Chapter 4. Exercise on slide 14.

Set up the Lagrangian:

$$L = w_1 x_1 + w_2 x_2 - \lambda(x_1^\rho + x_2^\rho - y^\rho)$$

Take FOCs:

$$\frac{\partial L}{\partial x_1} = w_1 + \lambda\rho x_1^{\rho-1} = 0$$

$$\frac{\partial L}{\partial x_2} = w_2 + \lambda\rho x_2^{\rho-1} = 0$$

$$\frac{\partial L}{\partial \lambda} = x_1^\rho + x_2^\rho - y^\rho = 0$$

Using the first two FOCs, we can write x_i^ρ as:

$$x_1^\rho = w_1^{\frac{\rho}{\rho-1}}(\lambda\rho)^{-\frac{\rho}{\rho-1}}$$

$$x_2^\rho = w_2^{\frac{\rho}{\rho-1}}(\lambda\rho)^{-\frac{\rho}{\rho-1}}$$

Now we can plug x_i^ρ into the third FOC:

$$x_1^\rho + x_2^\rho = y^\rho$$

$$w_1^{\frac{\rho}{\rho-1}}(\lambda\rho)^{-\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}(\lambda\rho)^{-\frac{\rho}{\rho-1}} = y^\rho$$

And write $(\lambda\rho)^{-\frac{\rho}{\rho-1}}$ as:

$$(\lambda\rho)^{-\frac{\rho}{\rho-1}} = \left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right)^{-1} y^\rho$$

Now we can plug $(\lambda\rho)^{-\frac{\rho}{\rho-1}}$ into our expression for x_i^ρ above. This allows to solve for x_i in terms of exogenous variables only:

$$x_1^\rho = w_1^{\frac{\rho}{\rho-1}}(\lambda\rho)^{-\frac{\rho}{\rho-1}}$$

$$x_1^\rho = w_1^{\frac{\rho}{\rho-1}}\left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right)^{-1} y^\rho$$

And hence we can write:

$$x_1(\boldsymbol{w}, y) = w_1^{\frac{1}{\rho-1}}\left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right)^{-\frac{1}{\rho}} y$$

$$x_2(\boldsymbol{w}, y) = w_2^{\frac{1}{\rho-1}}\left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right)^{-\frac{1}{\rho}} y$$