3.

Return Generating Models

3. Return Generating Models

- Estimation risk versus model risk
- Constant correlation models
- Single-factor models
- Fundamental analysis
- Multi-factor models

3.1 Estimation versus model risk

- Learning Objectives
- Estimation Risk
- Model Risk
- Questions

Learning objectives

- state how much data is needed to perform mean-variance portfolio analysis,
- discuss the problem with obtaining the data,
- be able to explain the notion of *estimation risk* in the context of MVT
- identify the main types of return generating models
- be able to explain the notion of *model risk* associated with return generating models.

Problems with mean-variance analysis

- Suppose we work with n assets.
- To apply the techniques of MVT, we need the $n \times n$ elements of the variance-covariance matrix of the returns, and the n expected returns .
- This requires estimation of

$$\underbrace{n}_{\bar{R}_i} + \underbrace{n}_{\sigma_i} + \underbrace{n(n-1)/2}_{\sigma_{ij}}$$

parameters.
- The number of parameters grows with the square of the number of assets.
  - If $N = 10$, need 65 numbers.
  - If $N = 100$, need 5150 numbers.
  - If $N = 1000$, need 506000 numbers.

Where do we get the numbers from?

Historical Approach:

- Use historical time series to estimate the necessary parameters.
- For this to be reliable, you need many more data points than numbers to estimate. Where would we get that much data?
- We are generally interested in 1-year time horizons. How many years back can we go?
- Markets are not qualitatively the same very far back and most companies are not that old or very similar to what they were.
- We could use shorter time horizons and scale. But evidence exists that short term behaviour qualitatively different.

OBS: MVT inputs are from the future distribution of returns at T. NOT from the distribution of past realised returns.

Professional estimates

Analysts Approach:

- One could instead use professional estimates.
- Not clear that the technique is reliable.
- You would need an awfully large number of analysts to estimate so many parameters.
- The cost of employing so many analysts would outweigh the benefits.

Estimation Risk

Estimation of MVT inputs is only a problem to the extent that MVT results are sensitive to the parameters.

- Unfortunately MVT is extremely sensitive to parameter estimation.
- This is know as estimation risk.
- Relatively small estimation errors lead to:
  - Wrong assessment of the investment opportunity set
  - Wrong deduction of tangent portfolios
  - Wrong efficient frontier

$$\Downarrow$$

Construction of inefficient portfolios.

Q: In the presence of estimation risk are we better of applying MVT or just using naive portfolio construction?

Model Risk

Model Approach:

- One could instead rely on the use models .
- Models use simplifying assumptions to reduce the number of the necessary estimates ⇒ reduces estimation risk.
- However, the assumptions may be nonrealistic – model risk – and may also lead to :
  - Wrong assessment of the investment opportunity set
  - Wrong deduction of tangent portfolios
  - Wrong efficient frontier

⇓

It may also lead to construction of inefficient portfolios

OBS: There is a estimation risk versus model risk tradeoff.

Return Generating Models

We are going to look at three types of return-generating models:

- Constant correlation models (CCM)
- Single factor models (SFM)
- Multi-factor models (MFM)

Their advantages are:

- Reduction of the number of parameters.
- Allow for a ranking of assets that permits three-step procedures to get tangent portfolios, even when shotselling is not allowed!

Questions

1. What are the data problems with mean-variance analysis?
2. Explain what is estimation risk.
3. Explain what is model risk.
4. Why to we have a model versus estimation risk tradeoff when using return generating models for MVT?

3.2 Constant Correlation Models

- Learning objectives
- The average correlation
- Constant correlation models
- Finding tangent portfolios under CCM

Average correlation models

- Biggest amount of parameters to estimate are covariances, or equivalently correlation coefficients,
- It would be nice if we did not need to determine the $n(n-1)/2$ different correlation coefficients.
- What if, instead, we focus our estimation process only on the average correlation

$$\rho = \frac{\sum_{i=1}^{n}\sum_{j>i}^{n}\rho_{ij}}{\frac{n(n-1)}{2}}$$

- And, when applying MVT we assume all correlations equal to ρ.

Constant correlation models

CCM Assumption

We suppose all asset correlations in the market are equal to the average market correlation:

$$\rho_{ij} = \rho \qquad \text{for all} i, j \text{with} i \neq j$$

The number of parameters to estimate reduces to

- $\bar{R}_i, i = 1, 2, \ldots, n$
- $\sigma_i, i = 1, 2, \ldots, n$
- ρ
- That accounts to $2n + 1$ parameters.
- So, for constant correlation models, the number of parameters grows linearly with the number of assets, rather than quadratically.

CCM: ranking of risky assets

- In general, MVT does not allow for any ranking of risky assets.
- The good news is that – if we were willing to accept the constant correlation assumption – ranking of risky assets is possible.

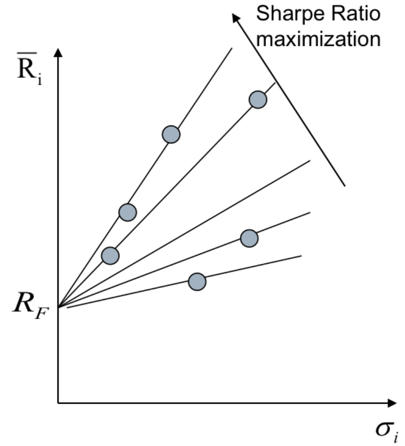
Theorem

Under the CCM assumption, if a given risky asset belongs to the tangent portfolio, then all risky assets with higher Sharpe Ratio also belong to the tangent portfolio.

- This result considerably simplifies the calculations for finding tangent portfolios, even when shortselling is not allowed.
  $\Rightarrow$ It all reduces to a three-step procedure.

