Microeconomics

Chapter 7
Utility maximization

Fall 2023

Consumer behavior

In the **theory of the firm**, the demand (and other) functions were derived from a model of profit-maximizing or cost-minimizing behavior and a specification of the underlying technological constraints.

In the **theory of the consumer**, we will derive demand (and other) functions by considering a model of utility-maximizing behavior coupled with a description of the underlying economic constraints.

Consumer preferences

If we want to analyze utility-maximizing behavior, we first need to discuss **consumer preferences**. And to discuss preferences we need to define **consumption bundles** and a **consumption set**.

Consumption bundle: a list of consumption goods, described by the vector **x** in R_{+}^{k}, where k is the number of different goods, and element $x_i \geq 0$ reflects the specific consumption for good $i = 1, ..., k$.

Consumption set: the set of all possible consumption bundles **x** that a consumer can hypothetically choose. This set is denoted by X,

$$X = \{\mathbf{x} \text{ in } R_{+}^{k} : \mathbf{x} \text{ can be hypothetically chosen}\}.$$

Consumer preferences

The consumer is assumed to have **preferences** on **x** in X. We assume that those preferences are represented by a binary relationship.

Let $\mathbf{x} \neq \mathbf{y}$ be two bundles in X, then the binary relationships are:

$\mathbf{x} \succ \mathbf{y}$	**x** is better than **y**
$\mathbf{x} \succeq \mathbf{y}$	**x** is weakly better than **y**
$\mathbf{x} \sim \mathbf{y}$	indifferent between **x** and **y**
$\mathbf{x} \preceq \mathbf{y}$	**x** is weakly worse than **y**
$\mathbf{x} \prec \mathbf{y}$	**x** is worse than **y**

Assumptions on preferences

1. Completeness: for all $\mathbf{x}$ and $\mathbf{y}$ in X, either $\mathbf{x} \succeq \mathbf{y}$ or $\mathbf{y} \succeq \mathbf{x}$ or both.

This ensures that the consumer can make comparisons: The consumer has the ability and information to evaluate alternatives.

2. Transitivity: for all $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ in X, if $\mathbf{x} \succeq \mathbf{y}$ and $\mathbf{y} \succeq \mathbf{z}$, then $\mathbf{x} \succeq \mathbf{z}$.

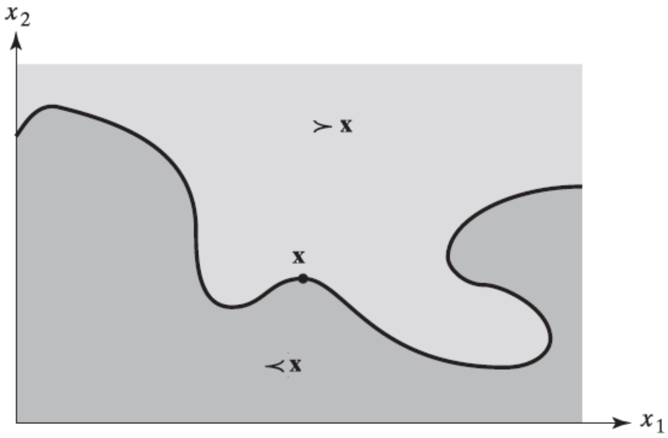
This ensures that choices are consistent: pairwise comparisons are linked together in a consistent way.

Completeness and Transitivity imply that the consumer can completely rank any finite number of $\mathbf{x}$ in X, from best to worst, possibly with ties.

3. Continuity: for all $\mathbf{y}$ in X, the sets $\{\mathbf{x} : \mathbf{x} \succeq \mathbf{y}\}$ and $\{\mathbf{x} : \mathbf{x} \preceq \mathbf{y}\}$ are closed sets.

This ensures the absence of discontinuous consumption behavior. Continuity mostly speaks to the mathematical aspects of representing preferences by a utility function.

Consumer preferences in a graph



The graph above reflects a hypothetical set of preferences over bundles $\mathbf{x} = (x_1, x_2)$ that satisfy the three assumptions: Completeness, Transitivity, and Continuity (and local non-satiation).

