**Lisbon University**
**Lisbon School of Economics and Management**
Ms in Economics, Mathematical Finance and
Monetary and Financial Economics

## <u>Mathematical Economics</u> – 1st Semester - 2023/2024

More exercises - Group III

1. The initial value problem $\begin{cases} y' = \sqrt{y-1} \\ y(0) = 1 \end{cases}$ admits two different solutions.

   (a) Find explicitly the two different solutions.

   (b) Explain why this does not contradict the Existence and Uniqueness Theorem for ordinary differential equations.

2. Assume that $a > b > 0$ and the Initial value problem (IVP) where $p$ is a function of $t$

$$p' = ap - bp^2, \quad p(0) = p_0 > 0$$

   (a) With respect to the differential equation above, what is the name of $a/b$? (This IVP is called the logistic law).

   (b) Write the unique solution of the differential equation, say $p(t)$.

   (c) Write the domain of $p$ when $p_0 < a/b$.

   (d) Write the domain of $p$ when $p_0 > a/b$.

   (e) Compute $\lim_{t\to+\infty} p(t)$.

3. Consider the differential equation ($y$ is a function of $x$):

$$y'' - 4y' + 4y = e^{2x}$$

   (a) Show that $y(x) = \frac{1}{2}x^2e^{2x}$ is a solution of the differential equation.

   (b) Write the general solution of the differential equation.

4. Consider the following differential equation where $y$ is a function of $x$:

$$y'' + 3y' = 0$$

   (a) Find the solutions of the differential equations whose graph in the plane $(y, y')$ lie in a line.

   (b) Find the solution of the differential equation such that $y(0) = 1$, $y'(0) = -3$. Represent it in the plane $(t, y)$

5. Consider the following differential equation ($y$ is a function of $t$)

$$y'' + 4y' + 5y = 1.$$

(a) Find a particular solution of the differential equation.

(b) Find the general solution of the differential equation.

(c) If $y(0) = 6/5$ and $y'(0) = -2$, trace the graph of the unique solution of the differential equation.

6. Consider the following differential equation and write its general solution.

(a)

$$\begin{cases} \dot{x} = x + 2y \\ \dot{y} = x + y \end{cases}$$

(b)

$$\begin{cases} \dot{x} = -x \\ \dot{y} = x - y \end{cases}$$

(c)

$$\begin{cases} \dot{x} = 2x + y \\ \dot{y} = -2x + 4y \end{cases}$$

7. Find the linearisation of the following systems around their equilibria. Classify the equilibria according to their Lyapunov stability.

(a)

$$\begin{cases} \dot{x} = x + x^2 + xy^2 \\ \dot{y} = y + y^3 \end{cases}$$

(b)

$$\begin{cases} \dot{x} = e^{x+y} - y \\ \dot{y} = -x + xy \end{cases}$$

(c)

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\sin x - y \end{cases}$$

8. Consider the following differential equation:

$$\begin{cases} \dot{x} = 1 - 6y + x^2 \\ \dot{y} = 1 - 2y - x^2 \end{cases}$$

(a) Find the equilibria and determine their Lyapunov stability.

(b) Draw the null-clines.

9. Consider the following differential equation in $\mathbb{R}^3$:

$$\begin{cases} \dot{x} = -3x - 4y + x^2yz \\ \dot{y} = 4x - 3y - xy^2z \\ \dot{z} = z + xyz^2 \end{cases} \tag{1}$$

(a) Show that $(0, 0, 0)$ is an **unstable** equilibrium (according to Lyapunov).

(b) Draw the phase portrait associated to (1) in a small neighbourhood of $(0,0,0)$.

(c) In the phase portrait of (b), locate the **local stable** and the **local unstable** manifolds of $(0,0,0)$.

(d) Suppose that $p \in W^u_{loc}(0,0,0)$. What is the $\alpha$-limit set of $p$?

(e) Suppose that $p \in W^s_{loc}(0,0,0)$. What is the $\omega$-limit set of $p$?