Universidade de Lisboa
Instituto Superior de Economia e Gestão
Msc in Economics, Mathematical Finance,
Monetary and Financial Economics

Mathematical Economics – 1st Semester - 2023/2024

Special Assessment - 05th of September 2024

Duration: $(120 + \varepsilon)$ minutes, $|\varepsilon| \leq 30$

Version A

Name: ....

Student ID #: ....

Part I

- Complete the following sentences in order to obtain true propositions. The items are independent from each other.
- There is no need to justify your answers.

(a) *(6)* The (maximal) domain of $f : D_f \to \mathbb{R}$ defined by

$$f(x, y) = \ln(y^2 - x)$$

is the set

$D_f =$ ....

and its representation in the plane (x, y) is

(b) *(7)* With respect to the set

$$\Omega = \left\{(x,y) \in \mathbb{R}^2 : x \in [1,2] \ \wedge \ 0 < y \leq 5^x\right\}$$

we may conclude that $\left(\frac{3}{2}, 25\right)$ is an .... point of Ω and

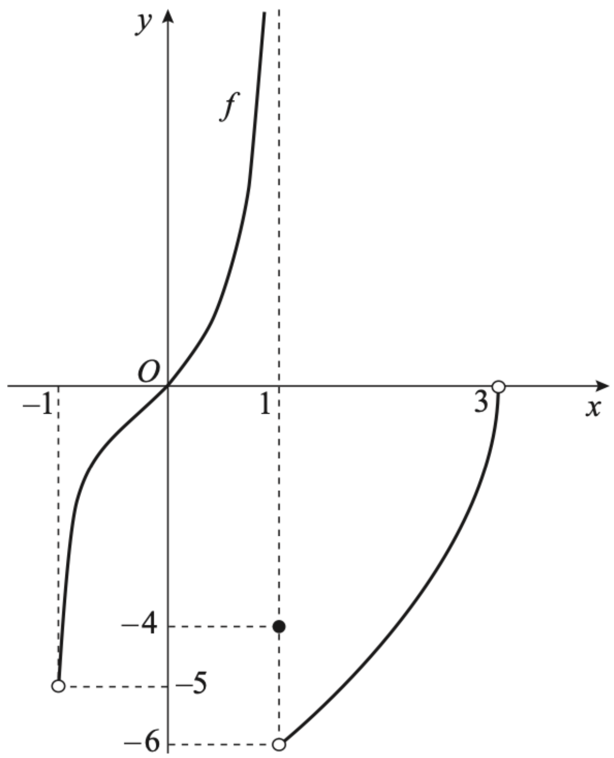
$\partial\Omega = fr(\Omega) =$ ....

Since Ω is not closed, then Ω is **not** ....

(c) *(6)* The graph of $f :]-1,3[\to \mathbb{R}$ is below. We know that

$$\lim_{x\to 1^-} f(x) = +\infty, \qquad f(1) = -4$$

and $(x_n)_{n\in\mathbb{N}}$ is a sequence in $]1,3[$ such that $\lim\limits_{n\in\mathbb{N}} x_n = 1$.



Then

$$\lim_{n\in\mathbb{N}} f(x_n) = \text{.........}$$

(d) *(4)* With respect to a map $f : \mathbb{R}^2 \to \mathbb{R}$, one knows that

$$\nabla f(x,y) = (6xy, 3x^2 + 1).$$

If $f(x,y)$ does not have constant terms in both components, then

$$f(x,y) = \text{..}$$

(e) *(8)* The graphical representation of the correspondence $H\colon [0,1] \rightrightarrows \mathbb{R}$ defined by

$$H(x) = \begin{cases} [\sqrt{x}, 1-2x] & x < \frac{1}{4} \\ [0, 2x], & \frac{1}{4} \leq x \leq \frac{1}{2} \\ \{1\} & x > \frac{1}{2} \end{cases}$$

is:

The set of fixed points of H are explicitly given by: ....

