Mathematical Economics – Exam 01/02/2024 – Version A.

Part I

a) $D_f = \{(x,y) \in \mathbb{R}^2 : y \geq x^2 \wedge x^2+y^2 > 0 \wedge x^2+y^2 \neq 1\}$

b) interior / accumulation / adherent

$\partial\Omega = \{(x,y) \in \mathbb{R}^2 : (x=1 \wedge y \geq 1) \vee (x=2 \wedge y \geq \frac{1}{2}) \vee (y=\frac{1}{x} \wedge x \in [1,2])\}$

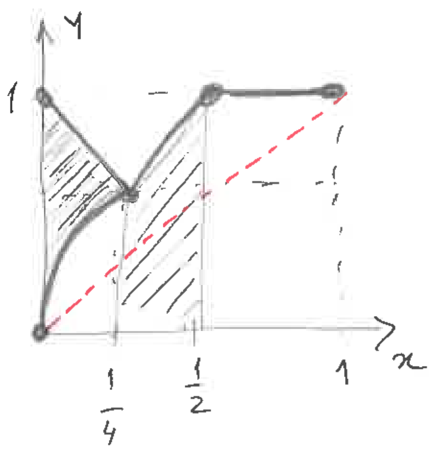
compact

c) 3

d) $\cos(x^2+y)$

e) 2

f)



$\{0\} \cup [\frac{1}{4}, \frac{1}{2}] \cup \{1\}$

g) $f(x) = x.$

h) $\begin{pmatrix} -5 & 0 \\ 0 & -3 \end{pmatrix}$ for instance.

i) $$\begin{cases} \dfrac{1}{x+1} - \dfrac{1}{2}\mu_1 + \mu_2 = 0. \\ 1 - \mu_1 + \mu_3 = 0 \\ \mu_1 x = 0, \ \mu_2 y = 0 \\ \mu_3\left(\frac{1}{2}x + y - 3\right) = 0. \\ \mu_1, \mu_2, \mu_3 \geq 0 \end{cases}$$

(...)

j) $$\begin{cases} y' = -16\cos(4x) \\ y(\pi/8) = 0 \end{cases} \quad \text{OR} \quad \begin{cases} y' = -16y \\ y(\pi/8) = 0. \end{cases}$$

k) Malthus.

$p(t) = 10\,e^{-2t}, t \in \mathbb{R}_0^+$

l) increasing.

m) $b = 0$

$$\lim_{t \to +\infty} P(t) = 3$$

n) $\begin{cases} \dot{x} = 3x \\ \dot{y} = 8y \end{cases}$ unstable

Hartman Grobman

o) $\det A > 0$, $Tr(A) < 0$

p) $H(t, x, u, p) = \ln u + p(x - u)$

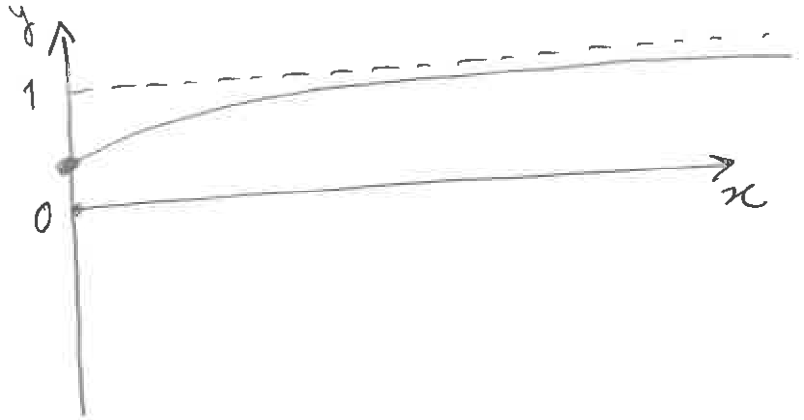
$$\frac{\partial H}{\partial u} = 0 \iff \frac{1}{u} - p = 0 \Rightarrow p = \frac{1}{u} \Rightarrow u = \frac{1}{p}$$

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial p} \\ \dot{p} = -\frac{\partial H}{\partial x} \end{cases} \Rightarrow \begin{cases} \dot{x} = x - \frac{1}{p} \\ \dot{p} = -p \end{cases}$$

Part II

1)

a) $f(x) = \frac{x+1}{x+1} - \frac{1/2}{x+1} = 1 - \frac{1/2}{x+1}$



$\lim\limits_{x \to +\infty} f(x) = 1$

b) • $[0,+\infty[$ is closed $\Rightarrow$ $[0,+\infty[$ is complete

• $f'(x) = \frac{(x+1)-(x+1/2)}{(x+1)^2} = \frac{1/2}{(x+1)^2}$

$|f'(x)| < 1$, $\forall x \in \mathbb{R}_0^+$

$\Downarrow$

f is a contraction

Then $f|_{[0,+\infty[}$ satisfies the Banach fixed point theorem.

$$f(x) = x \iff \frac{x + 1/2}{x+1} = x \iff \cancel{x} + \frac{1}{2} = x^2 + \cancel{x}$$

$$\iff x^2 = \frac{1}{2} \iff x = \pm \frac{\sqrt{2}}{2}$$

$$\mathrm{Fix}(f) = \left\{ \frac{\sqrt{2}}{2} \right\} \checkmark$$

2) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$(x, y) \longrightarrow y(x+2)^2 + xy^2 - yx^2$

a) $f(x,y) = y(x^2 + 4x + 4) + xy^2 - yx^2 =$

$= \cancel{yx^2} + 4xy + 4y + xy^2 - \cancel{yx^2}$

$= 4xy + 4y + xy^2$

$$\nabla f(x,y) = (4y + y^2 ; 4x + 4 + 2xy)$$

$$\nabla f(x,y) = \bar{0} \iff \begin{cases} 4y + y^2 = 0 \\ 4x + 4 + 2xy = 0 \end{cases} \iff \begin{cases} y(4+y) = 0 \\ \_\_\_\_ \end{cases}$$

$$\iff \begin{cases} y = 0 \\ x = -1 \end{cases} \checkmark \quad \begin{cases} y = -4 \\ x = 1 \end{cases}$$

