Lisbon School of Economics & Management
Universidade de Lisboa

Universidade de Lisboa
Instituto Superior de Economia e Gestão
Msc in Economics, Mathematical Finance,
Monetary and Financial Economics

Mathematical Economics – 1st Semester - 2023/2024

Repeat Assessment - 1st of February 2024

Duration: $(120+\varepsilon)$ minutes, $|\varepsilon| \leq 30$

Version A

Name: ....

Student ID #: ....

Part I

- Complete the following sentences in order to obtain true propositions. The items are independent from each other.
- There is no need to justify your answers.

(a) *(4)* The (maximal) domain of $f : D_f \to \mathbb{R}$ defined by

$$f(x,y) = \frac{\sqrt{y-x^2}}{\ln(x^2+y^2)}$$

is the set

$D_f =$ ....

(b) *(7)* With respect to the set

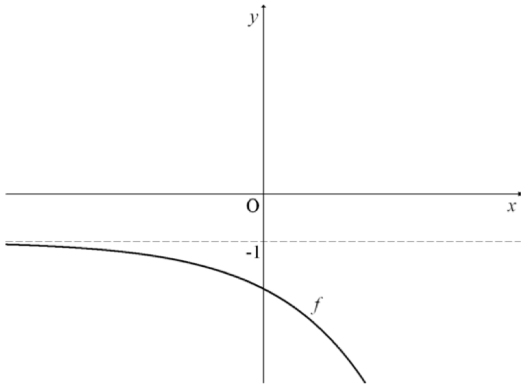
$$\Omega = \left\{ (x,y) \in \mathbb{R}^2 : x \in [1,2] \ \wedge \ \frac{1}{x} \leq y \right\}$$

we may conclude that $\left(\frac{3}{2}, 3\right)$ is an .... point of Ω and

$\partial\Omega = fr(\Omega) =$ ....

Since Ω is unbounded, then Ω is **not** ....

(c) *(4)* Let $g : \mathbb{R}^2 \to \mathbb{R}$ be defined by $g(x,y) = -y^2 + \cos(x)$ and $f : \mathbb{R} \to \,]-\infty, -1[$ be the map whose graphical representation is given by:



Then

$$\lim_{(x,y)\to(0,-\infty)} \frac{-3}{(f \circ g)(x,y)} = \ldots\ldots\ldots$$

(d) *(4)* With respect to a map $f : \mathbb{R}^2 \to \mathbb{R}$, one knows that

$$\nabla f(x,y) = (-2x\sin(x^2+y); -\sin(x^2+y)).$$

If $f(x,y)$ does not have constant terms in both components, then

$$f(x,y) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

(e) *(4)* If $u_n = \left(\ln\left(1 - \frac{1}{2n+1}\right)^{n+\frac{1}{2}}; \frac{1}{n}\right)$, $n \in \mathbb{N}$ and $f(x,y) = -2x + \cos y$, then

$$\lim_{n\to+\infty} f(u_n) = \ldots\ldots$$

(f) *(8)* The graphical representation of the correspondence $H\colon [0,1] \rightrightarrows \mathbb{R}$ defined by

$$H(x) = \begin{cases} [\sqrt{x}, 1-2x] & x < \frac{1}{4} \\ [0, 2x], & \frac{1}{4} \le x \le \frac{1}{2} \\ \{1\} & x > \frac{1}{2} \end{cases}$$

is:

The set of fixed points of H are explicitly given by: $\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

(g) *(4)* Let $f : [0,1] \to [0,1]$ be a continuous map. Then the equation .... has at least one solution, as a consequence of the Brouwer Fixed Point Theorem.

(h) *(4)* With respect to the C^2 map $f : \mathbb{R}^2 \to \mathbb{R}$, we know that $\nabla f(3,2) = (0,0)$ and

$$H_f(3,2) = \begin{pmatrix} -5 & \ldots\ldots \\ 0 & \ldots\ldots \end{pmatrix}.$$

Then, we conclude that $(3,2)$ is a local **maximizer** of f.