Finding Tangent Portfolios

Step 1: ranking by $SR_i = \frac{\bar{R}_i - R_f}{\sigma_i}$



Example: step 1

- **Example:**

$R_F = 5\%$ $\rho = 0.5$ (Average correlation)

Security No. i	Expected Return $\bar{R}_i$	Excess Return $\bar{R}_i - R_F$	Standard Deviation σ_i	Excess Return to Standard Deviation $\frac{(\bar{R}_i - R_F)}{\sigma_i}$
1	29	24	3	8.0
2	19	14	2	7.0
3	29	24	4	6.0
4	35	30	6	5.0
5	14	9	2	4.5
6	21	16	4	4.0
7	26	21	6	3.5
8	14	9	3	3.0
9	15	10	5	2.0
10	9	4	2	2.0
11	11	6	4	1.5
12	8	3	3	1.0

Finding Tangent Portfolios

Step 2: cut-off

- If shortselling is allowed, we know all assets belong to the tangent portfolio and

$$C^* = C_n = \frac{\rho \sum_{i=1}^{n} \left(\frac{\bar{R}_i - R_f}{\sigma_i} \right)}{1 - \rho + n\rho}$$

- C^* is called the cut-off level and it tells us to take:
  - long positions in all assets with $SR_i > C^*$,
  - no investment if it happens $SR_i = C*$, and
  - short positions in all assets with $SR_i < C^*$.

Finding Tangent Portfolios

Step 2: cut-off

- If shortselling is NOT allowed, we do not know how many assets belong to T. So, in Step 2, we need to proceed iteratively, starting from the asset with the highest SR and moving downwards.
- We start by considering T has only one asset, $k = 1$, then $k = 2$, $k = 3$, etc.
- The cut-off C^* is defined

$$C^* = C_k = \frac{\rho \sum_{i=1}^{k} \left(\frac{\bar{R}_i - R_f}{\sigma_i} \right)}{1 - \rho + k\rho} \quad \text{for } k \text{ s.t.} \quad \begin{cases} SR_i > C^* & i = 1, \cdots, k \\ SR_i < C^* & i = k+1, \cdots, n \end{cases}$$

  and we stop at the first k that verifies the condition above.
- Only assets with $SR_i > C^*$ are included in the tangent portfolio T, so, k is its number of assets.

Example: step 2

- **Example (cont)**

Security No. i	$\frac{\rho}{1-\rho+i\rho}$	$\sum_{j=1}^{i} \frac{\bar{R}_j - R_F}{\sigma_j}$	C_i	$\frac{\bar{R}_i - R_F}{\sigma_i}$
1	$\frac{1}{2}$	8	$\frac{8}{2} = 4$	8
2	$\frac{1}{3}$	15	$\frac{15}{3} = 5$	7
3	$\frac{1}{4}$	21	$\frac{21}{4} = 5.25$	6
4	$\frac{1}{5}$	26	$\frac{26}{5} = 5.2$	5
5	$\frac{1}{6}$	30.5	$\frac{30.5}{6} = 5.08$	4.5
6	$\frac{1}{7}$	34.5	$\frac{34.5}{7} = 4.93$	4
7	$\frac{1}{8}$	38	$\frac{38}{8} = 4.75$	3.5
8	$\frac{1}{9}$	41	$\frac{41}{9} = 4.56$	3
9	$\frac{1}{10}$	43	$\frac{43}{10} = 4.3$	2
10	$\frac{1}{11}$	45	$\frac{45}{11} = 4.09$	2
11	$\frac{1}{12}$	46.5	$\frac{46.5}{12} = 3.88$	1.5
12	$\frac{1}{13}$	47.5	$\frac{47.5}{13} = 3.65$	1

C*=5.25 *(shortselling not allowed)*

C*=3.65 *(shortselling allowed)*

Finding Tangent Portfolios

Step 3: composition

- Given the cut-off level, we have the following formula for the entries of our usual vector Z
$$z_i = \frac{1}{(1-\rho)\sigma_i}\left(\frac{\bar{R}_i - R_f}{\sigma_i} - C^*\right),$$
note C^* differs depending on wether we can or cannot shortsell.
- Recall $Z = \lambda X_T$, for λ constant and X_T the vector of weights T,
  - Shortselling allowed: $x_i^T = \frac{z_i}{\sum_{j=1}^n z_j}$
  - Shortselling allowed *a la* Lintner: $x_i^T = \frac{z_i}{\sum_{j=1}^n |z_j|}$
  - Shortselling not allowed: $x_i^T = \frac{z_i}{\sum_{j=1}^k z_j}$

Example: step 3

For our example we get

$$X_T = \begin{pmatrix} 89.54\% \\ 103.41\% \\ 36.25\% \\ 13.87\% \\ 26.15\% \\ 5.35\% \\ -1.58\% \\ -13.47\% \\ -20.44\% \\ -51.11\% \\ -33.28\% \\ -54.68\% \end{pmatrix} \quad X_T^{\text{Lintner}} = \begin{pmatrix} 19.94\% \\ 23.02\% \\ 8.07\% \\ 3.09\% \\ 5.82\% \\ 1.19\% \\ -0.35\% \\ -3.00\% \\ -4.55\% \\ -11.38\% \\ -7.41\% \\ -12.17\% \end{pmatrix} \quad X_T^{\text{no short}} = \begin{pmatrix} 46.32\% \\ 44.21\% \\ 9.47\% \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Example: model risk

Consider the following information concerning three risk assets:

	$\bar{R}$	σ	Correlations		
A	10%	7%	1	-0.4	0.7
B	15%	15%	-0.4	1	0
C	20%	30%	0.7	0	1

For simplicity assume short selling is allowed. Consider the lending rate $R_f^p = 3\%$ and the borrowing rate $R_f^a = 7\%$. Determine:

1. The efficient frontier and the two tangent portfolios T and $T2$.
2. Repeat the same task under the Constant Correlation Model (CCM) assumption.
3. Compare as assess model risk in the context of this example.

Example model risk: tangent portfolios

In this Example the average correlation is $\rho = 0.1$, so for the CCM that is the constant correlation parameter.

