Decision Making and Optimization

Master in Data Analytics for Business

Lisbon School of Economics & Management
Universidade de Lisboa

2024-2025

Duality and Sensitivity Analysis

The Dual problem

Construction of the Dual problem

max			min
constraint i type	$\leq$ $\geq$ $=$	≥ 0 ≤ 0 $\in \mathbb{R}$	variable i
variable j	≥ 0 ≤ 0 $\in \mathbb{R}$	$\geq$ $\leq$ $=$	constraint j
matrix A			matrix A^T
o.f. coefic.			RHS
RHS			o.f. coefic.

Example

PRIMAL

$$
\begin{aligned}
\max \quad & z = 4x_1 + 5x_2, \\
\text{s. to:} \quad & 4x_1 + 6x_2 \leq 24, \\
& 2x_1 + x_2 \leq 6, \\
& x_1 - x_2 \leq 1, \\
& x_1 \leq 2, \\
& x_1, x_2 \geq 0,
\end{aligned}
$$

DUAL

$$
\begin{aligned}
\min \quad & w = 24y_1 + 6y_2 + y_3 + 2y_4, \\
\text{s. to:} \quad & 4y_1 + 2y_2 + y_3 + y_4 \geq 4, \\
& 6y_1 + y_2 - y_3 \geq 5, \\
& y_1, y_2, y_3, y_4 \geq 0,
\end{aligned}
$$

Example

PRIMAL

$$
\begin{aligned}
\max \quad & z = 4x_1 + 5x_2, \\
\text{s. to:} \quad & 4x_1 + 6x_2 \leq 24, \\
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DUAL

$$
\begin{aligned}
\min \quad & w = 24y_1 + 6y_2 + y_3 + 2y_4, \\
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& 6y_1 + y_2 - y_3 \geq 5, \\
& y_1, y_2, y_3, y_4 \geq 0,
\end{aligned}
$$

Primal/Dual Relations

primal variables are connected with dual constraints

dual variables are connected with primal constraints

dual objective function coefficients are connected with the primal rhs

- The i-th shadow price (optimal value of dual variable i) represents the proportion of the change in the optimal value of the primal due to an increase in the i-th right-hand-side

The dual of the dual is the primal.

Fundamental properties

- The value of the o.f. for any feasible solution to the maximization problem is not greater than the value of the o.f. corresponding to a solution to the minimization problem
- If x and y are feasible solutions to a pair of primal and dual problems that give equal values to the respective o.f., then x and y are optimal solutions to the respective primal and dual problems
- For any pair of dual problems, the existence of finite optimal solutions for one of them guarantees the existence of finite optimal solutions for the other, and the respective optimal values of the o.f. coincide
- A LP problem has a finite optimal solution if and only if there are feasible solutions for the primal and the dual
- If for one of the problems there is an unbounded solution, then the other does not has feasible solutions

Obtain the Dual Optimal Solution

Obtain the Dual Optimal Solution

Knowing the Optimal Primal Solution

- Graphically: to find y_i solve LP with $\Delta b_i = +1$, then $y_i = \Delta z$
- Simplex optimal table: y_i are the values in the z row of the i-th constraint slack variable
- Sensitivity report of the Excel Solver: "Shadow Price" column.
- Complementarity primal/dual relations:
  If a constraint in (P) is not binding, then the dual variable is zero
  If a variable in (P) is non-zero, then the dual constraint is binding

Example: the RM model

The optimal solution of the (PRIMAL) model

$$\begin{aligned} \max \quad & z = 4x_1 + 5x_2, \\ \text{s. to:} \quad & 4x_1 + 6x_2 \leq 24, \\ & 2x_1 + x_2 \leq 6, \\ & x_1 - x_2 \leq 1, \\ & x_1 \leq 2, \\ & x_1, x_2 \geq 0 \end{aligned}$$

is $x^* = (\frac{3}{2}, 3) = (1.5, 3)$
with value $z^* = 21$

the augmented optimal solution is
$x^* = (\frac{3}{2}, 3, 0, 0, \frac{5}{2}, \frac{1}{2})$
$= (1.5, 3, 0, 0, 2.5, 0.5)$

