Universidade de Lisboa
Instituto Superior de Economia e Gestão
BsC in Economics, Finance and Management

Mathematics II – 1st Semester - 2023/2024

Regular Assessment - 4th of January 2024

Duration: $(120 + \varepsilon)$ minutes, $|\varepsilon| \leq 30$

Version C

Name: ....

Student ID #: ....

Part I

- Complete the following sentences in order to obtain true propositions. The items are independent from each other.
- There is no need to justify your answers.

(a) *(4)* If $A : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map such that $\det A = 0$ and $A(1, -2) = (-2, 4)$, then the eigenvalues of A are .... In particular, A is not ....

(b) *(7)* The (maximal) domain of $f : D_f \to \mathbb{R}$ defined by

$$f(x, y) = \frac{\ln(x - y^2 - 1)}{\sqrt{y}}$$

is the set D_f =..
and its planar representation in the cartesian plane (x, y) is:

(c) *(4)* The symmetric matrix associated to the quadratic form

$$Q(x,y,z) = 3x^2 + ...xz - y^2 - 5yz$$

is

$$\begin{pmatrix} 3 & ..... & 2 \\ ..... & ..... & .... \\ 2 & ..... & .... \end{pmatrix}$$

(d) *(7)* With respect to the set

$$\Omega = \{(x,y) \in \mathbb{R}^2 : |x-5| < 1 \ \wedge \ 4 \leq y < 7\} \cup \{(0,0)\},$$

we may conclude that $(0,0)$ is not a/an .... point of Ω,

$\overline{\Omega} = cl(\Omega) =$ ....,

and

$$\iint_{\Omega} 1 \quad \mathrm{dx\ dy} = ...$$

(e) *(6)* The continuous map $f(x,y) = \dfrac{1}{x^2+y^2}$ defined in $\mathbb{R}^2 \backslash \{(0,0)\}$ has a global maximum and a global minimum when restricted to the set

$$M = \left\{(x,y) \in \mathbb{R}^2 \backslash \{(0,0)\} : ....\right\}$$

This is a consequence of ....'s Theorem (since f is continuous and M is compact).

(f) *(4)* Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R} \backslash \{3\} \to \mathbb{R}$ be defined by

$$f(x,y) = x + y^2 \quad \text{and} \quad g(x) = 2 + \frac{5}{x-3}$$

Then

$$\lim_{(x,y)\to(-1,3)} [g \circ f(x,y)] = ...$$

(g) *(4)* If $u_n = \left(\left(1 + \dfrac{3}{n}\right)^n, \dfrac{\pi}{n}\right)$, $n \in \mathbb{N}$ and $f(x,y) = \ln x + \cos y$, then

$$\lim_{n\to+\infty} f(u_n) = ...$$

(h) *(6)* The map $f(x,y) = x^5\sqrt{y}$ is positively homegeneous of degree .... In this case, when $y > 0$, the *Euler identity* says that (**compute** explicitly the derivatives)

....

(i) *(6)* If $f(x,y) = x^2y$, $x(t) = e^t$ and $y(t) = \sin t^2$, by the *Chain rule* we get:

$$\frac{df}{dt}(t) = \text{..}$$

(j) *(4)* The gradient vector of $f : \mathbb{R}^2 \to \mathbb{R}$ is given by $(2xy^3, 3x^2y^2 - \cos y)$. If $f(x,y)$ does not have constant terms in both components, then

$$f(x,y) = \text{..}$$

(k) *(4)* With respect to the C^2 map $f : \mathbb{R}^2 \to \mathbb{R}$, we know that $\nabla f(3,2) = (0,0)$ and

$$H_f(3,2) = \begin{pmatrix} \text{......} & 0 \\ 0 & \text{......} \end{pmatrix}.$$

Then, $f(3,2)$ is a local maximum of f.

(l) *(4)* With respect to a C^2 map $f : \mathbb{R}^2 \to \mathbb{R}$, we know that $\dfrac{\partial^3 f}{\partial x^2 \partial y}(x,y) = 5x^6y^7$. Then we may conclude that:

$$\frac{\partial^4 f}{\partial x^2 \partial y^2}(x,y) = \text{............}$$

(m) *(5)* The Taylor expansion of order 2 of $f(x,y) = e^{-y}$ at the point $(0,0)$ is given by

....

(n) *(4)* The following equality holds

$$\int_0^2 \int_0^{x^4} \cos(8x+3y)\ \mathrm{dy}\ \mathrm{dx} = \int_{\ldots\ldots\ldots}^{\ldots\ldots\ldots} \int_{\ldots\ldots\ldots}^{\ldots\ldots\ldots} \cos(8x+3y)\ \mathrm{dx}\ \mathrm{dy}.$$

(o) *(4)* The map $y(x) = 5e^{-3x}$, $x \in \mathbb{R}$, is a solution of the IVP $\begin{cases} \dot{y} = \ldots\ldots \\ y(\ldots\ldots) = \dfrac{5}{e^3} \end{cases}$.