Additional assumptions on preferences

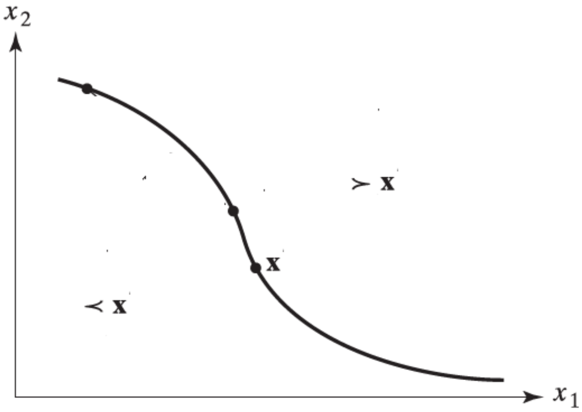
4. Monotonicity (strict): if $\mathbf{x} \geq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$ then $\mathbf{x} \succ \mathbf{y}$.

This implies that "at least as much of every good, and strictly more of some good(s), is strictly better". If free disposal of unwanted goods is allowed, then this assumption seems harmless.

5. Convexity (strict): given $\mathbf{x} \neq \mathbf{y} \neq \mathbf{z}$ in X, if $\mathbf{x} \succeq \mathbf{z}$ and $\mathbf{y} \succeq \mathbf{z}$, then $t\mathbf{x} + (1-t)\mathbf{y} \succ \mathbf{z}$ for all $0 < t < 1$.

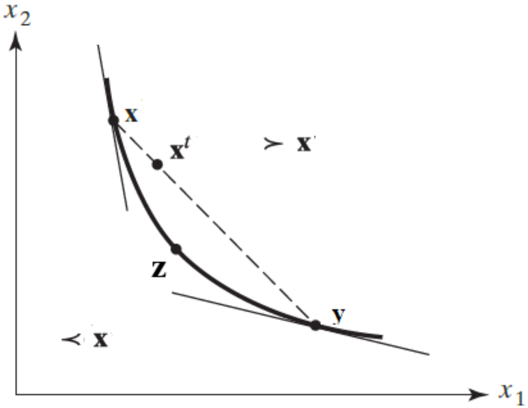
This implies that a consumers prefers a "balanced" consumption bundle instead of an "extreme" bundle: The weighted average (or mixture) of bundles $t\mathbf{x} + (1-t)\mathbf{y}$ is preferred to the extreme bundle $\mathbf{z}$.

Consumer preferences in a graph



The graph above reflects a hypothetical set of preferences over bundles $\mathbf{x} = (x_1, x_2)$ that satisfy the four assumptions: Completeness, Transitivity, Continuity, and Monotonicity.

Consumer preferences in a graph



The graph above reflects a hypothetical set of preferences over bundles $\mathbf{x} = (x_1, x_2)$ that satisfy the five assumptions: Completeness, Transitivity, Continuity, Monotonicity, and Convexity.

From preferences to utility function

Debreu (1954) has shown that under the three assumptions Completeness, Transitivity and Continuity, there exists a continuous **utility function** that represent those preferences.

Existence of a utility function: if preferences are Complete, Transitive, and Continuous, there exists a utility function $u(\mathbf{x}) : R_+^k \rightarrow R$ that represents those preferences. This implies that there exists a utility function $u(\mathbf{x})$ that satisfies $u(\mathbf{x}) \geq u(\mathbf{y}) \leftrightarrow \mathbf{x} \succeq \mathbf{y}$.

A utility function is a convenient way to describe preferences. For instance, with a utility function we can use the method of Lagrange to maximize $u(\mathbf{x})$ by choosing the optimal consumption bundles subject to economic constraints.

The additional assumptions Monotonicity and Convexity guarantee that the utility function has the shape we are used to. This makes sure that the SOCs of the constrained maximization problem are met.

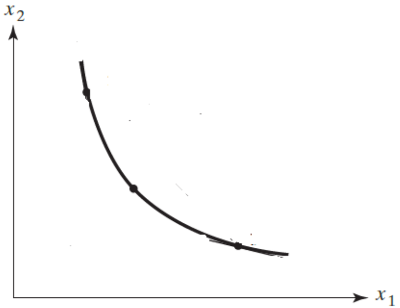
Indifference curve

A utility function $u(\mathbf{x})$ is often represented by an indifference curve. This is simply a level set for the utility function.

Indifference curve: all consumption bundles that give utility level u_0,

$$I(u_0) = \{\mathbf{x} : u(\mathbf{x}) = u_0\}.$$

An indifference curve for the consumer is analogous to the isoquant for the firm.



The ordinal character of utility functions

The only relevant feature of a utility function is its **ordinal character**: if some function $u(\mathbf{x})$ represents a consumer's preferences, then so will the function $u'(\mathbf{x}) = u(\mathbf{x}) + 5$. That is, if $u(\mathbf{x})$ satisfies $u(\mathbf{x}) \geq u(\mathbf{y}) \leftrightarrow \mathbf{x} \succeq \mathbf{y}$ then $u'(\mathbf{x})$ also satisfies $u'(\mathbf{x}) \geq u'(\mathbf{y}) \leftrightarrow \mathbf{x} \succeq \mathbf{y}$.

