Regular Assessment - Mathematics II

4th January 2024

(I1)

1st part

Part I

a) 0; −2; invertible

b) $D_f = \{(x,y) \in \mathbb{R}^2 : x - y^2 - 1 > 0 \wedge y > 0\}$

c) $Q(x,y,z) = 3x^2 + 4xz - y^2 - 5yz$

$$\begin{pmatrix} 3 & 0 & 2 \\ 0 & -1 & -5/2 \\ 2 & -5/2 & 0 \end{pmatrix}$$

d) accumulation / interior

$\overline{\Omega} = \{(x,y) \in \mathbb{R}^2 : |x-5| \le 1 \wedge 4 \le y \le 7\} \cup \{(0,0)\}$

$\iint_\Omega dx\,dy = 6.$ (área)

e) $M = \{(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\} : (x-2)^2 + (y-2)^2 \le 1\}$ (for instance)

Weierstrass.

I2

f) 3

g) 4

h) $\frac{11}{2}$ $\quad x \cdot \frac{\partial f}{\partial x}(x,y) + y \frac{\partial f}{\partial y}(x,y) = \frac{11}{2} x^5 \sqrt{y}$

$$\Leftrightarrow x \cdot (5x^4 \sqrt{y}) + y\left(\frac{x^5}{2\sqrt{y}}\right) = \frac{11}{2} x^5 \sqrt{y}$$

i) $$\frac{df}{dt}(t) = 2xy\Big|_{(x(t),y(t))} e^t + x^2\Big|_{(x(t),y(t))} 2t \cos t^2 =$$

$$= 2e^t \sin t^2 \cdot e^t + e^{2t}\, 2t \cos t^2$$

$$= 2e^{2t} \sin t^2 + 2t e^{2t} \cos t^2$$

j) $x^2 y^3 - \sin y$

k) $H_f(3,2) = \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix}$ for instance

l) $\frac{\partial^4 f}{\partial x^2 \partial y^2}(x,y) = 35x^6 y^6$

m) $1 - y + \frac{y^2}{2}$

I.3

n) $\int_0^2 \int_0^{x^4} \cos(8x+3y)\,dy\,dx = \int_0^{16} \int_{\sqrt[4]{y}}^{2} \cos(8x+3y)\,dx\,dy$

o) $\begin{cases} \dot{y} = -3y \\ y(1) = \frac{5}{e^3} \end{cases}$ or $\begin{cases} \dot{y} = -15e^{-3x} \\ y(1) = \frac{5}{e^3} \end{cases}$

p) 4; 5; $L(x,y,\lambda) = y^2 - \lambda\left((x-5)^2 + y^2 - 4\right)$

q) decreasing

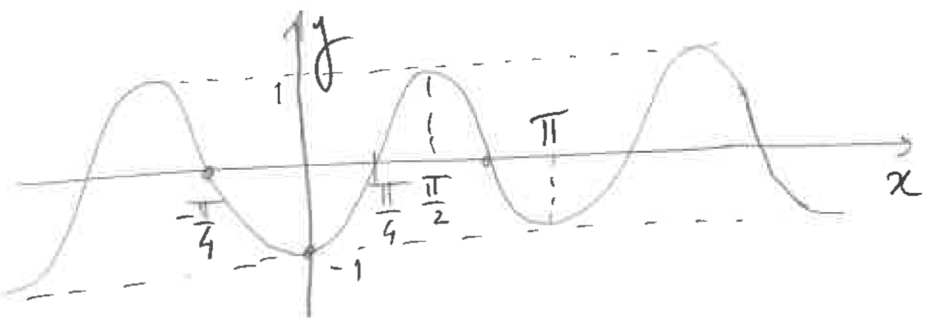
$p(t) = 10e^{3t}$

r) $y'' + 4y = 0 \Leftrightarrow y(x) = C_1 \cos 2x + C_2 \sin 2x$

$y(0) = -1 \Rightarrow C_1 = -1$

$y(\pi/4) = 0 \Rightarrow C_2 = 0$

$\therefore y(x) = -\cos(2x)$



1a) $A = \begin{bmatrix} \alpha & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$\Delta_1 = \alpha$

$\Delta_2 = \alpha - 1$

$\Delta_3 = 3\Delta_2 = 3(\alpha - 1)$

$\alpha \in [0,1[\implies \left.\begin{matrix} \Delta_1 > 0 \\ \Delta_2, \Delta_3 < 0 \end{matrix}\right\} \implies Q$ is undefined

$\alpha \in]1,+\infty[\implies \left.\begin{matrix} \Delta_1 > 0 \\ \Delta_2 > 0 \\ \Delta_3 > 0 \end{matrix}\right\} \implies Q$ is positively defined

b) $P(\lambda) = \det \begin{pmatrix} \alpha-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{pmatrix}$

$= (3-\lambda)\left[(\alpha-\lambda)(1-\lambda) - 1\right]$

$P(2) = 0 \iff 1 \cdot \left[(\alpha-2)\cdot(-1) - 1\right] = 0 \iff$

$\iff \alpha = 1$

$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff \begin{cases} -x + y = 0 \\ 3z = 0 \end{cases} \iff \begin{cases} x = +y \\ z = 0 \end{cases}$

$E_2 = \{(a, +a, 0),\ a \in \mathbb{R}\setminus\{0\}\}$

2a) $f(0,1) = 0$

$$\lim_{(x,y)\to(0,1)} \frac{x(x+y-1)}{\sqrt{x^2+(y-1)^2}} = \lim_{(x,y)\to(0,0)} \frac{x(x+y)}{\sqrt{x^2+y^2}}$$

$$0 \le \left| \frac{x(x+y)}{\sqrt{x^2+y^2}} \right| = \frac{|x| \cdot |x+y|}{\sqrt{x^2+y^2}} \le \frac{\sqrt{x^2+y^2}\,|x+y|}{\sqrt{x^2+y^2}} = |x+y|$$