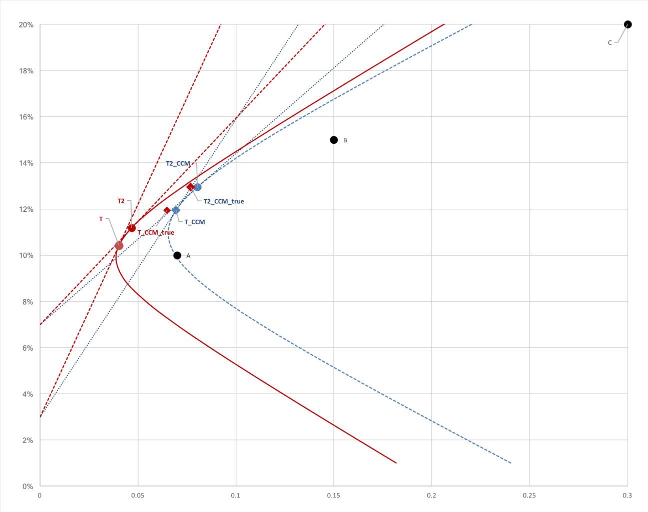
- Tangent Portfolios

$$X_T = \begin{pmatrix} 82.31\% \\ 27.01\% \\ -9.32\% \end{pmatrix} \quad X_T^{CCM} = \begin{pmatrix} 68.60\% \\ 24.37\% \\ 7.37\% \end{pmatrix}$$

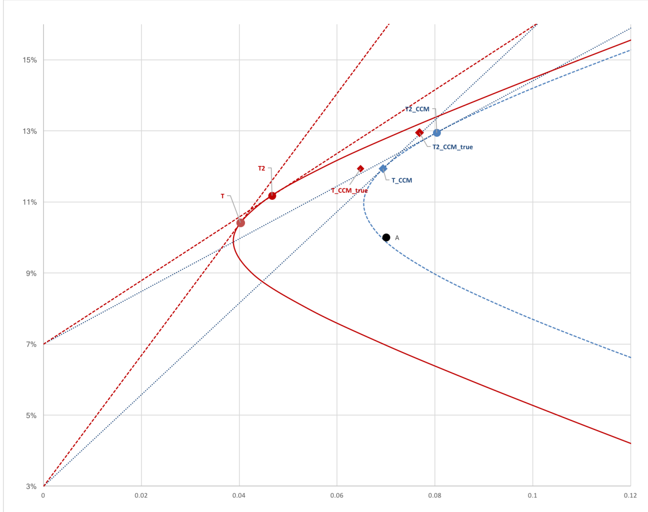
$$X_{T2} = \begin{pmatrix} 72.19\% \\ 32.05\% \\ -4.24\% \end{pmatrix} \quad X_{T2}^{CCM} = \begin{pmatrix} 53.76\% \\ 33.50\% \\ 12.74\% \end{pmatrix}$$

It is clear the tangent portfolios are not the same, under the true correlation structure and assuming a constant correlation. Indeed, using the CCM we would end up recommending non-efficient portfolios.

Example model risk: EF



Example: model risk: EF zoom



Questions

- What is the main assumption underlying CCMs?
- What is the total number of parameters one needs to estimate when using a CCM?
- When is it appropriate to use a CCM?
- When we use the average correlation as the constant correlation in a CCM, is it possible to rank individual assets? How?
- Explain the three-step procedure to find tangent portfolios.

3.3 Single Factor Models

- Learning objectives
- One-factor models
- The beta
- Parameter Estimation
- Finding Tangent Portfolios
- Questions

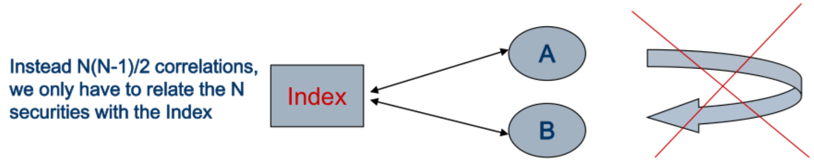
Learning objectives

- define a single factor model mathematically,
- compare the amount of data required for a single factor model with the general MVT setup,
- derive expected return, variance, and covariance.
- define specific, systematic and diversifiable risk,
- discuss and compute variances of large portfolios. factor models.
- Find alpha and beta given times series of returns.
- Derive the relationship between idiosyncratic risk and beta estimation.
- Discuss and use Blume's technique for improving beta estimation.

One-factor models

A very popular approach is to use a one-factor model in which all correlated movement comes a single source.

- One-factor means one common factor.
- In practice, many times this factor is an index.



- This is quite different from having one factor in total.

Mathematical formulation of one-factor model

- We set

$$R_i = \alpha_i + \beta_i R_m + e_i,$$

  where R_m is the return on the market. α_i, β_i, are constants and the variables e_i have mean zero, and are uncorrelated with the market.
- This does not say anything until we make the crucial additional assumption:

$$\mathbb{E}(e_i e_j) = 0, \text{ for } i \neq j,$$

  that is the variables e_i are not correlated with each other.

SFM – Assumption

There is a common factor that is able of capturing all possible dependence between any two assets, and we are able to identify it.

Data reduction

We now have to estimate the parameters:

- $\alpha_i, i = 1, 2, \ldots, n$
- $\beta_i, i = 1, 2, \ldots, n$
- $\bar{R}_m$,
- σ_m^2,
- $\sigma_{e_i}, i = 1, 2, \ldots, n$.
- That accounts to $3n + 2$ parameters.
- For one-factor models, the number of parameters grows linearly with the number of assets, rather than quadratically.
- The zero correlation assumption for the terms e_i vastly reduces the amount of data required.

