Lisbon School of Economics & Management
Universidade de Lisboa

Universidade de Lisboa
Instituto Superior de Economia e Gestão
BsC in Economics, Finance and Management

Mathematics II – 1st Semester - 2022/2023

Appeal Assessment - 2nd of February 2023

Duration: $(120 + \varepsilon)$ minutes, $|\varepsilon| \leq 30$

Version A

Name: ....

Student ID #: ....

Part I

- Complete the following sentences in order to obtain true propositions. The items are independent from each other.
- There is no need to justify your answers.

(a) *(6)* If $A : \mathbb{R}^3 \to \mathbb{R}^3$ is a non-invertible linear map such that

$$A(1,0,0) = (1,0,0) \qquad \text{and} \qquad A(0,0,3) = (0,0,3),$$

then the eigenvalues of A are .... The algebraic multiplicity of ..... is 1.

(b) *(6)* The quadratic form Q associated to the symmetric matrix $\begin{pmatrix} \ldots\ldots & 3 \\ \ldots\ldots & \ldots\ldots \end{pmatrix}$ is given by

$$Q(x,y) = 9x^2 + \ldots\ldots xy + y^2,$$

which is .... defined.

(c) *(5)* The (maximal) domain of $f : D_f \to \mathbb{R}$ is the set

$$D_f = \{(x,y) \in \mathbb{R}^2 : (x > 0 \wedge y > 0) \vee (x < 0 \wedge y < 0)\}.$$

One possible analytical expression for f is

$$f(x,y) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

(d) *(7)* With respect to the set

$$\Omega = \{(x,y) \in \mathbb{R}^2 : 1 < x^2 + y^2 \leq 3\} \cup \{(0,0)\},$$

we may conclude that $(0,0)$ is not a/an .... point of Ω,

$int(\Omega) = \{(x,y) \in \mathbb{R}^2 :$ ....$\}$,

and

$$\iint_\Omega 1 \quad \mathrm{dy\ dx} = \text{........}$$

(e) *(4)* The continuous map $f(x,y) = x^2 - \sin(xy)$ when restricted to

$M = \big\{(x,y) \in \mathbb{R}^2 :$ ....$\big\}$

has a global maximum and a global minimum. This is a consequence of ....'s Theorem.

(f) *(3)* The map $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) =$ .... is continuous but not differentiable at $x = 2$.

(g) *(6)* If $f(x,y) = \dfrac{x^2 y}{x^2 - y^2}$ where $x \neq \pm y$, then

$$\text{..........} = \lim_{(x,y)\to(0,0),y=x^2+x} f(x,y) \neq \lim_{(x,y)\to(0,0),y=3x} f(x,y) = \text{..........},$$

which means that f

....

(h) *(4)* If $u_n = \left(\left(1+\frac{1}{n}\right)^n, \sqrt{n} - \sqrt{n-1}\right)$ is a sequence in $\mathbb{R}^2$ and $f(x,y) = \ln x + 2\cos y$, then

$$\lim_{n\to+\infty} f(u_n) = \text{..........}$$

(i) *(5)* The map $f(x,y) = \ln\left(\frac{(x+y)^2}{xy}\right)$ is positively homegeneous of degree ....
In this case, the *Euler identity* says that (**do not** compute explicitly the derivatives)

....

(j) *(6)* If $f(x,y) = x^2 y^3$, $x(t) = te^t$ and $y(t) = 1 + t^2$, by the *Chain rule* we get:

$\dfrac{df}{dt}(t) =$ ....

(k) *(3)* With respect to the map $f : \mathbb{R} \to \mathbb{R}$, we know that $f'(3) = 0$ and $f''(3) > 0$. Then $f(3)$ is a local .... of f.

(l) *(4)* The following equality holds

$$\int_0^2 \int_0^{x^2} e^{x+3y} \text{ dy dx} = \int_{........}^{........} \int_{........}^{........} e^{x+3y} \text{ dx dy}.$$

(m) *(3)* The differential of order 2 of the map $f(x, y) = \sqrt{3}x + 4y$ at the point $(0, 0)$ is given by

$$D_2 f(0,0)(h_1, h_2) = ...h_1^2 + ...h_1 h_2 + ...h_2^2$$

(n) *(4)* The map $y(x) = \cos(x)$, $x \in \mathbb{R}$, is a solution of the IVP

$$\begin{cases} y'' = ... \\ y(......) = \sqrt{2}/2 \\ y'(0) = 0 \end{cases}$$

(o) *(3)* The absolute maximum of $f(x, y) = y$ when restricted to the set

$$M = \{(x, y) \in \mathbb{R}^2 : (x - 4)^2 + y^2 = ...\}$$

occurs at $(x, y) = (......, \sqrt{7})$.

