Lisbon University
Lisbon School of Economics and Management
Ms in Economics, Mathematical Finance and
Monetary and Financial Economics

Mathematical Economics – 1st Semester - 2023/2024

Exercises - Group III

1. Complete the following sentences:

   (a) The map $y(x) = \dfrac{1}{x}$, $x \in \mathbb{R}^-$, is a solution of the IVP $\begin{cases} \dot{y} = \ldots\ldots \\ y(\ldots\ldots) = -2 \end{cases}$

   (b) The .... law (associated to a given population of size p that depends on the time $t \geq 0$) states that

   $$p' = kp, \qquad k \in \mathbb{R} \text{ (parameter)}.$$

   If $p(0) = 10$ and $k = -1$, then $p(10) = \ldots\ldots\ldots$ and $\lim_{t \to +\infty} p(t) = \ldots\ldots\ldots\ldots$

   (c) The graph of the solution of the IVP $\begin{cases} y' = 3x \\ y(0) = 2 \end{cases}$ is

   (y is a function of x)

   (d) The logistic law (associated to a given population of size p that depends on the time $t \geq 0$) states that

   $$p' = ap - bp^2, \qquad a, b \in \mathbb{R}$$

   where a/b may be seen as the ....of the population.
   If $p(0) = 1000$, $a = 1$ and $b = 0.002$, then the solution of the previous differential equation is monotonic ....
   If $a = 3$, $b = 0$ and $p(0) = 1000$, the solution of the ODE is

   ...., where $t \in \mathbb{R}^+$.

   (e) The graph of the solution of the IVP $\begin{cases} y'' - 4y = 0 \\ y'(0) = 0 \\ y(0) = 2 \end{cases}$ is

   (y is a function of x)

2. Find the general solution of (x is a function of t)

   (a) $x' + 2x = 8$

   (b) $x' + 3x = e^t$

   (c) $x' + 2tx = e^{-t^2}$

(d) $x' + 2tx = 4t$

(e) $4t^2x' + 8tx = -12\sin(3t)$

3. For each 1st order linear ODE of previous Exercise solve the IVP with the initial condition

   (a) $x(0) = 0$

   (b) $x(0) = -1$

   (c) $x(0) = 1$

   (d) $x(0) = -2$

4. Solve the following IVP: $\begin{cases} y' + 2xy = x \\ y(0) = 3/2 \end{cases}$ and trace the graphs of the integrant factor and the solution.

5. Consider the following model of economic growth in a developing country,

$$X(t) = \sigma K(t), \quad K'(t) = \alpha X(t) + H(t)$$

   where $X(t)$ is the total domestic product per year, $K(t)$ the capital stock, $H(t)$ the net inflow of foreign investment per year, all measured at time instant t. Assume that $H(t) = H_0 e^{\mu t}$.

   (a) Derive a differential equation for $K(t)$ and find the solution given that $K(0) = K_0$.

   (b) If the size of the population $N(t) = N_0 e^{\rho t}$, compute $x(t) = X(t)/N(t)$ which is the domestic product per capita.

   (c) Assuming that $\mu = \rho$ and $\rho > \alpha\sigma$ compute $\lim_{t\to+\infty} x(t)$.

6. Find the solutions of the following IVP

   (a) $tx' = (1-t)x$ with $x(1) = 1/e$

   (b) $x' = t/x$ with $x(\sqrt{2}) = 1$

   (c) $x' = (x-1)(x+1)$ with $x(0) = 0$

   (d) $y' = xy - x$ where y is a function of x

   (e) $e^{x^4} yy' = x^3(9 + y^4)$ where y is a function of x

7. Solve the following IVP: $\begin{cases} y' = x^2 e^{2y} \\ y(0) = 0 \end{cases}$ and indicate the maximal domain of definition.

8. Consider the following IVP $\begin{cases} y' = x^2(y-2)^4 \\ y(0) = a \in \mathbb{R} \end{cases}$.

   Solve the IVP for (i) $a = 2$ and (ii) $a = 0$.