$|x| \le \sqrt{x^2+y^2}$

As $(x,y)\to(0,0)$: $0 \to 0$ and $|x+y| \to 0$

∴ By the squeezing theorem, we conclude that

$$\lim_{(x,y)\to(0,0)} \frac{x(x+y)}{\sqrt{x^2+y^2}} = 0 = f(0,1)$$

∴ f is continuous at $(0,1)$

2b) $$D_{(v_1,v_2)} f(0,1) = \lim_{t\to 0} \frac{f((0,1)+t(v_1,v_2)) - f(0,1)}{t}$$

$$= \lim_{t\to 0} \frac{f(tv_1, 1+tv_2)}{t} = \lim_{t\to 0} \frac{tv_1(tv_1+tv_2)}{t\sqrt{t^2v_1^2+t^2v_2^2}}$$

$$= \lim_{t\to 0} \begin{cases} \dfrac{v_1 (v_1+v_2)}{\sqrt{v_1^2+v_2^2}} & t>0 \\[2ex] \dfrac{-v_1 (v_1+v_2)}{\sqrt{v_1^2+v_2^2}} & t<0 \end{cases}$$

because $\sqrt{t^2 v_1^2 + t^2 v_2^2} = |t| \sqrt{v_1^2+v_2^2}$

the limit (above) exists and is zero, when

$$v_1 (v_1+v_2) = -v_1 (v_1+v_2)$$

$$\iff \boxed{v_1 = 0} \lor \boxed{v_1 = -v_2.}$$

Answer: $(0, v_2)$, $v_2 \in \mathbb{R}\setminus\{0\}$
$(v_1, -v_1)$, $v_1 \in \mathbb{R}\setminus\{0\}$.

3 $f(x,y) = (1-x)y + \ln g(x)$.

$$\nabla f(x,y) = \left(-y + \frac{g'(x)}{g(x)} ; 1-x\right)$$

$$\nabla f(x,y) = \bar{0} \iff \begin{cases} x = 1 \\ y = 0 \end{cases}$$

$(1,0)$. — critical point.

II ④

$$H_f(x,y) = \begin{pmatrix} \dfrac{g''(x)\,g(x) - [g'(x)]^2}{g^2(x)} & -1 \\ -1 & 0 \end{pmatrix}$$

$$H_f(1,0) = \begin{bmatrix} \dfrac{g''(1)\cdot \cancel{g(1)}}{g^2(1)} & -1 \\ -1 & 0 \end{bmatrix}$$

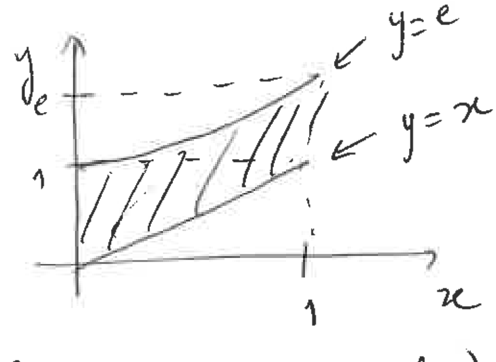
$$\left.\begin{aligned} \Delta_1 &= \frac{g''(1)}{g(1)} \\ \Delta_2 &= -1 \end{aligned}\right\}$$

H_f is undefined

$\Downarrow$

(1,0) is a saddle point.

4a)



(not respecting scales)

4b) $\int_0^1 \int_x^{e^x} y^2 + x \, dy \, dx =$

$$= \int_0^1 \left[\frac{y^3}{3} + xy\right]_x^{e^x} dx$$

$$= \int_0^1 \frac{e^{3x}}{3} + xe^x - \frac{x^3}{3} - x^2 \, dx$$

$$= \left[\frac{e^{3x}}{9} + xe^x - e^x - \frac{x^4}{12} - \frac{x^3}{3}\right]_{x=0}^{1} =$$

$$= \frac{e^3}{9} + e^1 - e^1 - \frac{1}{12} - \frac{1}{3} - \frac{1}{9} + 1 =$$

$$= \frac{e^3}{9} + \frac{17}{36}.$$

Remark: $\int \underbrace{x}_{g}\,\underbrace{e^x}_{f'} \, dx = xe^x - \int e^x \, dx = xe^x - e^x$

↑

Integration by parts

5

II 6

a) $y(x) = \sin(2x)$

$y'(x) = 2\cos 2x$

$y''(x) = -4\sin 2x$

$y''(x) - \alpha y(x) = -4\sin(2x) - \alpha \cdot \sin(2x)$

$= (-4-\alpha)\sin(2x)$

Since $y(x) = \sin(2x)$ is a solution, then $-4-\alpha = 5$

$\Leftrightarrow -9 = \alpha$

b) $y''(x) + 9y(x) = 5\sin(2x)$

$P(\lambda) = \lambda^2 + 9$ $\quad P(\lambda) = 0 \Leftrightarrow$

$\Leftrightarrow \lambda = \pm 3i$

Solution of the homogeneous equation:

$y(x) = C_1 \cos(3x) + C_2 \sin(3x)$, $C_1, C_2 \in \mathbb{R}$

general solution of (1) $\qquad D = \mathbb{R}$

$y(x) = C_1 \cos(3x) + C_2 \sin(3x) + \sin(2x)$. $\quad C_1, C_2 \in \mathbb{R}$