(f) *(4)* Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ a contracting linear map whose matrix is given by $\begin{pmatrix} 0.5 & 0 \\ 0 & -0.7 \end{pmatrix}$. Then, the equation $f(x, y) = (x, y)$ has a unique solution as a consequence of the .... Theorem.

(g) *(4)* With respect to the C^2 map $f : \mathbb{R}^2 \to \mathbb{R}$, we know that $\nabla f(3, 2) = (0, 0)$ and

$$H_f(3,2) = \begin{pmatrix} 1 & \ldots\ldots \\ 0 & \ldots\ldots \end{pmatrix}.$$

Then, we conclude that $(3, 2)$ is a **saddle-point** of f.

(h) *(4)* Consider the following utility **maximisation** problem. There are two goods and an agent with income 3 who wishes to maximize the utility:

$$U(x,y) = \ln(x+1) + y, \quad x \geq 0, \quad y \geq 0$$

subject to a budget constraint $\frac{1}{2}x + y \leq 3$. The necessary conditions to solve the utility maximization problem are:

$$\begin{cases} \text{...............} - \frac{1}{2}\mu_1 + \mu_2 = 0 \\ \text{...............} - \mu_1 + \mu_3 = 0 \\ \mu_1 x = 0, \ \mu_2 y = 0 \\ \text{...............................} = 0 \\ \mu_1, \ \mu_2, \ \mu_3 \text{..........} 0 \\ \frac{1}{2}x + y \leq 3 \\ x \geq 0, y \geq 0 \end{cases}$$

(i) *(4)* The map $y(x) = e^{4x}$, $x \in \mathbb{R}$, is a solution of the IVP $\begin{cases} y'' = \text{......} \\ y(\text{......}) = 1 \end{cases}$.

(j) *(5)* The .... law (associated to a given population of size p that depends on the time $t \geq 0$) states that

$$p' = kp, \qquad k \in \mathbb{R}.$$

If $p(0) = 2$ and $k = 3$, then the **solution** of the previous differential equation is given **explicitly** by

...., where $t \in \mathbb{R}_0^+$.

(k) *(3)* Assuming that y depends on x, any solution of the differential equation $y' = -e^x$ is monotonically ....

(l) *(6)* Consider the following Initial Value Problem (P depends on $t \in \mathbb{R}$):

$$P'' + 3P' + 2P = 0, \quad P(0) = 0, \quad P'(0) = -3.$$

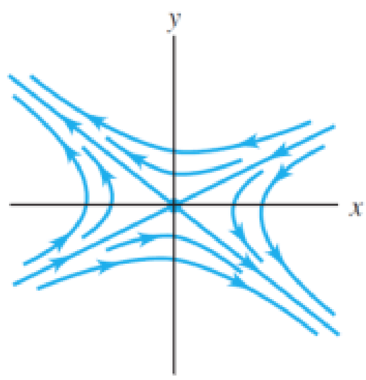
Then $\lim_{t \to +\infty} P(t) =$ ....

(m) *(7)* The linearisation of

$$(*)\begin{cases} \dot{x} = -3x + xy^2 \\ \dot{y} = 8y - yx^2 \end{cases} \qquad \text{around } (0,0) \qquad \text{is} \qquad (**)\begin{cases} \dot{x} = \ldots\ldots\ldots\ldots \\ \dot{y} = \ldots\ldots\ldots\ldots \end{cases}$$

With respect to the Lyapunov's stability, we may conclude that (0,0) is .... Furthermore, ...-.........................Theorem says that, in a small neighbourhood of $(0,0)$, the dynamics of (*) and (**) are "qualitatively" the same (topologically conjugated).

(n) *(4)* The phase portrait of $\dot{X} = AX$, where $A \in M_{2\times 2}(\mathbb{R})$ and $X \in \mathbb{R}^2$, is given by:



Then the following **inequality** holds: $\det(A)$............