No interpretation should be given to the actual numbers that are given by $u(\mathbf{x})$, only to the ordering of those numbers.

The above implies that the **utility function is invariant to positive monotonic transformations**: let $u(\mathbf{x})$ be a utility function that represents a consumer's preferences. Then $g(u(\mathbf{x}))$ also represents that consumer's preferences if $g : R \to R$ is a positive monotonic transformation: $u(\mathbf{x}) > u(\mathbf{y})$ implies $g(u(\mathbf{x})) > g(u(\mathbf{y}))$.

Exercise

Assume completeness, transitivity, continuity, monotonicity, and convexity.

1. Show that indifference curves must be downwards sloping.
2. Show that two indifference curves cannot cross.
3. Show that indifference curves become less steep as we move downward and to the right along them.

Marginal utility

Consider a setting with two goods, so that $u(\mathbf{x}) = u(x_1, x_2)$.

Marginal utility of good 1 or 2: how much does utility increase if we increase the consumption of good 1 or 2.

$$MU_i = \frac{\partial u(\mathbf{x})}{\partial x_i}, \quad \text{for } i = 1, 2.$$

Marginal rate of substitution

Marginal rate of substitution: How easy (or difficult) is it for a consumer to change between the consumption of x_1 and x_2 while keeping utility constant?

Let $x_2(x_1)$ be the indifference curve at utility level $u = u_0$, then:

$$MRS = \frac{\partial x_2(x_1)}{\partial x_1}.$$

$x_2(x_1)$ traces all bundles such that $u(x_1, x_2) = u_0$. Hence, $x_2(x_1)$ satisfies the identity $u(x_1, x_2(x_1)) = u_0$, so that the total derivative towards x_1 is zero:

$$\frac{\partial u(\mathbf{x})}{\partial x_1} + \frac{\partial u(\mathbf{x})}{\partial x_2} \frac{\partial x_2(x_1)}{x_1} = 0.$$

Hence, we can get an expression for the MRS without having to find $x_2(x_1)$:

$$MRS = \frac{\partial x_2(x_1)}{\partial x_1} = -\frac{\frac{\partial u(\mathbf{x})}{\partial x_1}}{\frac{\partial u(\mathbf{x})}{\partial x_2}} = -\frac{MU_1}{MU_2}.$$

The MRS for the consumer is analogous to the TRS for the firm.

Marginal rate of substitution

We can write that:

$$MRS = -\frac{MU_1}{MU_2}.$$

Imagine that $MU_1 = 2$ and $MU_2 = 1$. Then $MRS = -\frac{MU_1}{MU_2} = -\frac{2}{1} = -2$. Note that we can also reason this intuitively from $MRS = \frac{\partial x_2(x_1)}{\partial x_1}$:

- If consumer increases x_1 by 1, then utility increases by 2: $MU_1 = \frac{\Delta U}{\Delta x_1} = 2 \rightarrow \Delta U = 2 \times \Delta x_1 \rightarrow$ and $\Delta x_1 = 1$, so $\Delta U = 2$.
- Consumer needs to decrease x_2 by 2 as to keep utility constant: $MU_2 = \frac{\Delta U}{\Delta x_2} = 1 \rightarrow \Delta U = 1 \times \Delta x_2 \rightarrow$ and if $\Delta x_2 = -2$, then $\Delta U = -2$.
- Hence, $MRS = \frac{\Delta x_2}{\Delta x_1} = \frac{-2}{1} = -2$.

Exercise

1. Suppose $MU_1 = 9$ and $MU_2 = 3$ at utility level $u = u_0$. The consumer increases x_1 by 2. How much does the consumer need to decrease x_2 as to keep utility at level u_0?

2. Confirm that in the example above we have:

$$-\frac{MU_1}{MU_2} = \frac{\Delta x_2}{\Delta x_1}.$$

Consumer behavior

We will generally assume that consumers aim to maximize utility. More specifically, we will assume that the consumer will want to **choose the bundle x** from the **set of affordable alternatives** to **maximize utility**.