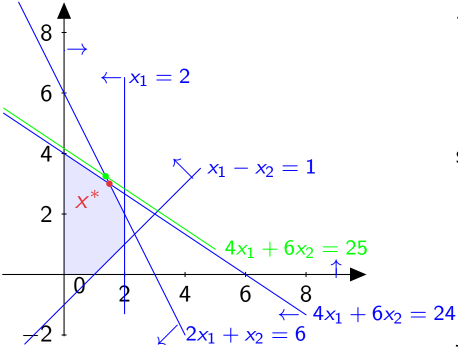
the binding constraints are

$$\begin{cases} 4x_1 + 6x_2 = 24, \\ 2x_1 + x_2 = 6, \end{cases}$$

we want to obtain the augmented optimal dual solution which is
$y^* = (\frac{3}{4}, \frac{1}{2}, 0, 0, 0, 0) = (0.75, 0.5, 0, 0, 0, 0)$

Graphically: value of y_1

The optimal value of dual variable y_1 represents the proportion of the change in the optimal value of the primal z^* due to an increase in the right-hand-side of the constraint $4x_1 + 6x_2 \leq 24$



$$\Delta b_1 = \hat{b_1} - b_1 = 1 \implies \hat{b_1} = 25$$

$$\begin{cases} 4x_1 + 6x_2 = 25, \\ 2x_1 + x_2 = 6, \end{cases}$$

solving we obtain

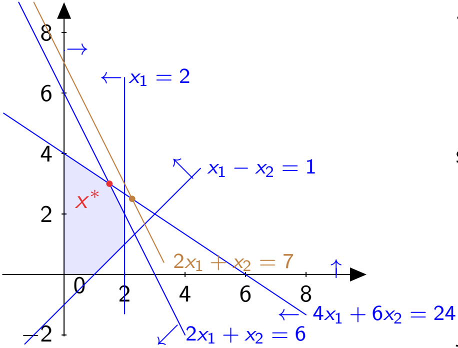
$$\begin{cases} x_1 = \frac{11}{8} \\ x_2 = \frac{13}{4}, \end{cases} \implies \hat{z}_1 = \frac{87}{4}$$

hence

$$y_1^* = \Delta z = \hat{z}_1 - z^* = \frac{87}{4} - 21 = \frac{3}{4}$$

Graphically: value of y_2

The optimal value of dual variable y_2 represents the proportion of the change in the optimal value of the primal z^* due to an increase in the right-hand-side of the constraint $2x_1 + x_2 \leq 6$



$$\Delta b_2 = \hat{b}_2 - b_2 = 1 \implies \hat{b}_2 = 7$$

$$\begin{cases} 4x_1 + 6x_2 = 24, \\ 2x_1 + x_2 = 7, \end{cases}$$

solving we obtain

$$\begin{cases} x_1 = \frac{9}{4} \\ x_2 = \frac{5}{2}, \end{cases} \implies \hat{z}_2 = \frac{86}{4}$$

hence

$$y_2^* = \Delta z = \hat{z}_2 - z^* = \frac{86}{4} - 21 = \frac{1}{2}$$

The Excel Solver

The Sensitivity Report: shadow price and reduced cost column

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B2	value x1	1,5	0	4	6	0,666666667
C2	value x2	3	0	5	1	3

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
D6	c1	24	0,75	24	12	4
D7	c2	6	0,5	6	0,666666667	2
D8	c3	-1,5	0	1	1E+30	2,5
D9	c4	1,5	0	2	1E+30	0,5