9. Determine the maximal interval of existence for the solutions of the following ODEs:

   (a) $x' = e^{-x}$

(b) $x' = \frac{1}{2x}$

10. Determine the phase portrait of the follows ODEs and classify the equilibria.

(a) $x' = ax$, with $a \neq 0$

(b) $x' = x - x^3$

(c) $x' = b + x$ with $b \in \mathbb{R}$

(d) $x' = (x+1)(x+2)$

(e) $x' = -x + x^3 + \lambda$ with $\lambda \in \mathbb{R}$

(f) $x' = 1 - \sin x$

11. Find the matrix P and determine the Jordan normal form for the following matrices

(a) $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

(b) $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

(c) $A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$

(d) $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

(e) $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

(f) $A = \begin{pmatrix} 0 & 4 \\ -5 & 4 \end{pmatrix}$

12. Find the solution of $X' = AX$ with $X(0) = X_0$ where

(a) $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(b) $A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$

(c) $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

(d) $A = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$

(e) $A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$

13. Solve the IVPs:

(a) $x'' = 4x$, with $x(0) = 2$ and $x'(0) = -1$.

(b) $x'' + x = 0$ with $x(0) = 0$ and $x'(0) = 1$.

(c) $x'' - 2x' + x = 0$ with $x(0) = 1$ and $x'(0) = 1$.

14. Consider the following IVP (y is a function of x):

$$\begin{cases} x^2y' + xy = x^3 \\ 3y(1) = 4 \end{cases}$$

Write the solution $y(x)$ of the IVP, identifying its maximal domain.

15. Solve the following IVPs

(a) $\begin{cases} x' = y + e^{-2x}, \\ y' = x + 1, \end{cases} \qquad x(0) = 1,\ y(0) = 2$

(b) $x'' + x' - 6x = 2$ with $x(0) = -1$ and $x'(0) = 1$.

16. For each of the following planar ODEs $X' = AX$

$$(i)\ A = \begin{pmatrix} -8 & -5 \\ 10 & 7 \end{pmatrix} \qquad (ii)\ A = \begin{pmatrix} 1/2 & -1/2 \\ 0 & 1 \end{pmatrix}$$

$$(iii)\ A = \begin{pmatrix} -1 & 1 \\ -1 & -3 \end{pmatrix} \qquad (iv)\ A = \begin{pmatrix} 4 & 1 \\ -4 & 0 \end{pmatrix}$$

$$(v)\ A = \begin{pmatrix} 5 & 4 \\ -10 & -7 \end{pmatrix} \qquad (vi)\ A = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}$$

(a) Find the Jordan normal form of A

(b) Compute the associated matrix P

(c) Compute solution of the associated IVP

(d) Sketch the phase portrait

17. Find the solution of the following 2nd order scalar ODEs,

(a) $x'' + bx = 0$ with $b > 0$ (harmonic oscillator)

(b) $x'' + ax' + bx = 0$ with $a, b > 0$ (damped harmonic oscillator)

For each case, discuss the phase portrait in the (x, x')-plane.

18. If $y = e^{2x}$ is a solution of the differential equation

$$y'' - \alpha y' + 10y = 0, \alpha \in \mathbb{R},$$

show that $\alpha = 7$ and find the general solution of the differential equation.

19. Find the real values of a and b for which e^{2x} and e^{-2x} are solutions of

$$y'' + ay' + by = 0, \alpha \in \mathbb{R}.$$

Write the general solution of the differential equation.

20. Solve the following IVP: $\begin{cases} y'' + 4y = 0 \\ y(0) = 0 \\ y(\pi) = 0 \end{cases}$

21. Solve the following differential equations:

(a) $y'' + 3y + 7y = 5e^{3x}$

(b) $y'' - 4y + 4y = 8x^2$

(c) $y'' - 3y + 2y = 20\sin(2x)$

(d) $y'' - 4y' + 4y = e^{2x}$