(i) *(4)* Consider the following utility **maximisation** problem. There are two goods and an agent with income 3 who wishes to maximize the utility:

$$U(x,y) = \ln(x+1) + y, \quad x \geq 0, \quad y \geq 0$$

subject to a budget constraint $\frac{1}{2}x + y \leq 3$. The necessary conditions to solve the utility maximization problem are:

$$\begin{cases} \ldots\ldots\ldots\ldots\ldots - \frac{1}{2}\mu_1 + \mu_2 = 0 \\ \\ \ldots\ldots\ldots\ldots\ldots - \mu_1 + \mu_3 = 0 \\ \\ \mu_1 x = 0, \ \mu_2 y = 0 \\ \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = 0 \\ \\ \mu_1, \ \mu_2, \ \mu_3 \ldots\ldots\ldots 0 \\ \\ \frac{1}{2}x + y \leq 3 \\ \\ x \geq 0, y \geq 0 \end{cases}$$

(j) *(4)* The map $y(x) = \cos(4x)$, $x \in \mathbb{R}$, is a solution of the IVP $\begin{cases} y'' = \ldots\ldots \\ y(\ldots\ldots) = 0 \end{cases}$.

(k) *(5)* The .... law (associated to a given population of size p that depends on the time $t \geq 0$) states that

$$p' = kp, \qquad k \in \mathbb{R}.$$

If $p(0) = 10$ and $k = -2$, then the **solution** of the previous differential equation is given **explicitly** by

...., where $t \in \mathbb{R}_0^+$.

(l) *(3)* Assuming that y depends on x, any solution of the differential equation $y' = e^{-x}$ is monotonically ....

(m) *(6)* For $b \in \mathbb{R}$, consider the following differential equation (P depends on $t \in \mathbb{R}$):

$$P'' + bP' + 5P = 15$$

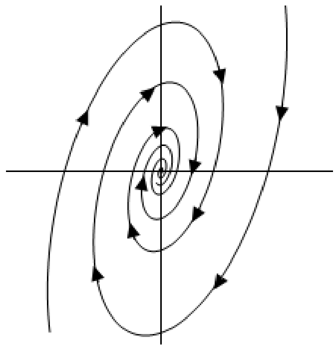
The solution P is periodic on t if and only if .... If $b = 2$, then $\lim_{t \to +\infty} P(t) =$ ....

(n) *(7)* The linearisation of

$$(*) \begin{cases} \dot{x} = 3x - xy^2 \\ \dot{y} = 8y - yx^2 \end{cases} \qquad \text{around } (0,0) \qquad \text{is} \qquad (**) \begin{cases} \dot{x} = \ldots\ldots\ldots\ldots \\ \dot{y} = \ldots\ldots\ldots\ldots \end{cases}$$

With respect to the Lyapunov's stability, we may conclude that (0,0) is .... Furthermore, ...-........................Theorem says that, in a small neighbourhood of $(0,0)$, the dynamics of (*) and (**) are "qualitatively" the same (topologically conjugated).

(o) *(4)* The phase portrait of $\dot{X} = AX$, where $A \in M_{2\times 2}(\mathbb{R})$ and $X \in \mathbb{R}^2$, is given by:



Then the following **inequalities** hold: $\det(A)$............ and $Tr(A)$............

(p) *(8)* Consider the following problem of optimal control where $x : [0, 10] \to \mathbb{R}$ is the *state* and $u : [0, 10] \to \mathbb{R}$ is the *control*:

$$\max_{u(t) \in \mathbb{R}} \int_0^{10} \ln(u(t))dt, \quad x'(t) = x(t) - u(t), \quad x(0) = 1 \quad \text{and} \quad x(10) = 1.$$

Then the Hamiltonian is given by (*specify the formulas to the case under consideration*):

$$H(t, x, u, p) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

The Pontryagin maximum principle says that the optimal control $u^\star$ should satisfy the equality .... The Hamiltonian equations are given by:

$$\begin{cases} \dot{x} = \ldots\ldots\ldots\ldots\ldots\ldots \\ \\ \dot{p} = \ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

Part II

- Give your answers in exact form. For example, $\frac{\pi}{3}$ is an exact number while 1.047 is a decimal approximation for the same number.
- In order to receive credit, you must show all of your work. If you do not indicate the way in which you solve a problem, you may get little or no credit for it, even if your answer is correct.

