Universidade de Lisboa
Instituto Superior de Economia e Gestão
Msc in Economics, Mathematical Finance,
Monetary and Financial Economics

<u>Mathematical Economics</u> – 1st Semester - 2023/2024

Regular Assessment - 17th of January 2024

Duration: $(120 + \varepsilon)$ minutes, $|\varepsilon| \leq 30$

Version A

Name: ....

Student ID #: ....

Part I

- Complete the following sentences in order to obtain true propositions. The items are independent from each other.
- There is no need to justify your answers.

(a) *(4)* The (maximal) domain of $f : D_f \to \mathbb{R}$ with analytical expression given by

$$f(x, y) = \frac{\ln(x - y^2 - 1)}{\sqrt{y}}$$

is the set $D_f = \{$...$\}$

(b) *(4)* The set of accumulation points of $\{(\cos(n\pi), \sin(n\pi)), n \in \mathbb{N}\} \subset \mathbb{R}^2$ is

....

(c) *(4)* Consider $C = \{(x, y) \in \mathbb{R}^2 : x^2 \leq y < 4 - x^2\}$. The topological border (frontier) of C is analytically defined by

$\partial C =$ ....

(d) *(4)* The map $f(x,y) = (x+2y)^2$ has a global maximum and a global minimum when restricted to the set

$$M = \{(x,y) \in \mathbb{R}^2 : \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\}$$

This is a consequence of ....'s Theorem because f is continuous (f is polynomial) and M is compact.

(e) *(4)* The equation $7x^{2024} + x^{2023} + x^{2022} - 1 = 0$ has at least one solution in the closed interval .... This is a consequence of the .... Theorem.

(f) *(4)* Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ and $g : \mathbb{R}\backslash\{3\} \to \mathbb{R}$ be defined by

$$f(x,y) = x + y^2 \quad \text{and} \quad g(x) = 2 + \frac{5}{x-3}$$

Then

$$\lim_{(x,y)\to(-1,3)} [g \circ f(x,y)] = \ldots\ldots\ldots$$

(g) *(3)* If $u_n = \left(\left(1 + \frac{3}{n}\right)^n, \frac{\pi}{n}\right)$, $n \in \mathbb{N}$ and $f(x,y) = \ln x + \cos y$, then

$$\lim_{n\to+\infty} f(u_n) = \ldots\ldots$$

(h) *(4)* The correspondence $H\colon [0,1] \rightrightarrows [0,1]$ defined by

$$H(x) = \begin{cases} \{4x(1-x)\}, & x < \frac{1}{2} \\ [0,1/4] \cup [3/4,1], & x = \frac{1}{2} \\ \{2x-1\} & x > \frac{1}{2} \end{cases}$$

has .... fixed points, although H **does not** satisfy the following assumption of the Kakutani Fixed Point Theorem: ....

(i) *(4)* The sets $I_1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < \ldots\ldots\}$ and $I_2 = \{(x,y) \in \mathbb{R}^2 : y = 4\}$ are non-empty, disjoint and .... The Hyperplane Separation Theorem says that they can be separated by the line defined by the equation ....

(j) *(4)* Consider the map $\mathcal{R}_\alpha$ geometrically described by a rotation around the origin of angle $\alpha \in [0, 2\pi[$, and $J = \{(x,y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 4\}$. Then $\mathcal{R}_\alpha(J)$ has fixed points if and only if ....

(k) *(4)* The gradient vector of $f : \mathbb{R}^2 \to \mathbb{R}$ is given by $(2xy^3, 3x^2y^2 - y)$. If $f(x, y)$ does not have constant terms in both components, then

$$f(x, y) = \text{..}$$

(l) *(3)* With respect to the C^2 map $f : \mathbb{R}^2 \to \mathbb{R}$, we know that $\nabla f(-2, 3) = (0, 0)$ and

$$H_f(-2, 3) = \begin{pmatrix} \cdots\cdots & 0 \\ 0 & \cdots\cdots \end{pmatrix}.$$

Then, $f(-2, 3)$ is a local minimum of f. Furthermore, if f is ...., then $f(-2, 3)$ is a global minimum of f.