(p) *(6)* The .... law (associated to a given population of size p that depends on the time $t \geq 0$) states that

$$p' = kp, \qquad k \in \mathbb{R} \text{ (parameter)}.$$

If $p(0) = 10$ and $k = -1$, then $p(10) =$ .... and $\lim_{t \to +\infty} p(t) =$ ....

(q) *(5)* The graph of the solution of the IVP $\begin{cases} y' = 3x \\ y(0) = 2 \end{cases}$ is

(y is a function of x)

Part II

- Give your answers in exact form. For example, $\frac{\pi}{3}$ is an exact number while 1.047 is a decimal approximation for the same number.
- In order to receive credit, you must show all of your work. If you do not indicate the way in which you solve a problem, you may get little or no credit for it, even if your answer is correct.

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1. For $\alpha \in \mathbb{R}\backslash\{0\}$, consider the following matrix $\mathbf{A} = \begin{bmatrix} 2\alpha & \alpha & \alpha \\ \alpha & 2\alpha & 0 \\ \alpha & 0 & 2\alpha \end{bmatrix}$.

   (a) Show that, for all $\alpha \neq 0$, the vector $(0, 1, -1)$ is an eigenvector of $\mathbf{A}$. It is associated to which eigenvalue?

   (b) For $\alpha = 1$ and $X \in \mathbb{R}^3$:

   (i) classify the quadratic form $Q(X) = X^T \mathbf{A}\ X$.

   (ii) solve the equation $X^T \mathbf{A}\ X = 0$.

2. Consider the map $f(x, y) = \begin{cases} \dfrac{2x^2y^2}{\sqrt{x^2+y^2}} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$.

   (a) Show that f is continuous in $\mathbb{R}^2$.

   (b) Compute the directional derivative of f at $(1, 1)$ along the vector $(0, 1)$.

3. Consider the map f defined in $\mathbb{R}^2$ as

$$f(x, y) = xy\, e^{x+y}$$

   (a) Identify and classify the critical points of f.

   (b) Show that f does not have global extrema (neither maximum nor minimum).

4. Let $\Omega \subset \mathbb{R}^2$ be the region defined by

$$\Omega = \{(x, y) \in \mathbb{R}^2 : \quad x \geq 0, \quad y \geq x, \quad y \leq 2 - x^2\}.$$

(a) Draw the set Ω in the plane (x, y).

(b) Compute $\iint_\Omega xy \;\, \mathrm{dx}\,\mathrm{dy}$.

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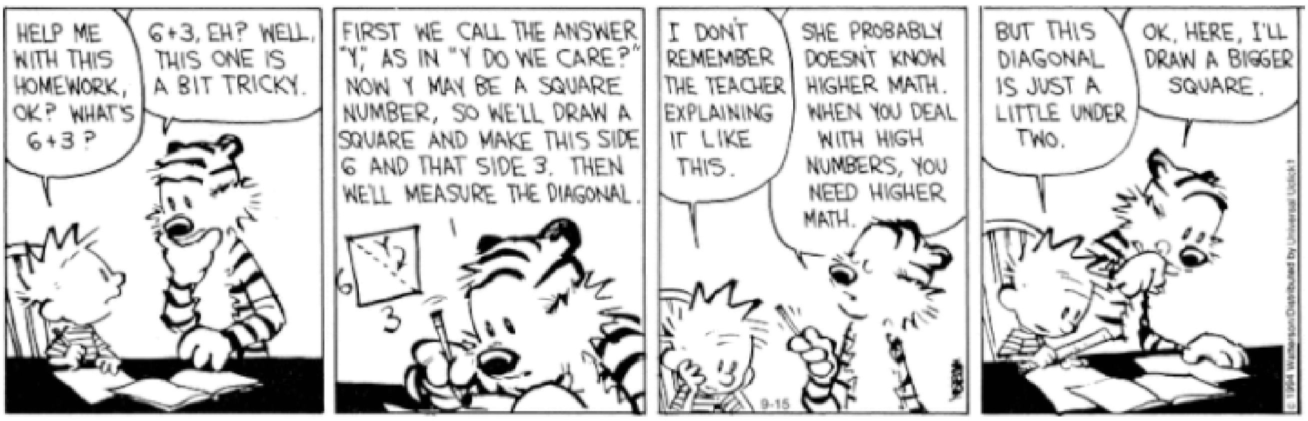
5. Consider the following IVP (y is a function of x):

$$\begin{cases} y'' + 2y' + y = x^2 \\ y'(0) = 1 \\ y(0) = 0 \end{cases}$$

Write the solution $y(x)$ of the IVP.



Score:

I	II.1(a)	II.1(b)	II.2(a)	II.2(b)	II.3(a)	II.3(b)	II.4(a)	II.4(b)	II.5
80	15	10+5	15	10	15	15	5	10	20