Mathematical Economics – 1st Assessment – Version A

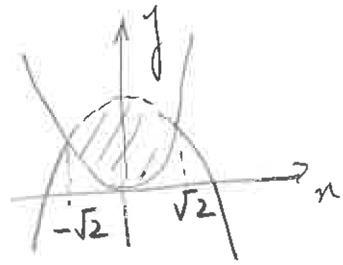
Part I

I)

a) $D_f = \{ (x,y) \in \mathbb{R}^2 : x - y^2 - 1 > 0 \wedge y > 0 \}$

b) $\{(-1,0), (1,0)\}$

c) $\partial C = \{ (x,y) \in \mathbb{R}^2 : (y = x^2 \vee y = 4 - x^2) \wedge x \in [-\sqrt{2}, \sqrt{2}] \}$



d) $M = \{ (x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \}$ (for instance)

Weierstrass

e) $[0,1]$

Intermediate Value

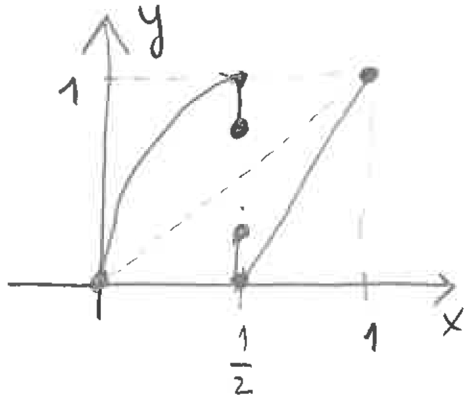
f) 3

g) 4

h) two

$\forall x \in [0,1]$, $f(x)$ is non-empty
and convex.

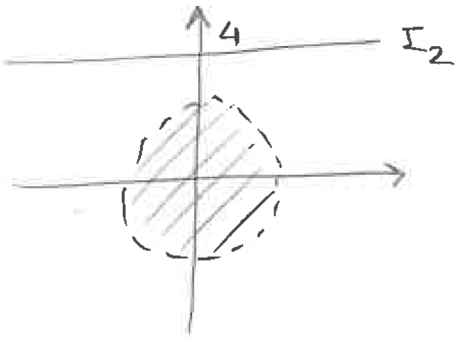
For example $f\left(\frac{1}{2}\right) = \underbrace{\left[0, \frac{1}{4}\right] \cup \left[\frac{3}{4}, 1\right]}_{\text{not convex}}$



i) . 1 (for instance)

. convex

$y = 2$



j) $\alpha = 0$.

k) $f(x,y) = x^2 y^3 - \frac{y^2}{2}$

l) $\begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix}$ for instance

convex

m) $$\begin{cases} 3x^2 - \lambda + \mu_1 = 0 \\ 3y^2 - \lambda + \mu_2 = 0 \\ 3z^2 - \lambda + \mu_3 = 0 \\ \mu_1, \mu_2, \mu_3 \leq 0 \\ \mu_1 x = \mu_2 y = \mu_3 z = 0. \\ x + y + z - 1 = 0 \\ x, y, z \geq 0 \end{cases}$$

$f(x,y,z) = x^3 + y^3 + z^3$

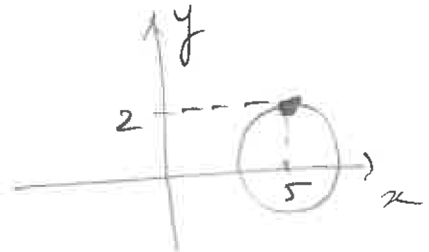
$g(x,y,z) = x + y + z - 1$

$l_1(x,y,z) = -x$

$l_2(x,y,z) = -y$

$l_3(x,y,z) = -z.$

n) $$\begin{cases} y' = -3y \\ y(1) = \frac{5}{e^3} \end{cases}$$ or $$\begin{cases} y' = -15e^{-3x} \\ y(1) = 5/e^3 \end{cases}$$



o) 4; $(5,2)$

$$\mathcal{L}(x,y,\lambda) = y - \lambda\left((x-5)^2 + y^2 - 4\right)$$

p) decreasing;

$$p(t) = p_0 e^{3t} = 1000\, e^{3t}, \quad t \in \mathbb{R}_0^+$$

q) $\begin{cases} \dot{x} = -x \\ \dot{y} = -8y \end{cases}$

$(0,0)$ is asymptotically stable (negative real eigenvalues).