Budget constraint

Budget constraint: Let m be the budget of a consumer, and let $\mathbf{p}$ in R_+^k be the vector of fixed prices for the k goods. Then we can define the budget constraint, or the set of affordable alternatives, as:

$$B = \{\mathbf{x} : \mathbf{px} = m\}.$$

Consider a setting with two goods, then we can write B as:

$$B = \{(x_1, x_2) : p_1 x_1 + p_2 x_2 = m\}.$$

Note that for any fixed m, we can think of B as a level set, and of the budget constraint as an **isobudget** line. The constraint gives us all bundles (x_1, x_2) that cost m:

$$B = \{(x_1, x_2) : x_2 = (\frac{m}{p_2}) - (\frac{p_1}{p_2}) x_1\}.$$

The intercept of the budget constraint $(\frac{m}{p_2})$ gives the budget in terms of the price of x_2. The slope of the budget line $(\frac{\partial x_2(x_1)}{\partial x_1} = \frac{p_1}{p_2})$ gives the **economic rate of substitution**: when x_1 increases, how much does x_2 need to decrease as to keep spending budget m.

Utility maximization

Now that we have introduced the utility function and the budget constraint, we can write the problem of utility maximization as:

$$\max_{\mathbf{x}} u(\mathbf{x}),$$

$$\text{such that } \mathbf{px} = m.$$

Since the only relevant feature about the utility function is its ordinal character, the problem of utility maximization can also be interpreted as the problem of finding the most preferred bundle.

The method of Lagrange with two goods

First, write down the Lagrangian,

$$\mathcal{L} = u(\mathbf{x}) - \lambda(p_1 x_1 + p_2 x_2 - m).$$

Second, differentiate $\mathcal{L}$ wrt each endogenous variable: x_1, x_2 and λ. The FOCs for an interior solution $\mathbf{x}^*$ set these derivatives to zero,

$$\begin{aligned} \frac{\partial u(\mathbf{x})}{\partial x_1} - \lambda p_1 &= 0, \\ \frac{\partial u(\mathbf{x})}{\partial x_2} - \lambda p_2 &= 0, \\ p_1 x_1 + p_2 x_2 - m &= 0. \end{aligned}$$

Third, since we have 3 unknown endogenous variables (x_1, x_2 and λ) and 3 FOCs, we can solve for the endogenous variables in terms of the exogenous variables (p_1, p_2 and m).

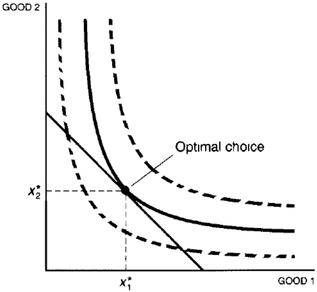
The method of Lagrange with two goods

Dividing the first two FOCs by each other gives us the following optimality condition:

$$\frac{p_1}{p_2} = \frac{\partial u(\mathbf{x}) / \partial x_1}{\partial u(\mathbf{x}) / \partial x_2}.$$

By analyzing the Lagrange method graphically we can develop an economic intuition for this condition and the method more generally.

The method of Lagrange graphically



The utility-maximizing consumer wants to find a **point on the budget constraint with maximal utility**: this is a point where the indifference curve is furthest to the northeast. This point $\mathbf{x}^*$ is characterized by the slopes of the two lines being equal, which is the optimality condition we have seen before:

$$\frac{p_1}{p_2} = \frac{\partial u(\mathbf{x}) / \partial x_1}{\partial u(\mathbf{x}) / \partial x_2}$$

The method of Lagrange with two goods

Recall that the optimality condition is:

$$\frac{p_1}{p_2} = \frac{\partial u(\mathbf{x})/\ \partial x_1}{\partial u(\mathbf{x})/\ \partial x_2}.$$

The LHS is the **economic rate of substitution** ($\frac{\partial x_2(x_1)}{\partial x_1} = \frac{p_1}{p_2}$): when x_1 increases, how much does x_2 need to decrease as to keep spending the same.

The RHS is the **marginal rate of substitution** ($\frac{\partial x_2(x_1)}{\partial x_1} = \frac{\partial u(\mathbf{x})/\ \partial x_1}{\partial u(\mathbf{x})/\ \partial x_2}$): when x_1 increases, how much does x_2 need to decrease as to keep utility constant.

Hence, the optimality condition tells us that at $\mathbf{x}^*$ the economic and marginal rate of substitution need to be equal. Imagine they are not:

$$\frac{p_1}{p_2} = \frac{2}{1} \neq \frac{1}{1} = \frac{\partial u(\mathbf{x})/\ \partial x_1}{\partial u(\mathbf{x})/\ \partial x_2},$$

then we can consume one unit less of x_1 and one unit more of x_2, so that utility stays constant, but we still have an additional euro to spend. This cannot be $\mathbf{x}^*$.