(m) *(5)* We apply the Karush-Kuhn-Tucker technique to the following problem. The minimizer of $f(x, y, z) = x^3 + y^3 + z^3$ on the simplex $\Delta = \{(x, y, z) \in \mathbb{R}^3_0 : x + y + z = 1\}$ should satisfy the following system:

$$\begin{cases} \text{...............} - \lambda + \mu_1 = 0 \\ \text{...............} - \lambda + \mu_2 = 0 \\ \text{...............} - \lambda + \mu_3 = 0 \\ \mu_1,\ \mu_2,\ \mu_3 \text{..........} 0 \\ \mu_1 x = 0,\ \mu_2 y = 0,\ \mu_3 z = 0, \\ \text{......................................} = 0 \\ x \geq 0, y \geq 0, z \geq 0 \end{cases}$$

(n) *(4)* The map $y(x) = 5e^{-3x}$, $x \in \mathbb{R}$, is a solution of the IVP $\begin{cases} \dot{y} = \text{......} \\ y(\text{......}) = \dfrac{5}{e^3} \end{cases}$.

(o) *(4)* The absolute maximum of $f(x, y) = y^2$ when restricted to the set

$$M = \{(x, y) \in \mathbb{R}^2 : (x - 5)^2 + y^2 = \text{.....}\}$$

occurs at $(x, y) = (\text{......}, 2)$. The associated Lagrangian map is defined by

$$\mathcal{L}(x, y, \lambda) = \text{..}$$

(p) *(6)* The logistic law (associated to a given population of size p that depends on the time $t \geq 0$) states that

$$p' = ap - bp^2, \qquad a, b \in \mathbb{R}^+_0$$

If $p(0) = 1000$, $a = 1$ and $b = 0.002$, then the solution of the previous differential equation is monotonic ....

If $a = 3$, $b = 0$ and $p(0) = 1000$, the solution of the ODE is

...., where $t \in \mathbb{R}_0^+$.

(q) *(5)* The linearisation of $\begin{cases} \dot{x} = -x + yx^{2024} \\ \dot{y} = -8y - xy^{2024} \end{cases}$ around $(0,0)$ is $\begin{cases} \dot{x} = \text{...........} \\ \dot{y} = \text{.............} \end{cases}$

With respect to the Lyapunov's stability, we may conclude that (0,0) is ....
....

(r) *(3)* With respect to the linear differential equation defined by $\dot{X} = AX$, where $A \in M_{2\times 2}(\mathbb{R})$ and $X \in \mathbb{R}^2$, we know that $\det A =$ .... Then, the origin is Lyapunov unstable.

(s) *(4)* Let $\varphi(t) = (x(t), y(t)) \in \mathbb{R}^2$ be a **non-constant** solution of $\begin{cases} \dot{x} = 4x \\ \dot{y} = -3y \end{cases}$ such that $\lim_{t\to+\infty} \varphi(t) = (0,0)$. One possible initial condition associated to φ might be (.....,.....).

(t) *(8)* Consider the following problem of optimal control where $x : [0,1] \to \mathbb{R}$ is the *state* and $u : [0,1] \to \mathbb{R}$ is the *control*:

$$\max_{u(t)\in\mathbb{R}} \int_0^1 (1 - tx(t) - u(t)^2)dt, \quad x'(t) = u(t), \quad x(0) = 1 \quad \text{and} \quad x(1) \in \mathbb{R}.$$

Then the Hamiltonian is given by (*specify the formulas to the case under consideration*):

$$H(t, x, u, p) = \text{..}$$

The Pontryagin maximum principle says that the optimal control $u^\star$ should satisfy the equality .... which is equivalent to $u(t) = p(t)/2$.