Essential results for single-factor models

1. $\bar{R}_i = \alpha_i + \beta_i \bar{R}_m$

$$\begin{aligned}\mathbb{E}(R_i) =& \mathbb{E}(\alpha_i + \beta_i R_m + e_i),\\ =& \alpha_i + \beta_i \mathbb{E}(R_m) + \underbrace{\mathbb{E}(e_i)}_{0},\\ =& \alpha_i + \beta_i \bar{R}_m,\end{aligned}$$

2. $\sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma_{e_i}^2$.

$$\begin{aligned}\mathrm{Var}(R_i) =& \mathrm{Var}(\alpha_i + \beta_i R_m + e_i)\\ =& \mathrm{Var}(\beta_i R_m) + \mathrm{Var}(e_i),\\ =& \beta_i^2 \mathrm{Var}(R_m) + \mathrm{Var}(e_i)\\ =& \beta_i^2 \sigma_m^2 + \sigma_{e_i}^2\end{aligned}$$

Covariance

3. $\sigma_{ij} = \beta_i \beta_j \sigma_m^2$, for $i \neq j$.

$$\begin{aligned}\mathrm{Cov}(R_i R_j) =& \mathbb{E}((R_i - \bar{R}_i)(R_j - \bar{R}_j)),\\ =& \mathbb{E}((\beta_i(R_m - \bar{R}_m) + e_i)(\beta_j(R_m - \bar{R}_m) + e_j))\\ =& \beta_i \beta_j \mathbb{E}((R_m - \bar{R}_m)^2) + \beta_i \underbrace{\mathbb{E}((R_m - \bar{R}_m)e_j)}_{0}\\ &+ \beta_j \underbrace{\mathbb{E}((R_m - \bar{R}_m)e_i)}_{0} + \underbrace{\mathbb{E}(e_i e_j)}_{0}.\end{aligned}$$

For $i \neq j$, the last 3 terms are zero from our assumption that e_i are uncorrelated with the market and each other.

Risk division

We have divided the risk fan asset i into two pieces.

$$\sigma_i^2 = \underbrace{\beta_i^2 \sigma_m^2}_{\text{systematic risk}} + \underbrace{\sigma_{e_i}^2}_{\text{specific risk}}$$

- The first part arises from exposure to the market and is called systematic risk.
- The remaining part is called specific risk, and is the part unique to the security and can be diversified away. It is also called :
  - alpha risk,
  - diversifiable risk,
  - unsystematic risk,
  - residual risk.

Beta

Clearly, for $i = m$

$$R_m = 0 + 1.R_m,$$

So the market portfolio has

$$\begin{aligned}\alpha_m =& 0,\\ \beta_m =& 1.\end{aligned}$$

OBS: So the level of beta is a measure of risk level compared to the common factor - systematic risk measure.

Using beta - example

- Suppose a stock i has a beta of $\beta_i = 3$.
- Suppose the market goes up 1%.
- What can we say about the stock's return?

$$R_i = \alpha_i + \beta_i R_m + e_i$$

- Assuming everything else constant, our best estimate without further knowledge is:

$$\frac{\partial R_i}{\partial R_m} = \beta_i$$
$$dR_i = \beta_i dR_m$$

i.e., R_i is expected to go up by 3% on average.

OBS: However, any value is possible because of specific part.

Risk of portfolios

Let us now consider a portfolio of n assets, with weights x_i. Its variance is

$$\sigma_p^2 = \text{Var}\left(\sum_{i=1}^n x_i R_i\right) = \mathbb{E}\left(\left(\sum_{i=1}^n x_i (R_i - \bar{R}_i)\right)^2\right)$$
$$= \mathbb{E}\left(\left(\sum_{i=1}^n x_i \beta_i (R_m - \bar{R}_m) + \sum_{i=1}^n x_i e_i\right)^2\right),$$
$$= \mathbb{E}\left(\left(\sum_{i=1}^n \beta_i X_i\right)^2 (R_m - \bar{R}_m)^2\right) + \sum_{i=1}^n x_i^2 \mathbb{E}(e_i^2),$$
$$= \left(\sum_{i=1}^n x_i \beta_i\right)^2 \sigma_m^2 + \sum_{i=1}^n x_i^2 \sigma_{e_i}^2$$
$$= (\beta_P)^2 \sigma_m^2 + \sigma_{eP}^2$$

Beta of large portfolios

- Let us consider large homogeneous portfolios.
- In that case, if we consider n assets are $x_i = \frac{1}{n}$ for $i = 1, \cdots, n$
- An the homogenous portfolio variance is

$$\sigma_H^2 = \left(\sum_{i=1}^n \frac{1}{n}\beta_i\right)^2 \sigma_m^2 + \sum_{i=1}^n \left(\frac{1}{n}\right)^2 \sigma_{e_i}^2$$
$$= \frac{\sum_{i=1}^n \beta_i}{n} + \frac{1}{n}\frac{\sum_{i=1}^n \sigma_{e_i}^2}{n}$$
$$= (\bar{\beta})^2 \sigma_m^2 + \frac{1}{n}\overline{\sigma_{e_i}^2},$$

where the term $\bar{\beta}$ is the average beta and $\overline{\sigma_{e_i}^2}$ is the average specific variance.
- As $n \to \infty$ we get, the second term goes to zero individual specific risks are bounded,

$$\lim_{n\to\infty} \sigma_H^2 = (\bar{\beta})^2 \sigma_m^2$$

Large portfolio limit

- By using a large portfolio of many different stocks, one can make the diversifiable risk disappear (which is why it's called that.)

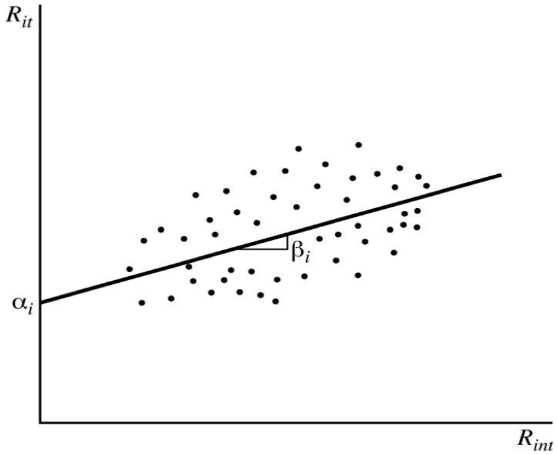
Number of Securities	Residual Risk (Variance) Expressed as a Percent of the Residual Risk Present in a One-Stock Portfolio with σ_{ei}^2 a Constant
1	100
2	50
3	33
4	25
5	20
10	10
20	5
100	1
1000	0.1

- The beta part, the undiversifiable risk, does not disappear.
- We expect risk premia for taking beta risk, but not diversifiable risk.

