \bar{R}\right)$ space for $n \geq 3$?
5. What shape does the efficient frontier take if there are $n \geq 3$ risky assets and no-risk-free asset in weight space and in $\left(\sigma, \bar{R}\right)$ space?
6. What shape does the efficient frontier take if there are $n \geq 3$ risky assets and a risk-free asset in weight space and in $\left(\sigma, \bar{R}\right)$ space
7. How does shortselling constraints affect the risky assets investment opportunity set?
8. What is the connection of Lintner definition of a portfolio with shortselling restrictions?

2.6 Portfolio Protection

- Learning objectives
- Safety criteria
- Roy criteria
- Kataoka criteria
- Telser criteria
- Mean-variance representation
- Questions

Learning objectives

- Understand the role of portfolio protection in portfolio management
- Identify and interpret the safety criteria of Roy, Kataoka and Telser
- For normally distributed returns and pre-defined market conditions :
  - represent safety criteria in the plane $(\sigma, \bar{R})$
  - determine and compare the optimal portfolios of Roy, Kataoka and Telser.

Safety criteria

- To evaluate portfolio risk we may be interested in knowing more than just its volatility.
- Many times criteria of some sort of portfolio protection are imposed by managers and/or investors.
- In typical situations one may wish to exclude from the analysis portfolios that do not satisfy some safety criteria.
- The notion of "safety" refer to a wish to limit risk of bad outcome:
  - minimize the probability of r eturns below a give threshold R_L;
  - maximize return in the worst α% worst scenarios;
  - exclude from the analysis all portfolios that have a probability higher than α% of r eturns below a given threshold R_L.

Safety criteria

- For a portfolio p, with $\Phi(\cdot)$ the distribution function of the portfolio returns R_p ., we have:

$$\begin{aligned} \Pr(R_p < R_L) &= \Pr\left(\frac{R_p - \bar{R}_p}{\sigma_p} < \frac{R_L - \bar{R}_p}{\sigma_p}\right) \\ &= \Pr\left(z < \frac{R_L - \bar{R}_p}{\sigma_p}\right) \\ &= \Phi\left(\frac{R_L - \bar{R}_p}{\sigma_p}\right) \ . \end{aligned}$$

Roy criterion

- An investor may wish to minimize the risk of returns below a pre-defined threshold R_L.
- According to this criterion the best portfolio is the one that solves:

$$\min_p \Pr(R_p < R_L)$$

- The threshold is pre-determined, it can take all sort of values:

$$R_L = \cdots, -10\%, \cdots, 0, \cdots, R_f, \cdots, 5\%, \cdots$$

-

Roy criterion

- No matter the distribution of portfolio returns Φ, we have

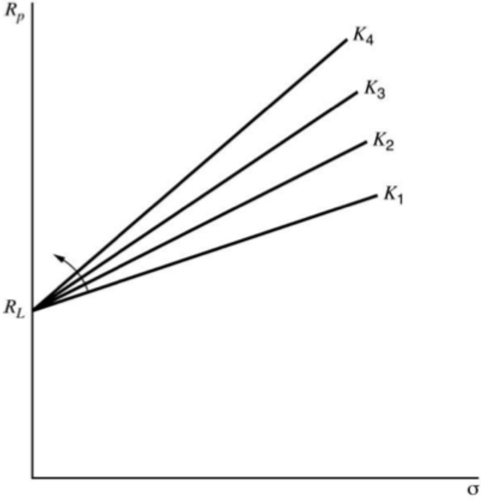
$$\min_p \Pr(R_p < R_L) \Leftrightarrow \min_p \Phi\left(\frac{R_L - \bar{R}_p}{\sigma_p}\right) \Leftrightarrow$$
$$\Leftrightarrow \min_p \frac{R_L - \bar{R}_p}{\sigma_p} \Leftrightarrow \max_p \frac{\bar{R}_p - R_L}{\sigma_p}$$

- Finding the safest portfolio according to Roy is, thus finding p that maximizes the ratio $\frac{\bar{R}_p - R_L}{\sigma_p}$.