(o) *(8)* Consider the following problem of optimal control where $x : [0, 10] \to \mathbb{R}$ is the *state* and $u : [0, 10] \to \mathbb{R}$ is the *control*:

$$\max_{u(t)\in\mathbb{R}} \int_0^{10} \ln(u(t))dt, \quad x'(t) = x(t) - u(t), \quad x(0) = 1 \quad \text{and} \quad x(10) = 1.$$

Then the Hamiltonian is given by (*specify the formulas to the case under consideration*):

$$H(t, x, u, p) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

The Pontryagin maximum principle says that the optimal control $u^\star$ should satisfy the equality .... The Hamiltonian equations are given by:

$$\begin{cases} \dot{x} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \\ \dot{p} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

Part II

- Give your answers in exact form. For example, $\frac{\pi}{3}$ is an exact number while 1.047 is a decimal approximation for the same number.
- In order to receive credit, you must show all of your work. If you do not indicate the way in which you solve a problem, you may get little or no credit for it, even if your answer is correct.

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1. Consider the following subsets of $\mathbb{R}^2$:

$$A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \quad \text{and} \quad B = \{(x, y) \in \mathbb{R}^2 : |y - 3| \leq 1\}.$$

   (a) Using the Hyperplane Separation Theorem, show that A and B may be separated by a line in $\mathbb{R}^2$.

   (b) Consider the map $f : A \to A$ given by:

$$f(x, y) = \left(\frac{y^2}{2}, \frac{x + y}{2}\right)$$

      (a) Using the Brouwer fixed point Theorem, show that f has a fixed point.

      (b) Compute the fixed point of f.

2. Consider the function $f(x, y) = y^2 - x^3 + x^2$.

   (a) Determine and classify all critical points of f. Show that f has no global extrema.

   (b) Determine the smallest value of f restricted to the set

$$M = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$

3. Determine the solution $y(x)$ of the differential equation

$$y'' + 2y' + y = e^{-x},$$

   satisfying the initial conditions $y(0) = 0$ e $y'(0) = 0$.

4. Consider the linear system in $\mathbb{R}^2$ given by (x and y depend on t):

$$\begin{cases} \dot{x} = 2x \\ \dot{y} = -3y \end{cases}$$

(a) Write the general form of the solution.

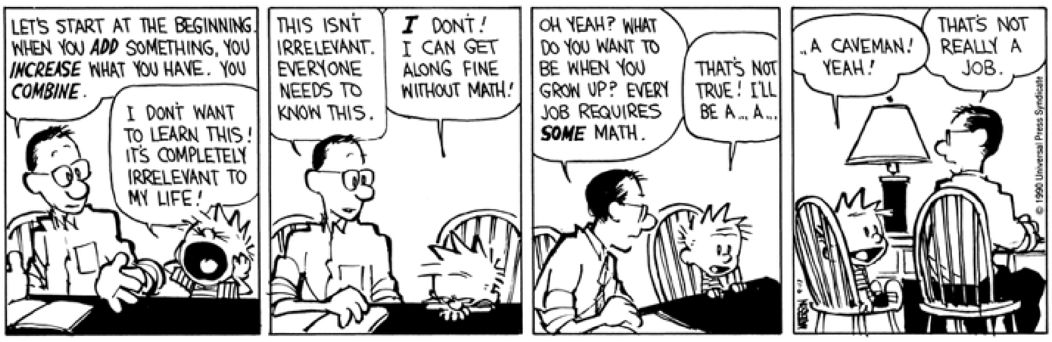
(b) Sketch the associated phase portrait.

5. Consider the following Problem on *Calculus of Variations*, where $x : [0,1] \to \mathbb{R}$ is a smooth function on t:

$$\min_{x(t)\in\mathbb{R}} \int_0^1 [2tx(t) + \dot{x}^2(t)]dt, \quad \text{with} \quad x(0) = 0 \quad \text{and} \quad x(1) \in \mathbb{R}.$$

(a) Write the corresponding Euler-Lagrange equation applied to the case under consideration.

(b) Find the solution of the problem.



Credits:

I	II.1(a)	II.1(b)	II.2(a)	II.2(b)	II.3	II.4(a)	II.4(b)	II.5(a)	II.5(b)
80	10	15	15	15	20	10	10	10	15