(p) *(6)* The absolute maximum of $f(x,y) = y^2$ when restricted to the set

$$M = \{(x,y) \in \mathbb{R}^2 : (x-5)^2 + y^2 = \ldots..\}$$

occurs at $(x,y) = (\ldots\ldots, 2)$. The Lagrangian map associated to the problem under consideration is:

$\mathcal{L}(x,y,\lambda) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

(q) *(6)* The logistic law (associated to a given population of size p that depends on the time $t \geq 0$) states that

$$p' = ap - bp^2, \qquad a > b \in \mathbb{R}_0^+$$

If $p(0) = 10$, $a = 1$ and $b = 0.1$, then the solution of the previous differential equation is monotonic ....

If $a = 3$, $b = 0$ and $p(0) = 10$, the solution of the previous differential equation is

...., where $t \in \mathbb{R}_0^+$.

(r) *(5)* Assuming that y depends on x, the graph of the solution of the IVP

$$\begin{cases} y'' = -4y \\ y(0) = -1 \\ y(\pi/4) = 0 \end{cases}$$

is

Part II

- Give your answers in exact form. For example, $\frac{\pi}{3}$ is an exact number while 1.047 is a decimal approximation for the same number.
- In order to receive credit, you must show all of your work. If you do not indicate the way in which you solve a problem, you may get little or no credit for it, even if your answer is correct.

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1. For $\alpha \in \mathbb{R}_0^+ \backslash \{1\}$, consider the following matrix $\mathbf{A} = \begin{bmatrix} \alpha & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.
   (a) Classify the quadratic form $Q(X) = X^T \mathbf{A} X$, $X \in \mathbb{R}^3$, as function of α.
   (b) Find the value of α for which 2 is an eigenvalue of $\mathbf{A}$. Compute the eigenvectors associated to this eigenvalue.

2. Consider the map $f(x,y) = \begin{cases} \dfrac{x(x+y-1)}{\sqrt{x^2+(y-1)^2}} & \text{if } (x,y) \neq (0,1) \\ 0 & \text{if } (x,y) = (0,1) \end{cases}$.
   (a) Show that f is continuous in $(0,1)$.
   (b) Find the directions (v_1, v_2) along with there exists directional derivative of f at $(0,1)$. In these cases, compute it.

3. Let $g : \mathbb{R} \to \mathbb{R}^+$ be a differentiable map such that $g'(1) = 0$. Identify and classify all critical points of
$$f(x,y) = (1-x)y + \ln(g(x))$$

4. Consider the set $\Omega = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1 \wedge x \leq y \leq e^x\}$.
   (a) Represent the set Ω in the cartesian plane (x,y).
   (b) Compute
$$\iint_\Omega (y^2 + x) \text{ dx dy}.$$

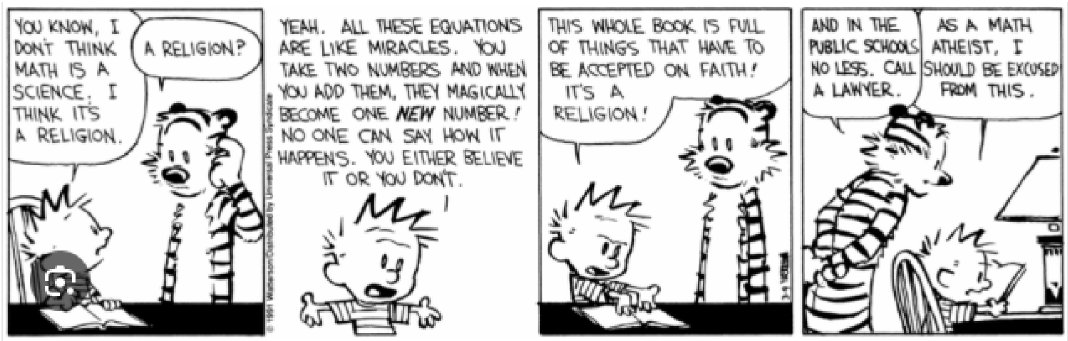
5. For $\alpha \in \mathbb{R}$, consider the following differential equation of order 2:

$$y''(x) - \alpha y(x) = 5\sin(2x) \tag{1}$$

for which $y(x) = \sin(2x)$ is a particular solution.

(a) Show that $\alpha = -9$.

(b) Write the general solution of (1), identifying its maximal domain.



Credits:

I	II.1(a)	II.1(b)	II.2(a)	II.2(b)	II.3	II.4(a)	II.4(b)	II.5(a)	II.5(b)
90	10	10	10	20	15	5	15	10	15