OBS: For $R_L = R_f$ what is this...

Roy criterion: MV representation

- The safest Roy portfolio is the one with highest slope.



Kataoka criterion

- Alternatively, one can define bad outcomes in terms of the likelihood of their occurrence.
- One may be worried about what happens in the $\alpha\%$ worst scenarios

$$\max_p R_L$$
$$\text{s.t.} \quad \Pr(R_p < R_L) \leq \alpha\%$$

- The focus this time is on what, unlikely bad scenarios, may mean.
- Note that the higher the R_L of a given portfolio the safer it is, in the senses losses are no as severe as in portfolios with a lower R_L.

Kataoka criterion -portfolio comparison

Kataoka criterion

- For any portfolio returns with distribution function Φ, we get

$$\Pr(R_p < R_L) \leq \alpha\% \quad \Leftrightarrow \quad \Phi\left(\frac{R_L - \bar{R}_p}{\sigma_p}\right) \leq \alpha\%$$

$$\Leftrightarrow \quad \frac{R_L - \bar{R}_p}{\sigma_p} \leq \Phi^{-1}(\alpha\%) \quad \Leftrightarrow \quad R_L \leq \Phi^{-1}(\alpha\%)\sigma_p + \bar{R}_p$$

$$\bar{R}_p \geq R_L - \Phi^{-1}(\alpha\%)\sigma_p$$

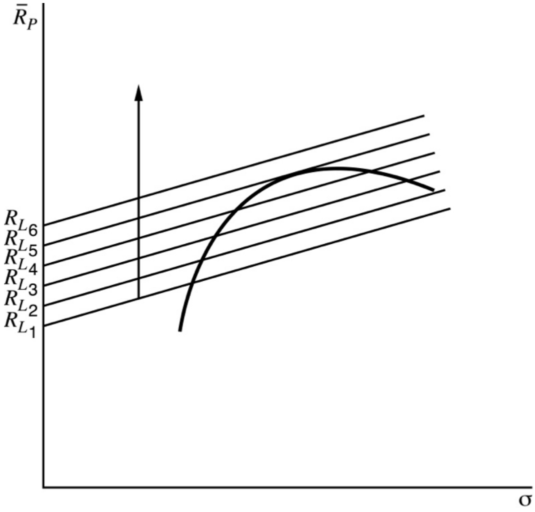
- I.e., for each portfolio p the best we can do is to choose

$$R_p = R_L - \Phi^{-1}(\alpha\%)\sigma_p$$

- Remember that in the plane $\left(\sigma, \bar{R}\right)$, these are represented by straight lines, where R_L are the $y-$crosses.

Kataoka criterion: MV representation

- The safest Kataoka portfolio is the one with highest $y-$cross.



Telser criterion

- If safety is defined *a la* Telser than one pre-defines both:
  - what are bad outcomes, fixing R_L
  - what is highest likelihood acceptable for those bad outcomes $\alpha\%$
- For given R_L and $\alpha\%$, acceptable portfolios are only those that verify

$$\Pr\left(R_p \leq R_L\right) \leq \alpha\%$$

- From all portfolios that satisfy the above condition and since risk has already been taken into account, Telser recommends to choose the one with the highest expected return.
- Telser criterion is thus

$$\begin{aligned} &\max_p \bar{R}_p \\ &\text{s.t.} \quad \Pr\left(R_p \leq R_L\right) \leq \alpha\% \end{aligned}$$