r) $\det A = -5$ for example

s) $(0,5)$ for instance

t) . $H(t,x,u,p) = f(t,x,u) + p \cdot g(t,x,u)$

$= \boxed{1 - tx - u^2 + p \cdot u}$

• $\frac{\partial H}{\partial u}(u^*) = 0 \Leftrightarrow \boxed{-2u^* + p = 0} \Leftrightarrow u^* = p/2$

• $\begin{cases} \dot{x} = \frac{\partial H}{\partial p} \\ \dot{p} = -\frac{\partial H}{\partial x} \end{cases} \Leftrightarrow \begin{cases} \dot{x} = u \\ \dot{p} = t \end{cases}$

• $\boxed{p(1) = 0}$ transversality condition

$$\ddot{x} = \dot{u} = \frac{\dot{p}}{2} = \frac{t}{2}$$

$$\dot{x} = \frac{t^2}{4} + k_1 \quad , k_1 \in \mathbb{R}$$

$$x = \frac{t^3}{12} + k_1 t + k_2 \quad , k_1, k_2 \in \mathbb{R}$$

Since $x(0) = 1$ and $p(1)=0$, then

$k_2 = 1$ and $k_1 = -1/4$?

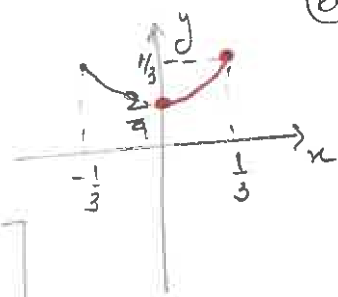
$$\boxed{p(1) = 0 \Rightarrow u(1) = 0 \Leftrightarrow \dot{x}(1) = 0 \Rightarrow +\frac{1}{4} + k_1 = 0 \quad k_1 = -\frac{1}{4}}$$

$\therefore$ $\boxed{x^*(t) = \frac{t^3}{12} - \frac{1}{4} t + 1}$ $\quad t \in [0,1]$

$\boxed{u^*(t) = \frac{t^2}{4} - \frac{1}{4}}$

H is $\boxed{\text{concave}}$

Part II



a) the restriction

$$f\big|_{[0,\frac{1}{3}]} : \left[0,\tfrac{1}{3}\right] \longrightarrow \left[\tfrac{2}{9} ; \tfrac{1}{3}\right]$$

Inverse

$$f^{-1} : \left[\tfrac{2}{9} ; \tfrac{1}{3}\right] \longrightarrow \left[0, \tfrac{1}{3}\right]$$

$$x \longrightarrow \sqrt{x - \tfrac{2}{9}}$$

$y = x^2 + \frac{2}{9}$

$y - \frac{2}{9} = x^2$

$\pm\sqrt{y - \frac{2}{9}} = x$

b) $f'(x) = 2x$

$$|f'(x)| = |2x| \le \frac{2}{3} \quad , \forall x \in \left[-\tfrac{1}{3}, \tfrac{1}{3}\right]$$

$$\Downarrow$$

f is a contraction in $D_f = \left[-\frac{1}{3}, \frac{1}{3}\right]$.

- $\left[-\frac{1}{3}, \frac{1}{3}\right]$ is a complete metric space because it is compact

Then by the Fixed Point theorem (Banach), f has a unique fixed point.

Fixed point:

$$x^2+\frac{2}{9}=x$$

$$\Leftrightarrow x^2-x+\frac{2}{9}=0$$

$$\Leftrightarrow x=\frac{1\pm\sqrt{1-4\times 2/9}}{2}$$

$$\Leftrightarrow x=\frac{1\pm\frac{1}{3}}{2} \Leftrightarrow x=\underbrace{\frac{2}{3}}_{\notin D_f} \vee \boxed{x=\frac{1}{3}}$$

$\Uparrow$ unique fixed point

2)

$$f(x,y)=(1-x)y+\ln g(x)=$$

$$=y-xy+\ln g(x)$$

$$\nabla f(x,y)=\left(-y+\frac{g'(x)}{g(x)}\,;\ 1-x\right)$$

$$\nabla f(x,y)=(0,0) \Leftrightarrow \begin{cases} y=0 \\ x=1 \end{cases}$$

$\underbrace{(1,0)}$ unique critical point.