Marshallian demand and indirect utility

Marshallian demand function $\mathbf{x}(\mathbf{p}, m)$: a function that gives us the optimal choice of consumption goods as a function of prices $\mathbf{p}$ and budget m.

How to get this function? From the FOCs of the Langrangian we can write $\mathbf{x}$ in terms of $(\mathbf{p}, m)$.

Note that the Marshallian demand function is just an “ordinary” demand function for consumption. It shows us how demand changes when prices change, while income is kept constant. Moreover, it is a function that can in principle be estimated with data on consumption, prices, and income.

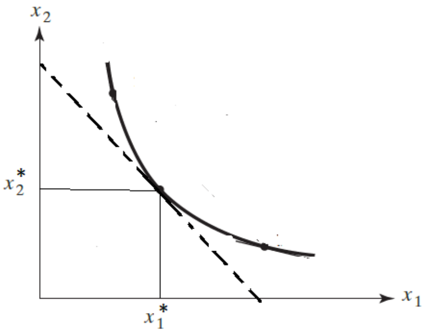
Indirect utility function $v(\mathbf{p}, m)$: a function that gives us the maximum utility achievable given prices $\mathbf{p}$ and budget m.

How to get this function? Substitute $\mathbf{x}(\mathbf{p}, m)$ into $u(\mathbf{x}) = u(\mathbf{x}(\mathbf{p}, m)) = v(\mathbf{p}, m)$

Exercise

The graph below indicates the optimal consumption bundle. Imagine p_1 increases.

1. Draw the new budget constraint.
2. Show the new Marshallian demand.



The Lagrange multiplier

Recall that the **Lagrange multiplier** λ measures how the optimal solution to the constrained maximization problem changes when the constraint is relaxed.

When we apply this interpretation of λ to the utility maximization problem, the optimal solution is the indirect utility function $v(\mathbf{p}, m)$ and the constraint is relaxed if we increase the budget m. Hence, the Lagrange multiplier measures the increase in utility when we are allowed to spend more.

The Lagrange multiplier

The proof for this interpretation of λ follows from the envelope theorem.

Consider the Lagrangian with two goods,

$$\mathcal{L}(\mathbf{p}, m, \mathbf{x}, \lambda) = u(\mathbf{x}) - \lambda(p_1 x_1 + p_2 x_2 - m).$$

First, note that:

$$\frac{\partial \mathcal{L}(\mathbf{p}, m, \mathbf{x}, \lambda)}{\partial m} = \lambda.$$

Second, substitute the Marshallian demand functions $\mathbf{x}(\mathbf{p}, m)$ and the Lagrange multiplier $\lambda(\mathbf{p}, m)$ into the Lagrangian to obtain the Lagrangian evaluated at the optimal point: $\mathcal{L}(\mathbf{p}, m, \mathbf{x}(\mathbf{p}, m), \lambda(\mathbf{p}, m)) = \mathcal{L}(\mathbf{p}, m)$. It turns out, this is equal to:

$$\begin{aligned}
\mathcal{L}(\mathbf{p}, m) =& u(\mathbf{x}(\mathbf{p}, m)) - \lambda(\mathbf{p}, m)(p_1 x_1(\mathbf{p}, m) + p_2 x_2(\mathbf{p}, m) - m), \\
=& u(\mathbf{x}(\mathbf{p}, m)), \\
=& v(\mathbf{p}, m).
\end{aligned}$$

The Lagrange multiplier

Third, use the logic of the envelope theorem to show that at the optimal point:

$$\frac{\partial \mathcal{L}(\mathbf{p}, m)}{\partial m} = \underbrace{\frac{\partial \mathcal{L}(\cdot)}{\partial m}}_{\text{direct effect}} + \underbrace{\frac{\partial \mathcal{L}(\cdot)}{\partial x_1}\frac{\partial x_1(\cdot)}{\partial m} + \frac{\partial \mathcal{L}(\cdot)}{\partial x_2}\frac{\partial x_2(\cdot)}{\partial m} + \frac{\partial \mathcal{L}(\cdot)}{\partial \lambda}\frac{\partial \lambda(\cdot)}{\partial m}}_{\text{indirect effect}},$$

$$= \frac{\partial \mathcal{L}(\cdot)}{\partial m},$$

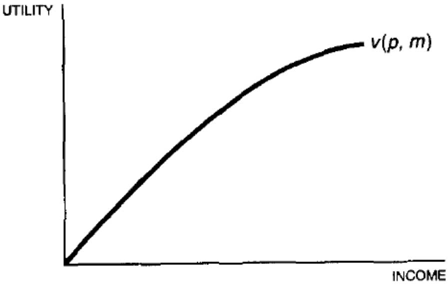
$$= \lambda(\mathbf{p}, m),$$

as the indirect effects are zero because of the FOCs of the Lagrangian.