Telser criterion: Gaussian returns

- For Gaussian returns, we already know

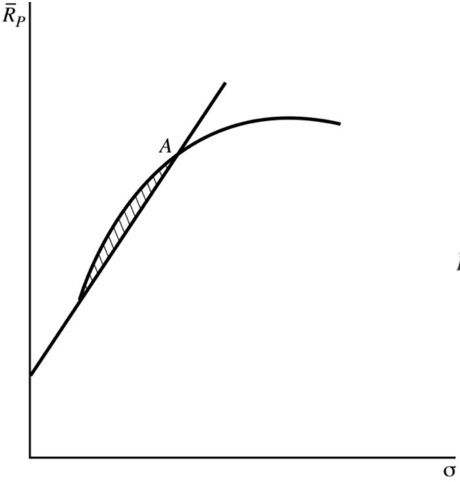
$$\Pr\left(R_p \leq R_L\right) \leq \alpha\% \Leftrightarrow R_L \leq \Phi^{-1}(\alpha\%)\sigma_p + \bar{R}_p \Leftrightarrow \bar{R}_p \geq \underbrace{R_L - \Phi^{-1}(\alpha\%)\sigma_p}_{\text{straight-line equation}}$$

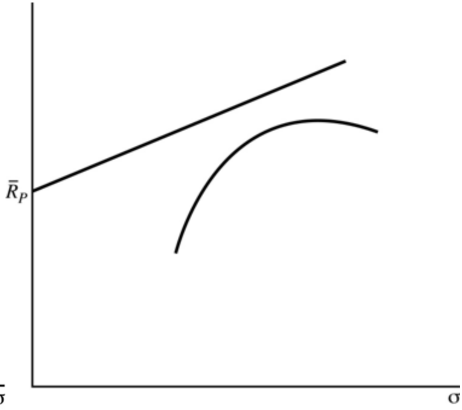
- In the $\left(\sigma, \bar{R}\right)$ plane, Telsser safe portfolios are those above a pre-determined straight-line since we fix both $y-$cross and slope.

Telser criterion: Gaussian returns MV representation

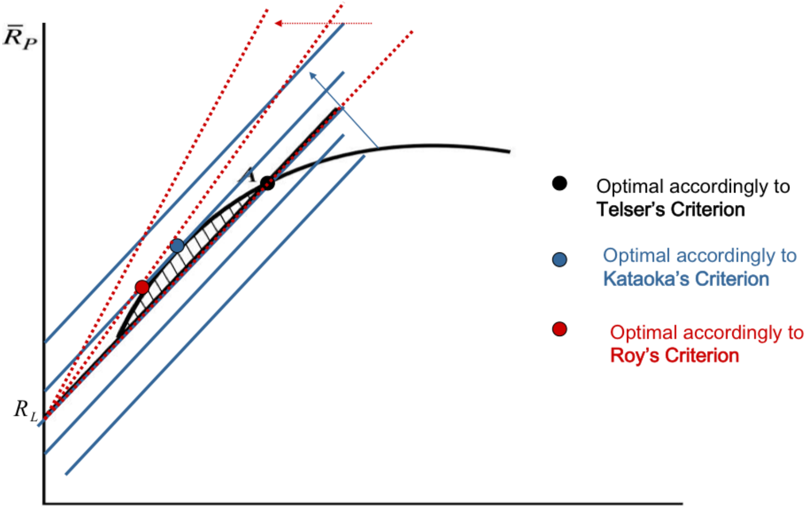
- Either we get:

A set of acceptable portfolios OR No acceptable portfolios





Safety-first criteria: MV comparison



Safety-first criteria: MV comparison

- The criteria definitions are independent of our market setup.
- I.e., for any investment opportunity set and associated efficient frontier, one can always determine the safest portfolios according to Roy, Kataoka and Telser.
- In the slides above the criteria were explained considering as investments opportunity sets of just risky assets.
- Whenever the riskless asset exists, some of the solutions to safety first criteria may be trivial.

Questions

- Why is portfolio protection important?
- What are the similarities and differences between the safety criteria of Roy, Kataoka and Telser?
- In general are safety criteria mean-variance efficient? Why or why not?
- For Gaussian returns, how to represent the Roy criterion in the $(\sigma, \bar{R})$ plane? What gets to be pre-determined?
- For Gaussian returns, how to represent the Kataoka criterion in the $(\sigma, \bar{R})$ plane? What gets to be pre-determined?
- For Gaussian returns, how to represent the Telser criterion in the $(\sigma, \bar{R})$ plane?
- For Gaussian returns, how to compare the safest portfolios of Roy, Kataoka and Telser?