Classification:

$$H_f(x,y)=\begin{pmatrix} \frac{g''(x)\,g(x)-g'(x)^2}{g(x)^2} & -1 \\ -1 & 0 \end{pmatrix}$$

$$H_f(1,0) = \begin{pmatrix} \frac{g''(1)}{g(1)} & -1 \\ -1 & 0 \end{pmatrix}$$

$\Delta_2 = -1 \implies (1,0)$ is a saddle-point

$\uparrow$

H_f is undefined.

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$$x^4 y' + 4x^3 y = \cos x$$

$$\underset{x \neq 0}{\iff} \quad y' + \frac{4}{x} y = \frac{\cos x}{x^4}$$

Integrant factor:

$$\mu' = \frac{4}{x}\mu \iff \frac{\mu'}{\mu} = \frac{4}{x} \iff$$

$$\iff \frac{d}{dx} \ln \mu(x) = \frac{4}{x}$$

$$\iff \ln \mu(x) = 4 \cdot \ln x$$

$$\iff \boxed{\mu(x) = x^4}$$

$$\frac{d}{dx}\left(x^4 \cdot y\right) = \frac{\cos x}{\cancel{x^4}} \cdot \cancel{x^4}$$

$$\Leftrightarrow \quad x^4 y = \operatorname{sen} x + C$$

$$\Leftrightarrow \quad y(x) = \frac{\operatorname{sen} x}{x^4} + \frac{C}{x^4}$$

Since $y(\pi) = \pi$, then $\frac{C}{\pi^4} = \pi \Leftrightarrow C = \pi^5$

$$\therefore \quad y(x) = \frac{\operatorname{sen} x}{x^4} + \frac{\pi^5}{x^4}, \quad x \in \mathbb{R}_0^+$$

$\boxed{4}$ $\begin{cases} \dot{x} = y \\ \dot{y} = 2x - y \end{cases} \Leftrightarrow \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

a) $P(\lambda) = (-\lambda)(-1-\lambda) - 2 = \lambda^2 + \lambda - 2$

$P(\lambda) = 0 \Leftrightarrow \lambda = \frac{-1 \pm \sqrt{1 - 4.(-2)}}{2} \Leftrightarrow \lambda = \frac{-1 \pm 3}{2}$

$\Leftrightarrow \boxed{\lambda = -2} \vee \boxed{\lambda = 1}$

<u>Eigenvectors:</u>

E_{-2}

$$\begin{pmatrix} +2 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} 2x+y=0 \\ \text{———} \end{cases}$$

$$\Leftrightarrow \begin{cases} y=-2x \\ \text{———} \end{cases} \qquad E_{-2} = \langle (1,-2) \rangle$$

E_1

$$\begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} x=y \\ \text{———} \end{cases}$$

$$E_1 = \langle (1,1) \rangle$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{-2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad c_1, c_2 \in \mathbb{R}$$

$$\boxed{\begin{cases} x(t) = c_1 e^{-2t} + c_2 e^{t} \\ y(t) = -2c_1 e^{-2t} + c_2 e^{t} \end{cases}} \qquad \begin{array}{c} c_1, c_2 \in \mathbb{R} \\ t \in \mathbb{R} \end{array}$$

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b) $\begin{cases} x(0) = -1 \\ y(0) = -3 \end{cases} \Leftrightarrow \begin{cases} c_1 + c_2 = -1 \\ -2c_1 + c_2 = -3 \end{cases}$