Parameter Estimation

We need to estimate the parameters α_i, β_i and σ^2_{ei}, knowing

$$R_i = \alpha_i + \beta_i R_m + e_i$$



Using linear regression

The historical Approach:

- Let $R_{i,t}$ and $R_{m,t}$ denote the returns of asset i and the market m for the t^{th} period.
- Take the sample size be T, i.e., $t = 1, \cdots, T$
- Compute average returns of asset i, $\bar{R}_i$, and market, $\bar{R}_m$.
- Find $\text{Cov}(R_i, R_m) = \frac{1}{T}\sum_{t=1}^{T}(R_{i,t} - \bar{R}_i)(R_{m,t} - \bar{R}_m)$.
- And $\sigma^2_m = \text{Var}(R_m) = \frac{1}{T}\sum_{t=1}^{T}(R_{m,t} - \bar{R}_m)^2$
- Then, β_i is determined by

$$\beta_i = \frac{\text{Cov}(R_i, R_m)}{\text{Var}(R_m)}.$$

Using linear regression

- Once we have β_i, we use $\mathbb{E}(R_i - (\alpha_i + \beta_i R_m)) = 0$, to determine α_i

$$\alpha_i = \mathbb{E}(R_i - \beta R_m) = \frac{1}{T}\sum_{t=1}^{T}(R_{i,t} - \beta R_{m,t}).$$

- Finally to get σ^2_{ei}, we use both α_i, β_i we can determine all e_{it}

$$e_{it} = R_{i,t} - (\alpha_i + \beta_i R_{m,t})$$

and

$$\sigma^2_{ei} = \text{Var}(e_i) = \frac{\sum_{t=1}^{T}(e_{i,t})^2}{T}$$

Commentary

- Linear regression almost always works, in the sense that it will always find a straight line through a cloud of points that minimizes the least-squares error.
- The residual generated will be uncorrelated with the market and will have zero expectation.
- However, the residuals may well have high correlation with each other.
- The model says that the residuals are uncorrelated with each other.
- So when they are highly correlated the model is a poor fit to the data, and one should consider carefully whether it should be used.

How good are the estimates?

- It's important to realize that the estimates can contain a lot of noise.
- Even if the model was wholly correct, our computation of the values of α and β would contain errors – estimation risk.
- Suppose we start with a true model with parameters as follows

α	1%,
β	2,
R_m standard deviation	2.5%,
e_i standard deviation	2%,
R_m mean	3%,

and generate a time series – a possible sample – and then measure the implied α and β.

Q:What happens?

Synthetic data

We give such a series:

market return	stock return
6.430%	14.48%
3.998%	8.89%
2.350%	5.57%
0.506%	2.43%
2.431%	5.44%
8.445%	14.79%
0.624%	-1.49%
2.484%	11.01%
0.151%	3.08%

Using this sample we get:

$$\alpha_i^{sample} = 1.73\% \quad , \quad \beta_i^{sample} = 1.77 \ .$$

Many times

Suppose we do this to get many different samples:

α^{sample}	β^{sample}
1.17	1.93
1.77	1.75
1.51	1.78
1.91	1.72
1.57	1.78
1.76	1.81
2.37	1.56
2.33	1.51
0.06	2.34
1.12	2.03
-1.08	2.65
-6.50	4.51

Estimation risk

- We see that there is a lot of variation. Some of the values of β are quite good but some are terrible. You would never know which is the case with real data. $\rightarrow$ there is estimation risk, even within a model.
- How can we get less noisy estimates? We need more data points. We can use monthly returns instead of yearly returns to estimate α and β. This will give us less noise but will provide a good estimate of month on month correlation rather than year on year correlation.
- These will be similar but there is evidence that they are different.

Portfolio effects

- The size of the idiosyncratic (specific) risk is an important factor in determining the accuracy of estimations of beta.
- In particular, if we take a portfolio that has had much of the idiosyncratic risk diversified away, the beta estimate will be much better.
- The more similar to the portfolio is to the market, the less residual risk there will be.

Beta bias

- If we observe a large sample of stocks and compute the beta for each, we will get a distribution of betas.
- Each of these betas will have an error.
- If the observed beta is very high then the high value is as likely to come from upwards error as from the true value being high.
- Similarly, for very low betas, it is as likely to come from a large downwards error as from the true value being low.
- This suggests that betas should have a tighter distribution than typically observed.

OBS: True betas should be closer to one than observed ones.

Blume Adjustment

If the distribution of observed betas is too wide then

- observed betas that are high should be overestimates,
- observed betas that are low should be underestimates

Measure betas of stocks in one period, β_1, and then remeasure them for the same stocks in the next period, β_2.

If we are correct then high betas should tend to go down, and low betas should tend to go up.

If we regress to get

$$\beta_2 = a\beta_1 + b,$$

then we should get $a < 1$.

Blume Adjustment

- Blume measured the value of β for a range of stocks in a number of periods.
- He then regressed the beta for each stock in each period against its beta in the preceding period.
- He found in each case that the straight line had slope less than one.

Period	Preceding period	Regression
July 33 - June 40	July 26 - June 33	$\beta_2 = 0.320 + 0.714\beta_1$
July 40 - June 47	July 33 - June 40	$\beta_2 = 0.265 + 0.750\beta_1$
July 47 - June 54	July 40 - June 47	$\beta_2 = 0.526 + 0.489\beta_1$
July 54 - June 61	July 47 - June 54	$\beta_2 = 0.343 + 0.677\beta_1$
July 61 - June 68	July 54 - June 61	$\beta_2 = 0.399 + 0.546\beta_1$

So, if the observed value in 54 to 61 was 1.4, then our best estimate from 61 to 68 is $\beta_2 = 0.399 + 0.546 \times 1.4 = 1.1634$

Critique of Blume Adjustment

- Blume's method is pretty *ad hoc*. It does not have a firm mathematical foundation, but is just a generally plausible idea.
- It also does not take into account that for stocks with low idiosyncratic risk, the beta will be more accurate, which suggests less scaling for such stocks.
- Another issue with the Blume method is that it does not ensure that a portfolio with $\beta = 1$ and no idiosyncratic risk, i.e. the market!, is mapped to a portfolio with $\beta = 1$.