2.7 International Diversification

- Learning objectives
- International correlations
- Exchange rate risk
- The world portfolio
- Questions

Learning Objectives

- Discuss the advantages and disadvantages of including foreign assets in portfolios.
- Compute domestic returns of a foreign asset.
- Understand how exchange risk affect the expected returns, variances and covariances of returns.
- Explain how international diversification may change the investment opportunity set and the associated efficient frontier.
- Define and determine the world portfolio.

International investments

Most portfolio managers have for decades routinely invested a large fraction of their portfolio in securities that were issued in other countries or in foreign currency.

Hence it is important to know how a world market will affect

- The allocation decision
- The investment opportunity set
- The efficient frontier
- The optimal portfolio decision

The allocation decision: international correlations

On the one hand, inclusion of foreign assets is good because

- It augments the investment opportunity set.
- Correlations across returns from different countries tend to be lower than domestic correlations. => from a diversification point of view, we want a portfolio with the lowest possible average correlation.

	Australia	Austria	Belgium	Canada	France	Germany	Hong Kong	Italy	Japan	Netherlands	Spain	Sweden	Switzerland	United Kingdom	United States
Australia															
Austria	0.279														
Belgium	0.304	0.459													
Canada	0.608	0.316	0.299												
France	0.400	0.505	0.677	0.465											
Germany	0.393	0.671	0.612	0.454	0.749										
Hong Kong	0.501	0.350	0.225	0.572	0.387	0.395									
Italy	0.248	0.358	0.396	0.361	0.487	0.495	0.231								
Japan	0.430	0.245	0.317	0.355	0.415	0.307	0.289	0.330							
Netherlands	0.480	0.578	0.738	0.514	0.758	0.740	0.424	0.429	0.432						
Spain	0.460	0.422	0.523	0.455	0.681	0.606	0.415	0.575	0.482	0.599					
Sweden	0.490	0.364	0.348	0.486	0.600	0.639	0.393	0.480	0.461	0.577	0.693				
Switzerland	0.363	0.530	0.610	0.410	0.598	0.537	0.327	0.304	0.465	0.697	0.567	0.494			
United Kingdom	0.543	0.519	0.577	0.460	0.642	0.594	0.437	0.313	0.474	0.722	0.602	0.523	0.494		
United States	0.505	0.281	0.504	0.709	0.534	0.489	0.491	0.301	0.348	0.592	0.530	0.466	0.523	0.646	
Average Correlation Coefficient				0.475											

The allocation decision: exchange rate risk

On the other hand, inclusion of foreign assets is bad because

- Foreign assets bear exchange rate risk.
- Exchange rates affect: expected returns, volatilities and even correlations.
- The same set of basic assets A, B, C, D may have very different representations in the planes:

$$(\sigma, \bar{R})^{€} \qquad (\sigma, \bar{R})^{\$} \qquad (\sigma, \bar{R})^{¥} \qquad \cdots$$

Investing in a foreign asset

Foreign assets can be understood as portfolios of

- The foreign currency
- The asset its self (as it would be seen by a domestic investor)

Take the case of an European investor, going long on a US stock:

$$W_0^{€} \rightarrow W_0^{\$} = W_0^{€} \times E_0^{\$/€} \rightarrow W^{\$} = (1+R^{\$})W_0^{\$} \rightarrow W^{€} = \frac{W^{\$}}{E^{\$/€}}$$

$$\begin{aligned}(1+R^{€})W_0^{€} &= W^{€} \\ &= \frac{W^{\$}}{E^{\$/€}} \\ &= \frac{(1+R^{\$})W_0^{\$}}{E^{\$/€}} \\ &= \frac{(1+R^{\$})W_0^{€} \times E_0^{\$/€}}{E^{\$/€}}\end{aligned}$$