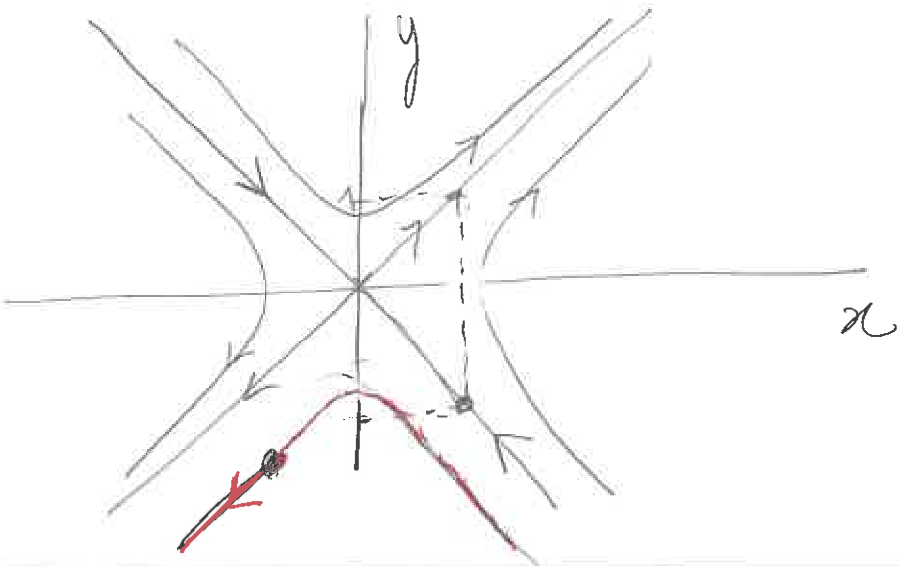
$\Leftrightarrow \begin{cases} c_1 = -1 - c_2 \\ -2(-1 - c_2) + c_2 = -3 \end{cases} \Leftrightarrow \begin{cases} \text{———} \\ 2 + 2c_2 + c_2 = -3 \end{cases}$

$\Leftrightarrow \begin{cases} \text{———} \\ 3c_2 = -5 \end{cases} \Leftrightarrow \begin{cases} c_1 = 2/3 \\ c_2 = -5/3 \end{cases}$

Particular Solution

$\begin{cases} x(t) = \frac{2}{3} e^{-2t} - \frac{5}{3} e^{t} \\ y(t) = -\frac{4}{3} e^{-2t} - \frac{5}{3} e^{t} \end{cases} \quad t \in \mathbb{R}$

c)



5 a)

$$F(t, x, \dot{x}) = x^2 + \dot{x}^2 - 1$$

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}} = 0$$

$$2x - \frac{d}{dt}(2\dot{x}) = 0 \iff 2x = 2\ddot{x}$$

$$\iff \boxed{\ddot{x} - x = 0} \quad //$$

b)

$$\ddot{x} - x = 0$$

$$P(\lambda) = \lambda^2 - 1$$

$$P(\lambda) = 0 \iff \lambda = \pm 1$$

general solution $x(t) = C_1 e^t + C_2 e^{-t}, C_1, C_2 \in \mathbb{R}$

$$\begin{cases} x(0) = 1 \\ x(1) = 0 \end{cases} \iff \begin{cases} C_1 + C_2 = 1 \\ C_1 e + C_2 e^{-1} = 0 \end{cases}$$

$$\iff \begin{cases} C_1 = 1 - C_2 \\ (1 - C_2) e + C_2 e^{-1} = 0 \end{cases} \iff$$

$$\iff \begin{cases} \overline{} \\ e - C_2 e + C_2 e^{-1} = 0 \end{cases} \iff$$

$$\begin{cases} \text{\_\_\_\_\_\_} \\ c_2(e^{-1}-e) = -e \end{cases} \Leftrightarrow \begin{cases} \\ c_2 = \dfrac{-e}{e^{-1}-e} \end{cases}$$

$$\Leftrightarrow \begin{cases} c_1 = 1 - \dfrac{e^2}{e^2-1} \\ c_2 = \dfrac{e^2}{e^2-1} \end{cases} \Leftrightarrow \begin{cases} c_1 = \dfrac{\cancel{e^2} - 1 - \cancel{e^2}}{e^2-1} \\ c_2 = \dfrac{e^2}{e^2-1} \end{cases}$$

$$\Leftrightarrow \begin{cases} c_1 = -\dfrac{1}{e^2-1} \\ c_2 = \dfrac{e^2}{e^2-1} \end{cases}$$

$$x^*(t) = -\frac{1}{e^2-1}e^{t} + \frac{e^2}{e^2-1}e^{-t}, \quad t \in \mathbb{R}$$

F is convex in $(x, \dot{x})$ $\quad H_F(x,\dot{x}) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ P.D

$$\Downarrow$$

x^* is the solution of the problem.

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