Since $\mathcal{L}(\mathbf{p}, m) = v(\mathbf{p}, m)$, we conclude that:

$$\frac{\partial \mathcal{L}(\mathbf{p}, m)}{\partial m} = \frac{\partial v(\mathbf{p}, m)}{\partial m} = \lambda(\mathbf{p}, m).$$

The indirect utility function



Indirect utility function $v(\mathbf{p}, m)$: maximum utility for each income m.

Since preferences satisfy monotonicity, indirect utility is increasing in m: If m increases, x increases, and so u increases. Hence, for every v in its range there is only one x in its domain. Hence, the indirect utility function has an inverse function. Write the inverse of $u = v(\mathbf{p}, m)$ as $m = v^{-1}(\mathbf{p}, u) = e(\mathbf{p}, u)$

Expenditure function $e(\mathbf{p}, u)$: minimum income required to achieve utility u.

Expenditure minimization

You can also get the expenditure function $e(\mathbf{p}, u)$ by solving the consumers' choice problem via expenditure minimization.

Expenditure minimization problem (EMP): find $\mathbf{x}^*$ that minimizes expenditure $\mathbf{px}$ subject to the utility constraint $u(\mathbf{x}) = u$.

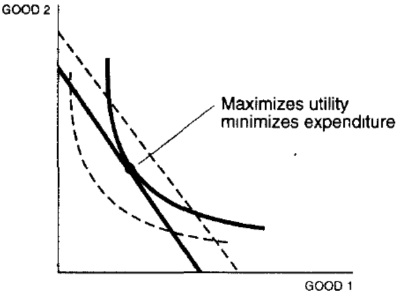
$$\min_{\mathbf{x}} \mathbf{px},$$
$$\text{such that } u(\mathbf{x}) = u.$$

Note that EMP is identical to cost minimization. So the Lagrange method, intuitions, and properties from Chapter 4 apply to EMP.

Utility maximization problem (UMP): find $\mathbf{x}^*$ that maximizes utility $u(\mathbf{x})$ subject to the budget constraint $\mathbf{px} = m$.

$$\max_{\mathbf{x}} u(\mathbf{x}),$$
$$\text{such that } \mathbf{px} = m.$$

EMP=UMP



EMP: Given the indifference curve, slide the budget constraint to the south-west region until it just touches the indifference curve.

UMP: Given the budget constraint, slide the indifference curve to the north-east region until it just touches the budget constraint.

Under the assumptions we made, EMP and UMP will find the same optimal point $\mathbf{x}^*$. This is referred to as **duality**.

The method of Lagrange for the EMP

This slide shows the method of Lagrange for the **EMP**. Note that the method, its intuition, and properties are identical to the cost minimization problem.

First, write down the Lagrangian,

$$\mathcal{L} = p_1 x_1 + p_2 x_2 - \lambda(u(\mathbf{x}) - u)$$

Second, differentiate $\mathcal{L}$ wrt each endogenous variable: x_1, x_2 and λ. The FOCs for an interior solution $\mathbf{x}^*$ set these derivatives to zero,

$$\begin{aligned} p_1 - \lambda \frac{\partial u(\mathbf{x})}{\partial x_1} &= 0, \\ p_2 - \lambda \frac{\partial u(\mathbf{x})}{\partial x_2} &= 0, \\ u(\mathbf{x}) - u &= 0. \end{aligned}$$

Third, since we have 3 unknown endogenous variables (x_1, x_2 and λ) and 3 FOCs, we can solve for the endogenous variables in terms of the exogenous variables (p_1, p_2 and u).

The Lagrange multiplier

Recall (again) that the **Lagrange multiplier** λ measures how the optimal solution to the constrained maximization problem changes when the constraint is relaxed.

When we apply this interpretation of λ to the expenditure minimization problem, the optimal solution is the expenditure function $e(\mathbf{p}, m)$ and the constraint is relaxed if we increase utility u. Hence, the Lagrange multiplier measures how expenditures (budget) increase when we are allowed additional utility.

Hicksian demand and expenditure function

Hicksian demand function $\mathbf{h}(\mathbf{p}, u)$: a function that gives us the optimal choice of consumption goods as a function of prices $\mathbf{p}$ and utility u.

How to get this function? From the FOCs of the Lagrangian we can write $\mathbf{x}$ in terms of $(\mathbf{p}, u)$.