The Simplex Optimal Table

$\bar{z}$ and basic var	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_2	0	1	$\frac{1}{12}$	$-\frac{1}{2}$	0	0	3
x_1	1	0	$-\frac{1}{8}$	$\frac{3}{4}$	0	0	$\frac{3}{2}$
x_5	0	0	$\frac{3}{8}$	$-\frac{5}{4}$	1	0	$\frac{5}{2}$
x_6	0	0	$\frac{1}{8}$	$-\frac{3}{4}$	0	1	$\frac{1}{2}$
$\bar{z}$	0	0	$\frac{3}{4}$	$\frac{1}{2}$	0	0	21

The optimal values of the dual solution can be found in the row of the Z:
The value of the dual decision variable y_i is the value in the Z row of the i-th constraint slack variable
The value of the dual slack variable corresponding to constraint j is the value of the j-th primal decision variable x_j

Complementarity solutions

complementary solutions verify complementarity primal/dual relations

Problem (P)		Dual of (P)
non-zero decision variable	$\Rightarrow$	binding constraint
non-binding constraint	$\Rightarrow$	decision variable equal to zero

If x and y are feasible solutions to a pair of primal and dual problems that give equal values to the respective o.f., and verify the complementarity solutions property, then x and y are optimal solutions to the respective primal and dual problems

Complementarity primal/dual relations

$\max \quad z = \sum_{j\in J} c_j x_j$ s. a: $\sum_{j\in J} a_{ij}x_j \le b_i, \quad i \in I$ $x_j \ge 0, \quad j \in J$	$\min \quad w = \sum_{i\in I} y_i b_i$ s. a: $\sum_{i\in I} a_{ij}y_i \ge c_j, \quad j \in J$ $y_i \ge 0, \quad i \in I$
$x_j^* > 0 \Rightarrow \sum_{i\in I} a_{ij}y_i^* = c_j$ $\sum_{j\in J} a_{ij}x_j^* < b_i \Rightarrow y_i^* = 0$	$y_i^* > 0 \Rightarrow \sum_{j\in J} a_{ij}x_j^* = b_i$ $\sum_{i\in I} a_{ij}y_i^* > c_j \Rightarrow x_j^* = 0$

If a variable in (P) is non-zero, then the dual constraint is binding

$$x_j^* \left(\sum_{i\in I} a_{ij}y_i^* - c_j\right) = 0, \qquad j \in J$$

If a constraint in (P) is not binding, then the dual variable is zero

$$\left(\sum_{j\in J} a_{ij}x_j^* - b_i\right) y_i^* = 0, \qquad i \in I$$

Complementarity primal/dual relations

as we know the primal solution, $x^* = (\frac{3}{2}, 3, 0, 0, \frac{5}{2}, \frac{1}{2})$, the complementarity primal/dual relations for our example are

$$\begin{cases} x_1^* \left(4y_1^* + 2y_2^* + y_3^* + y_4^* - 4\right) = 0 \\ x_2^* \left(6y_1^* + y_2^* - y_3^* - 5\right) = 0 \\ \left(24 - 4x_1^* - 6x_2^*\right) y_1^* = 0 \\ \left(6 - 2x_1^* - x_2^*\right) y_2^* = 0 \\ \left(1 - x_1^* + x_2^*\right) y_3^* = 0 \\ \left(2 - x_1^*\right) y_4^* = 0 \end{cases}$$

