1.1. (2 points)

$$f(tx_1, tx_2) = (tx_1)^{\alpha}(tx_2)^{\beta}$$

$$f(tx_1, tx_2) = t^{\alpha+\beta}x_1^{\alpha}x_2^{\beta}$$

$$f(tx_1, tx_2) = t^{\alpha+\beta}f(x_1, x_2)$$

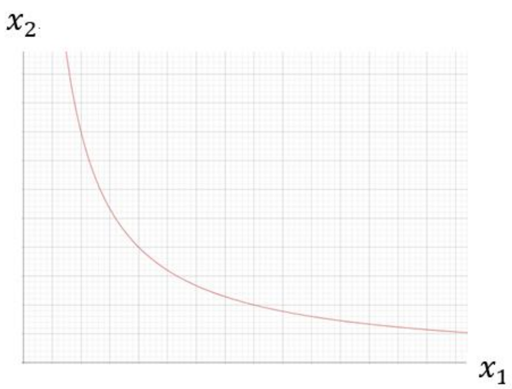
Hence, with $t > 1$, we can conclude that:

If $(\alpha + \beta) < 1$, then $f(tx_1, tx_2) < tf(x_1, x_2)$, hence DRTS.

If $(\alpha + \beta) = 1$, then $f(tx_1, tx_2) = tf(x_1, x_2)$, hence CRTS.

If $(\alpha + \beta) > 1$, then $f(tx_1, tx_2) > tf(x_1, x_2)$, hence IRTS.

1.2. (1 point)



1.3. (2 points)

$$TRS = -\frac{\frac{df}{dx_1}}{\frac{df}{dx_2}} = -\frac{\frac{1}{2}x_1^{-\frac{1}{2}}x_2^{\frac{1}{2}}}{\frac{1}{2}x_1^{\frac{1}{2}}x_2^{-\frac{1}{2}}} = -\frac{x_2}{x_1}$$

1.4. (2 points)

From both the formula of the TRS and the graph of the isoquant, one can conclude that:

(i) If x_2 is large and $x_1$is small, the TRS is strongly negative. This implies that if one increases x_1 from a small value, one can decrease x_2 by a lot and keep producing the same amount.

In contrast, **(ii)** if x_2 is small and $x_1$is large, the TRS is close to zero. This implies that if one increases x_2 from a small value, one can decrease x_1 by a lot and keep producing the same amount.

The economic intuition is that: if we already use a lot of x_2 (x_1) using more of x_2 (x_1) is not that productive. Indeed, if we already use a lot of x_2 (x_1), we can decrease x_2 (x_1) by a lot and only increase x_1 (x_2) by a little and keep producing the same. One can write that this reflects a preference for a “balanced” input bundle over an “extreme” input bundle.

2.1. (3 points)

No, one cannot test the WACM for this firm. The formula for the WACM is that:

$$\boldsymbol{w}^t \boldsymbol{x}^t \leq \boldsymbol{w}^t \boldsymbol{x}^s, \qquad \forall s, t \ \ with\, y^s \geq y^t.$$

Since we do not observe output y, we cannot test the WACM.

2.2. (3 points)

If we want to know how an optimized function (e.g., $f(a, x(a))$) changes when an exogenous variable changes (e.g., a), only the direct of this exogenous variable needs to be considered, even if the exogenous variable also enters the optimized function indirectly as part of the solution to the endogenous choice variables (e.g., $x(a)$).

In math, if we want to know how the optimized function changes when the exogenous variable changes, we take the derivative of the optimized function $f(a, x(a))$ towards the exogenous variable a. This is similar to taking the total derivative:

$$\frac{\partial f\big(a, x(a)\big)}{\partial a} = \frac{\partial f(a, x)}{\partial a} + \frac{\partial f(a, x)}{\partial x} \frac{\partial x(a)}{\partial a}$$

The first is a direct effect of a and the second an indirect of a (as it goes via x). And since the function $f(a, x(a))$ is optimized with respect to x, we know that the following FOC holds:

$$\frac{\partial f(a, x)}{\partial x} = 0$$

Hence, we have that only the direct effect remains:

$$\frac{\partial f\big(a, x(a)\big)}{\partial a} = \frac{\partial f(a, x)}{\partial a}$$

3.1. (4 points)

Set up the Lagrange

$$L = 4x_1 + 4x_2 - \lambda(x_1^{\frac{1}{2}} x_2^{\frac{1}{2}} - y_0)$$

Take FOCS

$$\frac{\partial L}{\partial x_1} = 4 - \lambda \frac{1}{2} x_1^{-\frac{1}{2}} x_2^{\frac{1}{2}} = 0$$

$$\frac{\partial L}{\partial x_2} = 4 - \lambda \frac{1}{2} x_1^{\frac{1}{2}} x_2^{-\frac{1}{2}} = 0$$

$$\frac{\partial L}{\partial \lambda} = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}} - y_0 = 0$$

Divide the first two FOCS by each other to get

$$1 = \frac{x_2}{x_1}$$

$$x_1 = x_2$$

Plug this into the third FOC

$$x_1^{\frac{1}{2}} x_1^{\frac{1}{2}} - y_0 = 0$$

$$x_2^{\frac{1}{2}} x_2^{\frac{1}{2}} - y_0 = 0$$

And solve for x_i in terms of exogenous variables to obtain the factor demand functions:

$$x_1 = y_0$$

$$x_2 = y_0$$

Finally, we can find the cost function by plugging the factor demand functions into the costs:

$$c(\boldsymbol{w}, y_0) = 4x_1 + 4x_2$$

$$c(\boldsymbol{w}, y_0) = 4y_0 + 4y_0$$

$$c(\boldsymbol{w}, y_0) = 8y_0$$

3.2. (3 points)

The Lagrange multiplier can either be find by (i) solving for λ in the problem above, or (ii) taking the derivative of the cost function towards y.

Via route (i) we can use the first FOC to get

$$\lambda \frac{1}{2} y_0^{-\frac{1}{2}} y_0^{\frac{1}{2}} = 4$$

$$\lambda \frac{1}{2} y_0^0 = 4$$

$$\lambda \frac{1}{2} = 4$$

$$\lambda = 8$$

Via route (ii) we get that the derivative of the cost function towards y is:

$$\frac{\partial c(\boldsymbol{w}, y_0)}{\partial y_0} = 8$$

Hence, we have that

$$\frac{\partial c(\boldsymbol{w}, y_0)}{\partial y_0} = \lambda = 8$$

Indeed, λ can be interpreted as the marginal costs. In this case, if we increase the production level y_0 by 1, then we increase the costs by 8. This makes sense: To produce one more unit of y we need one more unit of both x_1 and x_2 (see the factor demand functions derived in question 3.1), and the price of both x_1 and x_2 are 4.