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1. Consider the map $f : [0, +\infty[\to \mathbb{R}$ defined by

$$f(x) = \frac{x + 1/2}{x + 1}$$

   (a) Sketch the graph of f and compute $\lim_{x \to +\infty} f(x)$.

   (b) Show that f satisfies the hypotheses of the Banach fixed point Theorem and find the fixed point of f.

2. Consider the map $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = y(x + 2)^2 + xy^2 - yx^2$$

   (a) Identify the critical points of f and show that they are saddle-points. (**Suggestion:** First simplify the expression of $f(x, y)$)

   (b) Consider the set $M = \{(x, y) \in \mathbb{R}^2 : x^2 + (y + 2)^2 \leq 10\}$.

   Without performing computations:

   (i) show that $f|_M$ has a global maximum and a global minimum.

   (ii) show that the global extrema of $f|_M$ lie on the boundary of M.

3. For $\alpha \in \mathbb{R}$, consider the following differential equation of order 2:

$$y''(x) - \alpha y(x) = 5 \sin(2x) \tag{1}$$

   for which $y(x) = \sin(2x)$ is a **particular** solution.

   (a) Show that $\alpha = -9$.

   (b) Write the general solution of (1), identifying its maximal domain.

4. Consider the linear system in $\mathbb{R}^2$ given by (x and y depend on t):

$$\begin{cases} \dot{x} = -y \\ \dot{y} = -x \end{cases}$$

(a) Write the general form of the solution.

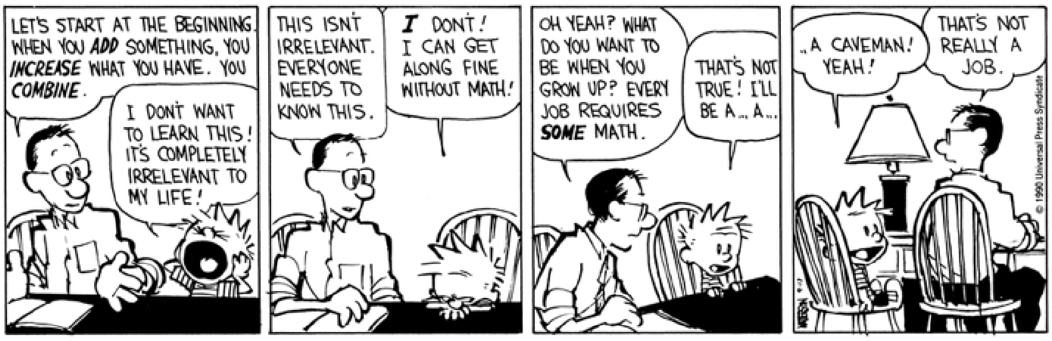
(b) Sketch the associated phase portrait.

5. Consider the following Problem on *Calculus of Variations*, where $x : [0,1] \to \mathbb{R}$ is a smooth function on t:

$$\min_{x(t)\in\mathbb{R}} \int_0^1 [2tx(t) + \dot{x}^2(t)]dt, \quad \text{with} \quad x(0) = 0 \quad \text{and} \quad x(1) \in \mathbb{R}.$$

(a) Write the corresponding Euler-Lagrange equation applied to the case under consideration.

(b) Find the solution of the problem.



Credits:

I	II.1(a)	II.1(b)	II.2(a)	II.2(b)	II.3(a)	II.3(b)	II.4(a)	II.4(b)	II.5(a)	II.5(b)
80	10	15	15	15	10	10	15	5	10	15