Critical points: $(-1, 0)$ and $(1, -4)$.

$$H_f(x,y) = \begin{pmatrix} 0 & 4+2y \\ 4+2y & 2x \end{pmatrix}$$

$$H_f(-1,0) = \begin{pmatrix} 0 & 4 \\ 4 & -2 \end{pmatrix} \qquad \left.\begin{matrix} \Delta_1 = 0 \\ \Delta_2 = -16 \end{matrix}\right\} \Rightarrow H_f \text{ und} \Downarrow (-1,0) \text{ saddle}$$

$$H_f(1,-4) = \begin{pmatrix} 0 & -4 \\ -4 & 2 \end{pmatrix} \qquad \left.\begin{matrix} \Delta_1 = 0 \\ \Delta_2 = -16 \end{matrix}\right\} \Rightarrow H_f \text{ und} \Downarrow (1,-4) \text{ saddle.}$$

b) i) M is a closed ball $\Rightarrow$ M is compact

f is continuous because it is polynomial

$\Downarrow$

then by Weierstrass theorem. $f|_M$ has a global max and a global min.

ii) $$M = \underbrace{\text{int}(M)}_{\text{open}} \dot\cup \partial M$$

If the global extrema would lie on int(M) (open) then they would have been detected in (a). Therefore $f|_M$ lie on the boundary of M.

3) a) $y(x) = \sin(2x)$

$y'(x) = 2\cos(2x)$

$y''(x) = -4\sin(2x)$

Since $y(x)$ is a solution of (1), then

$$-4\sin(2x) - \alpha . \sin(2x) = 5\sin(2x)$$

$$\Leftrightarrow -4 - \alpha = 5$$

$$\Leftrightarrow \alpha = -9. //$$

b) $y'' + 9y(x) = 5\sin(2x)$.

$P(\lambda) = \lambda^2 + 9 \qquad P(\lambda) = 0 \Leftrightarrow \lambda = \pm 3i$

general solution of (1):

$$y(x) = \underbrace{C_1 \cos(3x) + C_2 \sin(3x)}_{\text{general of the homogeneous}} + \boxed{\sin(2x)}$$

particular solution.

4. a) $\begin{cases} \dot{x} = -y \\ \dot{y} = -x \end{cases} \iff \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$

eigenvalues of A:

$$p(\lambda) = \det \begin{pmatrix} -\lambda & -1 \\ -1 & -\lambda \end{pmatrix} = \lambda^2 - 1$$

$$p(\lambda) = 0 \iff \lambda = \pm 1$$

eigenspaces.

E_1 $\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \boxed{x = -y}$

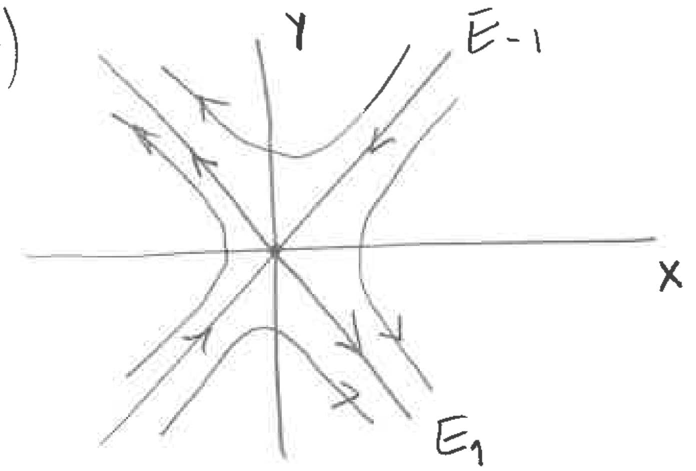
E_{-1} $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \boxed{x = y}$

general solution

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\Leftrightarrow \boxed{\begin{cases} x(t) = c_1 e^{t} + c_2 e^{-t} \\ y(t) = -c_1 e^{t} + c_2 e^{-t} \end{cases}} \quad , c_1, c_2 \in \mathbb{R}$$

b)



5 a) $F(t, x, \dot{x}) = 2tx + \dot{x}^2$

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}} = 0 \Leftrightarrow$$

$$\Leftrightarrow 2t - \frac{d}{dt}(2\dot{x}) = 0 \Leftrightarrow \boxed{t = \ddot{x}}$$

b) $\ddot{x} = t \Leftrightarrow \dot{x} = \frac{t^2}{2} + c_1$

$$\Leftrightarrow x(t) = \frac{t^3}{6} + c_1 t + c_2$$

Since $x(0) = 0$, then $c_2 = 0$.

Since $x(1)$ is free, then $\frac{\partial F}{\partial \dot{x}} = 0$

$$\Leftrightarrow 2\dot{x} = 0 \Leftrightarrow 2 \cdot \left(\frac{1}{2} + c_1\right) = 0$$

$$\Leftrightarrow c_1 = -1/2.$$

Solution: $\boxed{x(t) = \frac{t^3}{6} - \frac{1}{2} t}$ $t \in [0,1]$.

$\nabla F(x, \dot{x}) = (2t, 2\dot{x})$

$H_F = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \rightsquigarrow$ semi-positively defined (eigenvalues 0, 2)

$\Downarrow$

F is convex

$\Downarrow$

$x(\cdot)$ is the <u>min</u>