Investing in a foreign asset

Using $E^{\$/€} = 1/E^{€/\$}$ and $\mathrm{E}^{\mathrm{E}/\$} / E_0^{\mathrm{E}/\$} = (1 + R^{\mathrm{E}/\$})$:

$$1 + R^{€} = (1 + R^{\$})(1 + R^{€/\$})$$

The expected return in euros is thus

$$1 + \bar{R}^{€} = \underbrace{\mathbb{E}\left[(1 + R^{\$})(1 + R^{€/\$})\right]}_{\text{product} \rightarrow \text{covariance dependent}}$$

- The €– expected return ($\bar{R}^{€}$) of investing in a \$ – denominated asset, depends on the covariance between returns of exchange rates and returns in the foreign stock market.
- The same is true for variances any any covariances.

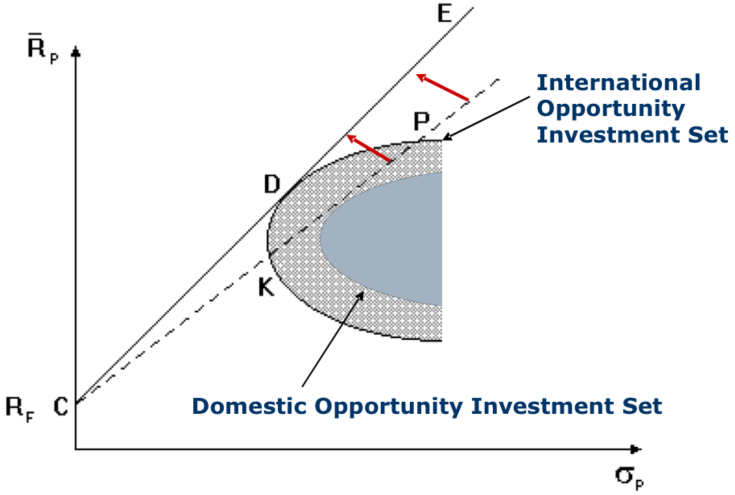
Investing in a foreign asset: Example

Taking the perspective of a US investor:

Stocks	Domestic Risk	Exchange Risk	Total Risk
Australia	13.94	8.66	17.92
Austria	24.80	10.59	24.50
Belgium	16.15	10.21	15.86
Canada	15.02	4.40	17.13
France	18.87	10.61	17.76
Germany	20.41	10.55	20.13
Hong Kong	29.75	0.43	29.79
Italy	24.55	11.13	25.29
Japan	22.04	12.46	25.70
Netherlands	16.04	10.59	15.50
Spain	22.99	11.18	23.27
Sweden	24.87	11.18	24.21
Switzerland	17.99	11.61	17.65
U.K.	14.45	10.10	15.59
United States	13.59	0.00	13.59
Equally Weighted Index (Non-U.S.)	21.57	10.03	23.43
Value-Weighted Index (Non-U.S.)			16.70

OBS: Notice that risk must be computed from the investor point of view, including exchange risk and its possible covariance with market risk.

The Investment Opportunity Set



The World Portfolio: Example

Again from the perspective of a US investor:

Area or Country	Percent of Total[a]
Austria	0.1%
Belgium	0.4%
Denmark	0.4%
Finland	1.6%
France	5.5%
Germany	4.3%
Ireland	0.2%
Italy	2.1%
Netherlands	2.5%
Norway	0.2%
Portugal	0.2%
Spain	1.3%
Sweden	1.6%
Switzerland	2.8%
U.K.	9.7%
Europe	**32.8%**
Australia	1.1%
Hong Kong	1.0%
Japan	12.6%
Malaysia	0.5%
New Zealand	0.1%
Singapore	0.4%
Pacific	**15.5%**
Canada	2.1%
United States	49.5%
North America	**51.6%**
Total	**100.0%**

Questions

- Explain how lower average correlations between assets denominated in different currencies may affect the allocation decision?
- How does the inclusion of foreign assets influence:
  - the determination of mean-variance inputs?
  - the investment opportunity set?
  - the efficient frontier?
- Will two investors facing the same set of assets denominated in a variety of currencies always choose the same world portfolio? Why or why not?