The Hicksian demand function is not an “ordinary” demand function. It shows us how demand changes when prices change, while utility is kept constant. However, if one wants to keep utility constant while prices change, one needs to change/compensate income.

This makes the Hicksian demand a theoretical concept, and often referred to as **compensated demand**. Moreover, since utility is not observed, the Hicksian demand function cannot be estimated.

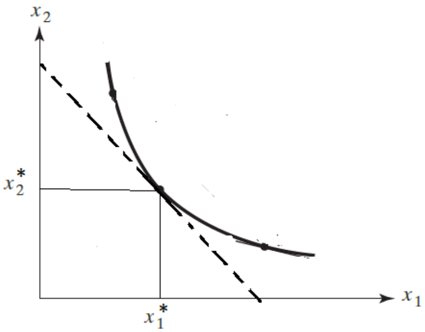
Expenditure function $e(\mathbf{p}, u)$: a function that gives us the minimum income required to achieve utility u at prices $\mathbf{p}$.

How to get this function? Substitute $\mathbf{h}(\mathbf{p}, u)$ into $\mathbf{px} = \mathbf{ph}(\mathbf{p}, u) = e(\mathbf{p}, u)$.

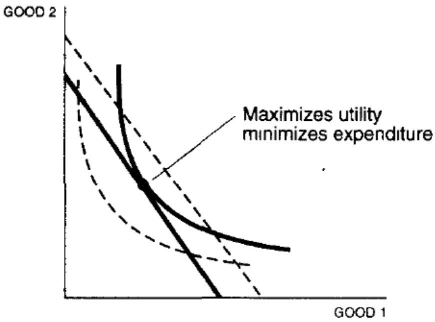
Exercise

The graph below indicates the optimal consumption bundle. Imagine p_1 increases.

1. Draw the new budget constraint.
2. Show the new Hicksian demand.



Two important identities



UMP: Let $\mathbf{x}(\mathbf{p}, m)$ and $v(\mathbf{p}, m)$ be the solutions.

EMP: Let $\mathbf{h}(\mathbf{p}, u)$ and $e(\mathbf{p}, u)$ be the solutions.

Duality ensures that (1) $\mathbf{x}(\mathbf{p}, m) = \mathbf{h}(\mathbf{p}, u)$ if $u = v(\mathbf{p}, m)$, and
(2) $\mathbf{h}(\mathbf{p}, u) = \mathbf{x}(\mathbf{p}, m)$ if $m = e(\mathbf{p}, u)$.

Two important identities

(1) $x_i(\mathbf{p}, m) = h_i(\mathbf{p}, v(\mathbf{p}, m))$: The Marshallian demand at income m is the same as the Hicksian demand at utility $v(\mathbf{p}, m)$.

(2) $h_i(\mathbf{p}, u) = x_i(\mathbf{p}, e(\mathbf{p}, u))$: The Hicksian demand at utility u is the same as the Marshallian demand at income $e(\mathbf{p}, u)$.

It is identity (2) that gives rise to the term compensated demand for the Hicksian demand: It is the Marshallian demand when income changes are arranged as to achieve some target level of utility.

Marshallian demand from indirect utility

If you were given the Marshallian demand $\mathbf{x}(\mathbf{p}, m)$, finding the indirect utility function is easy: just substitute the Marshallian demand into $u(\mathbf{x})$,

$$u(\mathbf{x}(\mathbf{p}, m)) = v(\mathbf{p}, m).$$

It turns out that if you know the indirect utility function, it also easy to find the Marshallian demand. This is what **Roy's identity** shows us.

Roy's identity

Roy's identity shows that we can find the Marshallian demand function from the indirect utility function as follows:

$$-\frac{\partial v(\mathbf{p}, m)/\ \partial p_i}{\partial v(\mathbf{p}, m)/\ \partial m} = x_i(\mathbf{p}, m).$$

In words, **Roy's identity** is that the Marshallian demand function can be found from the fraction that contains the derivative of the indirect utility function towards the price and income. This is similar to Hotelling's and Shephard's lemma, but then applied to utility maximization.