hence we have

Complementarity primal/dual relations

$$\begin{cases} x_1^* \ (4y_1^* + 2y_2^* + y_3^* + y_4^* - 4) = 0 \\ x_2^* \ (6y_1^* + y_2^* - y_3^* - 5) = 0 \\ (24 - 4x_1^* - 6x_2^*) \ y_1^* = 0 \\ (6 - 2x_1^* - x_2^*) \ y_2^* = 0 \\ (1 - x_1^* + x_2^*) \ y_3^* = 0 \\ (2 - x_1^*) \ y_4^* = 0 \end{cases} \Longrightarrow \begin{cases} x_1^* = \frac{3}{2} \Longrightarrow (4y_1^* + 2y_2^* + y_3^* + y_4^* - 4) = 0 \\ x_2^* = 3 \Longrightarrow (6y_1^* + y_2^* - y_3^* - 5) = 0 \\ (24 - 4x_1^* - 6x_2^*) = 0 \Longrightarrow y_1^* ?? \\ (6 - 2x_1^* - x_2^*) = 0 \Longrightarrow y_2^* ?? \\ (1 - x_1^* + x_2^*) = \frac{5}{2} \Longrightarrow y_3^* = 0 \\ (2 - x_1^*) = \frac{1}{2} \Longrightarrow y_4^* = 0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} 4y_1^* + 2y_2^* + y_3^* + y_4^* - 4 = 0 \\ 6y_1^* + y_2^* - y_3^* - 5 = 0 \\ - \\ - \\ y_3^* = 0 \\ y_4^* = 0 \end{cases} \Longleftrightarrow \begin{cases} 4y_1^* + 2y_2^* - 4 = 0 \\ 6y_1^* + y_2^* - 5 = 0 \\ y_3^* = 0 \\ y_4^* = 0 \end{cases} \Longleftrightarrow \begin{cases} y_1^* = \frac{3}{4} \\ y_2^* = \frac{1}{2} \\ y_3^* = 0 \\ y_4^* = 0 \end{cases}$$

Economic interpretation

Economic interpretation

Activity analysis

- n activities (decision variables)
- m resources (constrainsts)
- z total profit with the n activities
- x_j level of activity $j \in J = \{1, \ldots, n\}$
- c_j unitary profit of activity $j \in J$
- b_i availability of resource $i \in I = \{1, \ldots, m\}$
- a_{ij} amount of resource $i \in I$ spent per unit of activity $j \in J$

The dual model provides economic information about scarce resources.

Economic interpretation

Resources analysis

The dual model provides economic information about scarce resources.

The reduced cost of variable $j \in J$ is the shadow price of constraint $x_j \geq 0$

The shadow price or marginal value of the resource (constraint) $i \in I$

- is the value of the dual variable y_i associated with this constraint and
- represents the change in the optimal value z^* when one unit of this constraint is added to the RHS, keeping everything else the same, i.e.,
- it is the proportion of variation in the optimal profit z^* when the available quantity of resource i (b_i) is slightly increased
- also corresponds to the *fair price* that we would pay for adding one more unit to the RHS, available quantity of resource i (b_i), of this constraint

Economic interpretation

When an optimal solution exists

$$z^* = w^* = b_1 y_1 + b_2 y_2 + \cdots + b_m y_m$$

thus $b_i y_i$ is the contribution to the profit z^* of the available b_i units of resource i (in the primal)

consider a variation Δb_i in the resource i, then the new optimal value is

$$\bar{z} = b_1 y_1 + b_2 y_2 + \cdots + (b_i + \Delta b_i) y_i + \cdots + b_m y_m$$

$$\iff \bar{z} = b_1 y_1 + b_2 y_2 + \cdots + b_i y_i + \cdots + b_m y_m + \Delta b_i \; y_i = z^* + \Delta b_i \; y_i$$

$$\iff \Delta b_i \; y_i = \bar{z} - z^*$$

if $\Delta b_i y_i = 1$ then $\bar{z} - z^* = y_i$

Example

PRIMAL

$$\begin{aligned} \max \quad & z = 4x_1 + 5x_2, \\ \text{s. to:} \quad & 4x_1 + 6x_2 \le 24, \\ & 2x_1 + x_2 \le 6, \\ & x_1 - x_2 \le 1, \\ & x_1 \le 2, \\ & x_1, x_2 \ge 0, \end{aligned}$$

$$x^* = (\tfrac{3}{2}, 3, 0, 0, \tfrac{5}{2}, \tfrac{1}{2}) = (1.5, 3, 0, 0, 2.5, 0.5)$$
$$z^* = 21$$