Vasicek Adjustment

- Vasicek has suggested a Bayesian technique that corrects this deficiency, as with all Bayesian techniques it requires the user to have a *prior* distribution, that is a view on what distribution of βs is reasonable.
$$\beta_{2i} = \frac{\sigma^2_{\beta_{1i}}}{\sigma^2_{\beta_{1i}} + \sigma^2_{\bar{\beta}_1}}\bar{\beta}_1 + \frac{\sigma^2_{\bar{\beta}_1}}{\sigma^2_{\beta_{1i}} + \sigma^2_{\bar{\beta}_1}}\beta_{1i}$$
where $\bar{\beta}_1$ equal the average beta across a sample of stocks and $\sigma^2_{\bar{\beta}_1}$ is the variance of the distribution of historical estimates of beta over the same sample of stocks.
- Note $\sigma^2_{\bar{\beta}_1}$ is a measure of the uncertainty associated with the measurement of individual betas.
- For asset i, the Vasicek procedure involves taking a weighted average of $\bar{\beta}_1$ and the historical beta of asset i, β_{i1}.

Fundamental analysis

- Another approach is to consider where does β come from?
- What makes one company riskier than another?
- One view is that it comes from the fundamental characteristics of the firm.
- If we define beta in terms of these, we get a measure that is much more reactive.

Example: If the company risks up its strategy by being much more leveraged, e.g. issuing a large amount of debt, it will immediately be reflected in the beta; while with the historical method it would take time to filter through to the measured values.

Carrying out fundamental analysis

What sort of quantities could be regarded as important?

- Dividend to earnings ratio
- Asset growth, i.e., percentage increase
- Leverage, e.g. debt to equity ratio
- Earnings variability

How would we use fundamental characteristics in practice?

- Measure betas over some past period eg 5 years.
- Carry out linear regression of the betas against the fundamental characteristics.
- Use the regressed equation to predict the future beta for each firm.

Testing SFMs

- To test SFMs we can use the fact that, its assumptions imply

$$\rho_{ij} = \frac{\beta_i \beta_j \sigma_m^2}{\sigma_i \sigma_j} .$$

- Using this we can test betas through their perfomance as correlation coefficients' forecaster.
- Empirical evidence based upon correlations shows that:
  - Non adjusted betas underestimate average correlations.
  - Blume's Technique tens to overestimate average correlations.
  - Vasicek's Technique leads to both underestimated and overestimated estimatives, such that its final effect is not cristal clear.

SFM: ranking of risky assets

- Recall that in general, MVT does not allow for any ranking of risky assets.
- The good news is that – if we were willing to accept the SFM assumptions – ranking of risky assets is possible.

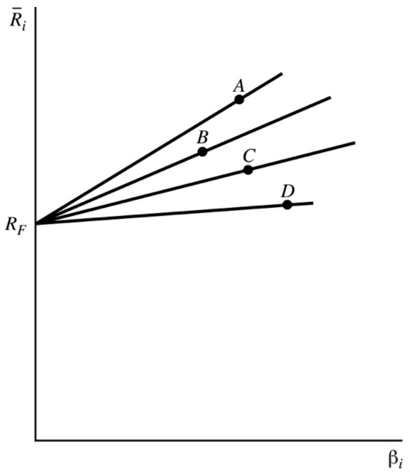
Theorem

Under the SFM assumptions, if a given risky asset belongs to the tangent portfolio, then all risky assets with higher Excess Return towards Beta (ERB) also belong to the tangent portfolio.

- This result considerably simplifies the calculations for finding tangent portfolios, even when shortselling is not allowed.
  $\Rightarrow$ It all reduces to a three-step procedure.

Finding Tangent Portfolios

Step 1: Ranking by $ERB_i = \dfrac{\bar{R}_i - R_f}{\beta_i}$



Example: step 1

Consider the following 10 risky assets ranked by excess return over beta (ERB), and suppose and $R_f = 5\%$ (both for lending and borrowing)

1 Security No. i	2 Mean Return $\bar{R}_i$	3 Excess Return $\bar{R}_i - R_F$	4 Beta β_i	5 Unsystematic Risk σ_{ei}^2	6 Excess Return over Beta $\frac{(\bar{R}_i - R_F)}{\beta_i}$
1	15	10	1	50	10
2	17	12	1.5	40	8
3	12	7	1	20	7
4	17	12	2	10	6
5	11	6	1	40	6
6	11	6	1.5	30	4
7	11	6	2	40	3
8	7	2	0.8	16	2.5
9	7	2	1	20	2
10	5.6	0.6	0.6	6	1.0

Finding Tangent Portfolios

Step 2: cut-off

- If shortselling is allowed, we know all assets belong to the tangent portfolio and

$$C^* = C_n = \frac{\sigma_m^2 \sum_{i=1}^{n} \frac{(\bar{R}_i - R_f)\beta_i}{\sigma_{ei}^2}}{1 + \sigma_m^2 \sum_{i=1}^{n} \frac{\beta_i^2}{\sigma_{ei}^2}}$$

- C^* is called the cut-off level and it tells us to take:
  - long positions in all assets with $SR_i > C^*$,
  - no investment if it happens $SR_i = C*$, and
  - short positions in all assets with $SR_i < C^*$.

Finding Tangent Portfolios

Step 2: cut-off

- If shortselling is NOT allowed, we do not know how many assets belong to T. So, in Step 2, we need to proceed iteratively, starting from the asset with the highest ERB and moving downwards.
- We start by considering T has only one asset, $k = 1$, then $k = 2$, $k = 3$, etc.
- The cut-off C^* is defined

$$C^* = C_k = \frac{\sigma_m^2 \sum_{i=1}^{k} \frac{(\bar{R}_i - R_f)\beta_i}{\sigma_{ei}^2}}{1 + \sigma_m^2 \sum_{i=1}^{k} \frac{\beta_i^2}{\sigma_{ei}^2}} \quad \text{for } k \text{ s.t. } \begin{cases} ERB_i > C^* & i = 1, \cdots, k \\ ERB_i < C^* & i = k+1, \cdots, n \end{cases}$$

  and we stop at the first k that verifies the condition above.
- Only assets with $SR_i > C^*$ are included in the tangent portfolio T, so, k is its number of assets.