Proof Roy's identity

Consider the Lagrangian with two goods,

$$\mathcal{L}(\mathbf{p}, m, \mathbf{x}, \lambda) = u(\mathbf{x}) - \lambda(p_1 x_1 + p_2 x_2 - m).$$

First, note that:

$$\begin{aligned}\frac{\partial \mathcal{L}(\mathbf{p}, m, \mathbf{x}, \lambda)}{\partial p_1} &= -\lambda x_1, \\ \frac{\partial \mathcal{L}(\mathbf{p}, m, \mathbf{x}, \lambda)}{\partial m} &= \lambda.\end{aligned}$$

Second, substitute the Marshallian demand functions $\mathbf{x}(\mathbf{p}, m)$ and the Lagrange multiplier $\lambda(\mathbf{p}, m)$ into the Lagrangian to obtain the Lagrangian evaluated at the optimal point: $\mathcal{L}(\mathbf{p}, m, \mathbf{x}(\mathbf{p}, m), \lambda(\mathbf{p}, m)) = \mathcal{L}(\mathbf{p}, m)$. It turns out, this is equal to:

$$\begin{aligned}\mathcal{L}(\mathbf{p}, m) &= u(\mathbf{x}(\mathbf{p}, m)) - \lambda(\mathbf{p}, m)(p_1 x_1(\mathbf{p}, m) + p_2 x_2(\mathbf{p}, m) - m), \\ &= u(\mathbf{x}(\mathbf{p}, m)), \\ &= v(\mathbf{p}, m).\end{aligned}$$

Proof Roy's identity

Third, use the logic of the envelope theorem to show that at the optimal point:

$$\frac{\partial \mathcal{L}(\mathbf{p}, m)}{\partial p_1} = \underbrace{\frac{\partial \mathcal{L}(\cdot)}{\partial p_1}}_{\text{direct effect}} + \underbrace{\frac{\partial \mathcal{L}(\cdot)}{\partial x_1}\frac{\partial x_1(\cdot)}{\partial p_1} + \frac{\partial \mathcal{L}(\cdot)}{\partial x_2}\frac{\partial x_2(\cdot)}{\partial p_1} + \frac{\partial \mathcal{L}(\cdot)}{\partial \lambda}\frac{\partial \lambda(\cdot)}{\partial p_1}}_{\text{indirect effect}}$$

$$= \frac{\partial \mathcal{L}(\cdot)}{\partial p_1} = -\lambda(\mathbf{p}, m) x_1(\mathbf{p}, m),$$

$$\frac{\partial \mathcal{L}(\mathbf{p}, m)}{\partial m} = \underbrace{\frac{\partial \mathcal{L}(\cdot)}{\partial m}}_{\text{direct effect}} + \underbrace{\frac{\partial \mathcal{L}(\cdot)}{\partial x_1}\frac{\partial x_1(\cdot)}{\partial m} + \frac{\partial \mathcal{L}(\cdot)}{\partial x_2}\frac{\partial x_2(\cdot)}{\partial m} + \frac{\partial \mathcal{L}(\cdot)}{\partial \lambda}\frac{\partial \lambda(\cdot)}{\partial m}}_{\text{indirect effect}}$$

$$= \frac{\partial \mathcal{L}(\cdot)}{\partial m} = \lambda(\mathbf{p}, m),$$

as the indirect effects are zero because of the FOCs of the Lagrangian.

Since $\mathcal{L}(\mathbf{p}, m) = v(\mathbf{p}, m)$, we conclude that:

$$-\frac{\partial \mathcal{L}(\mathbf{p}, m)/\,\partial p_i}{\partial \mathcal{L}(\mathbf{p}, m)/\,\partial m} = -\frac{\partial v(\mathbf{p}, m)/\,\partial p_i}{\partial v(\mathbf{p}, m)/\,\partial m} = x_i(\mathbf{p}, m).$$

Exercise

Consider the following utility maximization problem:

$$\max_{x_1, x_2} x_1^{\alpha} x_2^{1-\alpha},$$

$$\text{such that } p_1 x_1 + p_2 x_2 = m.$$

1. Find the Marshallian demand functions.

2. Find the indirect utility function.

3. Find $\lambda(\mathbf{p}, m)$. Show that the derivative of the indirect utility function towards m is equal to $\lambda(\mathbf{p}, m)$. Use this to provide an economic interpretation of $\lambda(\mathbf{p}, m)$.

4. Show Roy's identity for $x_1(\mathbf{p}, m)$.

Exercise

Consider the following expenditure minimization problem:

$$\min_{x_1, x_2} p_1 x_1 + p_2 x_2,$$
$$\text{such that } x_1^{\alpha} x_2^{1-\alpha} = u.$$

1. Find the Hicksian demand functions.

2. Find the expenditure function.

3. Plug the indirect utility function into the Hicksian demand function. What do you find? Plug the expenditure function into the Marshallian demand function. What do you find?

Homework exercises

Exercises: 7.4(a)-(b), 8.5, and exercises on the slides