The Hamiltonian equations are given by:

$$\begin{cases} \dot{x} = \text{.....................} \\ \\ \dot{p} = \text{.....................} \end{cases}$$

The *transversality condition* is given by ....

Integrating the differential equation above, using the initial conditions and the transversality condition, we conclude that

$$x^\star(t) = \text{................................} \quad \text{and} \quad u^\star(t) = \text{......................................}$$

Since H is .... in (x, u) then the above solutions are the sought solutions.

Part II

- Give your answers in exact form. For example, $\frac{\pi}{3}$ is an exact number while 1.047 is a decimal approximation for the same number.
- In order to receive credit, you must show all of your work. If you do not indicate the way in which you solve a problem, you may get little or no credit for it, even if your answer is correct.

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1. Consider the map $f : [-1/3, 1/3] \to \mathbb{R}$ defined by

$$f(x) = x^2 + \frac{2}{9}$$

   (a) Define the **inverse** of the restriction of f to the interval $[0, 1/3]$.
   (*Define the domain, the range and the analytical expression*).

   (b) Show that f satisfies the hypotheses of the Banach fixed point Theorem and find the fixed point of f.

2. Let $g : \mathbb{R} \to \mathbb{R}^+$ a differentiable map such that $g'(1) = 0$. Identify and classify all critical points of

$$f(x, y) = (1 - x)y + \ln(g(x))$$

3. Consider the following IVP (y is a function of x):

$$\begin{cases} x^4 y' + 4x^3 y = \cos x \\ y(\pi) = \pi \end{cases}$$

   Write the solution $y(x)$ of the IVP, identifying its maximal domain.

4. Consider the linear system in $\mathbb{R}^2$ given by (x and y depend on t):

$$\begin{cases} \dot{x} = y \\ \dot{y} = 2x - y \end{cases}$$

   (a) Write the general form of the solution.

   (b) Find the particular solution such that $x(0) = -1$ and $y(0) = -3$.

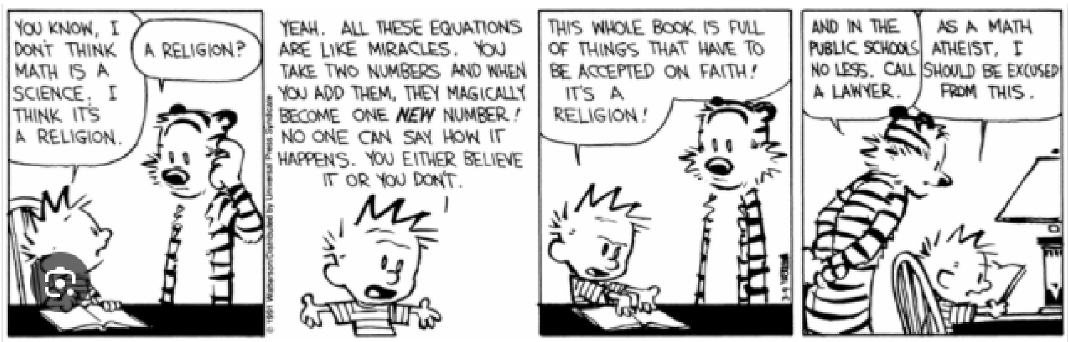
   (c) Sketch the phase portrait and locate the solution $(x(t), y(t))$ found in (b), $t \in \mathbb{R}$.

5. Consider the following Problem on *Calculus of Variations*, where $x : [0, 1] \to \mathbb{R}$ is a smooth function on t:

$$\min_{x(t) \in \mathbb{R}} \int_0^1 (x^2 + \dot{x}^2 - 1)dt, \quad \text{with} \quad x(0) = 1 \quad \text{and} \quad x(1) = 0.$$

(a) Write the corresponding Euler-Lagrange equation applied to the case under consideration.

(b) Find the solution of the problem.



Credits:

I	II.1(a)	II.1(b)	II.2	II.3	II.4(a)	II.4(b)	II.4(c)	II.5(a)	II.5(b)
85	10	15	20	20	15	5	5	10	15