Example: step 2

1 Security No. i	2 $\frac{(\bar{R}_i - R_F)}{\beta_i}$	3 $\frac{(\bar{R}_i - R_F)\beta_i}{\sigma_{ei}^2}$	4 $\frac{\beta_i^2}{\sigma_{ei}^2}$	5 $\sum_{j=1}^{i} \frac{(\bar{R}_j - R_F)\beta_j}{\sigma_{ej}^2}$	6 $\sum_{j=1}^{i} \frac{\beta_j^2}{\sigma_{ej}^2}$	7 C_i
1	10	2/10	2/100	2/10	2/100	1.67
2	8	4.5/10	5.625/100	6.5/10	7.625/100	3.69
3	7	3.5/10	5/100	10/10	12.625/100	4.42
4	6	24/10	40/100	34/10	52.625/100	5.43
5	6	1.5/10	2.5/100	35.5/10	55.125/100	5.45
		3/10	7.5/100	38.5/10	62.625/100	5.30
		3/10	10/100	41.5/10	72.625/100	5.02
8	2.5	1/10	4/100	42.5/10	76.625/100	4.91
9	2.0	1/10	5/100	43.5/10	81.625/100	4.75
10	1.0	0.6/10	6/100	44.1/10	87.625/100	4.52

C*=5.45 (shortselling not allowed)

C*=4.52 (shortselling allowed

Finding Tangent Portfolios

Step 3: composition

- Given the cut-off level, we have the following formula for the entries of our usual vector Z

$$z_i = \frac{\beta_i}{\sigma_{e_i}^2}\left(\frac{\bar{R}_i - R_f}{\beta_i} - C^*\right),$$

  note C^* differs depending on wether we can or cannot shortsell.
- Recall $Z = \lambda X_T$, for λ constant and X_T the vector of weights T,
  - Shortselling allowed: $x_i^T = \frac{z_i}{\sum_{j=1}^{n} z_j}$
  - Shortselling allowed *a la* Lintner: $x_i^T = \frac{z_i}{\sum_{j=1}^{n} |z_j|}$
  - Shortselling not allowed: $x_i^T = \frac{z_i}{\sum_{j=1}^{k} z_j}$

Example: step 3

Security	Short Sales Disallowed	Lintner Definition of Short Sales	Standard Definition of Short Sales
1	23.5	8.0	647.1
2	24.6	9.5	770.6
3	20.0	9.0	729.4
4	28.4	21.5	1741.2
5	3.5	2.7	217.6
6	0	−1.9	−152.9
7	0	−5.5	−447.1
8	0	−7.3	−594.1
9	0	−9.1	−741.2
10	0	−25.5	−2070.6

Theory questions

1. What is a single factor model?
2. Derive expressions for expected return, variance and covariance.
3. Define specific risk, systematic risk and diverifiable risk.
4. Derive expressions for the beta and variance of a portfolio.
5. What happens to the variance of a large portfolio as the number of assets goes to infinity?

Theory questions

6. Explain how to find SFM parameters using linear regressions on historical market data.
7. How can we improve the stability of α and β estimates when using a single-factor model?
8. How could you assess if a single factor model is suitable for fitting a covariance matrix?
9. If we linearly regress the betas in one period against a previous period, what properties would we expect of the coefficients found?
10. Describe Blume's adjustment and discuss briefly why it is plausible.
11. Explain the idea underlying the Vasicek's adjustment.
12. How can a SFM be tested? Explain.

3.3 Multi-factor Models

- Learning objectives
- Multi-factor models
- Properties
- Model Types

Learning objectives

- motivate the use of multi-factor models,
- define a multi-factor model,
  - derive and use expressions for the expected returns, variances and covariances,
  - state how many parameters are required,
  - classify sorts of multi-factor models,
- discuss how to compare models for covariance matrices, and state some results,

The need for multi-factor models

- A single-factor model may be too simplistic.
- Do we really believe that all correlation between stocks arises from the level of the market?
- Stocks in the same sector tend to have much more correlation than stocks from competing sectors.
- Stocks from different countries will have less correlation than stocks from the same country.
- One can attempt to introduce more complexity into the modelling by assuming return correlations can be explained my more than one common factor.

The multi-factor model

- Let us consider a a number of uncorrelated indices, I_j.

$$\text{Cov}(I_i, I_j) = 0 \quad \forall i, j$$

- We set

$$R_i = a_i + \sum_{k=1}^{K} b_{ik} I_j + c_i, \qquad (1.1)$$

  where the numbers a_i and b_{ij} are constants, while c_i are random but uncorrelated with the indices, I_k and with zero mean.
- As in the single-factor case, c_i expresses the idiosyncratic risk. We make the key simplifying assumption:

$$\mathbb{E}(c_i c_j) = 0.$$

- Without the independence assumptions – uncorrelated indices and specific components across assets – the model would not reduce complexity.

Model properties

Let us consider K indices, $k = 1, 2, \cdots, K$, all uncorrelated with one another. We denote $\text{Var}(I_k) = \sigma_k^2$ and $var(c_i) = \sigma_{c_i}^2$.

In the context of MFM, we have:

1. $\mathbb{E}(R_i) = a_i + \sum_{k=1}^{K} b_{ik}\mathbb{E}(I_k)$,
2. $\text{Var}(R_i) = \sum_{k=1}^{K} b_{ik}^2 \sigma_k^2 + \sigma_{c_i}^2$,
3. $\text{Cov}(R_i, R_j) = \sum_{k=1}^{K} b_{ik} b_{jk} \sigma_k^2$

Q: How to get to the above expressions?

Expectation

1. The first of these follows from the linearity of expectation:

$$\mathbb{E}(R_i) = \mathbb{E}(a_i) + \sum_{k=1}^{K} \mathbb{E}(b_{ik} I_k) + \mathbb{E}(c_i),$$
$$= a_i + \sum_{k=1}^{K} b_{ik} \mathbb{E}(I_k).$$

Variance

2. For variance, we need to compute the expectation of $(R_i - \mathbb{E}(R_i))^2$. This is equal to

$$\sigma_i^2 = \mathbb{E}\left(\left(c_i + \sum_{k=1}^{K} b_{ik}(I_k - \mathbb{E}(I_k))\right)^2\right).$$

We can discard all cross terms because of zero correlation, so we get

$$\sigma_i^2 = \mathbb{E}\left(c_i^2\right) + \sum_{k=1}^{K} b_{ik}^2 \mathbb{E}(((I_k - \mathbb{E}(I_k))^2)$$
$$= \sigma_{c_i}^2 + \sum_{k=1}^{K} b_{ik}^2 \sigma_k^2.$$

Covariance formula

3. For the covariance, we need to compute

$$\mathbb{E}((R_i - \mathbb{E}(R_i))(R_j - \mathbb{E}(R_j)).$$

This is equal to

$$\sigma_{ij} = \mathbb{E}\left(\left(c_i + \sum_{k=1}^{K} b_{ik}(I_k - \mathbb{E}(I_k))\right)\left(c_j + \sum_{v=1}^{K} b_{jl}(I_v - \mathbb{E}(I_v))\right)\right)$$

As with the variance, all cross terms disappear, and we obtain

$$\sigma_{ij} = \mathbb{E}\left(\sum_{k=1}^{k} b_{ik}(I_k - \mathbb{E}(I_k)) b_{jk}(I_k - \mathbb{E}(I_k))\right),$$

which is equal to

$$\sigma_{ij} = \sum_{k=1}^{L} b_{ik} b_{jk} \sigma_k^2.$$

Data requirements

We need less data than for a general model but more than for a single factor model. The model has

$$2n + 2K + Kn$$

parameters to calibrate.

OBS: *If $n \sim 500$, and $K \sim 10$, this is a lot better than the general model but a lot worse than the single factor model.*

Q: *For a fixed number of risky assets n, what is the maximum number of factors K that makes sense to consider?*

Types of multi-factor models

There are various different quantities, we could use as driving factors for our multi-factor model. These include

- Sector-based models.
- Macro-economic models.
- Statistical models.

One could also consider mixed models – models that are a mixture of the above types.

Sector-based models

- Sector-based models are intuitive and popular.
- One has an index for the overall market and then one for each industrial sector.
- Examples of sectors are
  - banks,
  - oil,
  - pharmaceuticals,
  - steel.
- One could also extend this model to have factors for each country, as well for each sector.

  ⇒ There is evidence that the market index accounts for about $30 - 50\%$ of the variability of stocks, and that introducing industrial sectors explains another 10%.

OBS: The factors mentioned above do not fit our model definition, since they are not uncorrelated (but this can be tackled.)

Macro-economic model

Assets' return correlations are driven by the wider economy.

Factors could include:

- price of oil,
- inflation,
- government bond yields,
- corporate bond yields (or spreads)
- economic growth.

Statistical factor models

- Rather than trying to find economically meaningful variables, one can instead apply statistical techniques to find the factors which explain most correlations.
- The standard technique is principal components analysis.
  Essentially this means finding eigenvalues and eigenvectors of the covariance matrix, one keeps the two or three largest eigenvectors and then discards the rest.

Evaluating multi-factor models

- Are multi-factor models worth the effort? How can we answer this question?
- One way is to observe that they give a better fit to the historical correlation matrix.
  - that is not necessarily a good thing
  - if that were our sole criterion, we would just use the historical correlation matrix
- Remember a theory is assessed by taking the predictions it makes and experimentally testing them.
  - The quality of its assumptions is not necessarily what matters the most.
- A better criterion is: How well does the model predict future correlations?
  - This will determine how well the model does at suggesting good investments when using mean-variance analysis.

The need for orthogonal factors

- We assumed that the indices driving our multi-factor model were uncorrelated.
- This is unrealistic as for real-life choice of indices, these will always be correlated indices:

$$I_1^*, I_2^*, \cdots, I_K^*$$

Examples:

- There is correlation between the US market index and the price of oil.
- There is correlation between the market and sector indices.
- etc.

The need for orthogonal factors

Theorem

For finite set of correlated indices, there is always an equivalent finite set of orthogonal factors.

- The standard procedure for removing correlation is the Gram-Schmidt orthogonalization procedure.
- Gram-Schmidt is an algorithm designed for inner-product spaces.
- Gram-Schmidt procedure allows for quick orthogonalisation of factors.
- In many situations we can also use linear regressions.

Example: two-factor models

Given a set of real-life (correlated indices) I_1^*, I_2^*, we can use linear regressions to get an equivalent set of uncorrelated indices I_1, I_2:

$$R_i = a_i^* + b_{i1}^* I_1^* + b_{i2}^* I_2^* + c_i$$

Orthogonal Indexes

$$I_1 = I_1^*$$

$$I_2^* = \gamma_0 + \gamma_1 I_1 + d_t$$

$$d_t = I_2^* - (\gamma_0 + \gamma_1 I_1)$$

$$I_2 = d_t = I_2^* - (\gamma_0 + \gamma_1 I_1)$$

$$R_i = a_i + b_{i1} I_1 + b_{i2} I_2 + c_i$$

$$a_i = (a_i^* + b_{i2}^* \gamma_0)$$

$$b_{i1} = (b_{i1}^* + b_{i2}^* \gamma_1)$$

$$b_{i2} = b_{i2}^*$$

Theory questions

1. State the definition of multi-factor models for stock returns.
2. Derive the expected returns, variances and covariances of returns in multi-factor models for stock returns.
3. What are the advantages of multi-factor models over single-factor models for stock returns?
4. How many parameters are there in a multi-factor model?
5. Give three different ways to choose factors for a multi-factor model.