DUAL

$$\begin{aligned} \min \quad & w = 24y_1 + 6y_2 + y_3 + 2y_4, \\ \text{s. to:} \quad & 4y_1 + 2y_2 + y_3 + y_4 \ge 4, \\ & 6y_1 + y_2 - y_3 \ge 5, \\ & y_1, y_2, y_3, y_4 \ge 0, \end{aligned}$$

$$y^* = (\tfrac{3}{4}, \tfrac{1}{2}, 0, 0, 0, 0) = (0.75, 0.5, 0, 0, 0, 0)$$
$$w^* = 21$$

Economic interpretation

$z^* = 21$	the total profit is 21 monetary units
$x_1^* = 1.5$	production of 1.5 units of P1 (interior paint)
$x_2^* = 3$	production of 3 units of P2 (exterior paint)
$x_3^* = 0$	the resource M1, raw material M1, is exhausted
$x_4^* = 0$	the resource M2, raw material M2, is exhausted
$x_5^* = 2.5$	the proportion of paints is satisfied, but not equal
$x_6^* = 0.5$	daily max production level is satisfied

$w^* = 21$	internal valuation of production and resources
$y_1^* = 0.75$	internal valuation of the resource M1
$y_2^* = 0.5$	internal valuation of the resource M2
$y_3^* = 0$	internal valuation of the paints proportion
$y_4^* = 0$	internal valuation of the daily max production level
$y_5^* = 0$	loss of opportunity to produce P1
$y_6^* = 0$	loss of opportunity to produce P1

Economic interpretation

The following interpretation is only valid as long as the binding constraints are the same (first and second in this example), i.e. the basis is the same.

- $x_1 = 1.5$ and $y_5 = 0$ ($\Leftrightarrow 4y_1 + 2y_2 + y_3 + y_4 \geq 4$) : 1.5 tons of paint P1 are produced, so its opportunity loss is zero
- $x_2 = 3$ and $y_6 = 0$ ($\Leftrightarrow 6y_1 + y_2 - y_3 \geq 5$) : 3 tons of paint P2 are produced, so its opportunity loss is zero

Note: in the case $x_j = 0$ and $y_j = u \geq 0$ the interpretation is that product Pj is not produced because its opportunity loss is non-zero ($= u$), meaning that the internal valuation of the primal constraints (on the resources needed to produce one unit of P1 and others) is higher than its unit profit, so producing one unit of P1 would cause a decrease in the profit of u

Economic interpretation

- $x_3 = 0$ ($\Leftrightarrow 4x_1 + 6x_2 \leq 24 \;\leftarrow$ M1) and $y_1 = 0.75$: the available raw material has been exhausted, it is a scarce resource because its internal valuation is non-zero; this resource has been internally valued by 0.75 units which means that for each additional unit of raw material M1 the profit z increases by 0.75 u.m..
- $x_4 = 0$ ($\Leftrightarrow 2x_1 + x_2 \leq 6 \;\leftarrow$ M2) and $y_2 = 0.5$: the available raw material has been exhausted, it is a scarce resource because its internal valuation is non-zero; this resource has been internally valued by 0.5 units which means that for each additional unit of raw material the profit increases by 0.5 u.m..
- $x_5 = 2.5$ ($\Leftrightarrow x_1 - x_2 \leq 1 \;\leftarrow$ paint proportion) and $y_3 = 0$: the paint proportion level has not been attained by 2.5 units so its internal valuation is zero.
- $x_6 = 0.5$ ($\Leftrightarrow x_1 \leq 2 \;\leftarrow$ max production) and $y_1 = 0$: the max production level has not been attained by 0.5 units so its internal valuation is zero.

Sensitivity analysis

Sensitivity analysis

- Changes at the RHS
- Changes to the objective function coefficients
- Add a new constraint
- Add a new variable

Changes at the RHS

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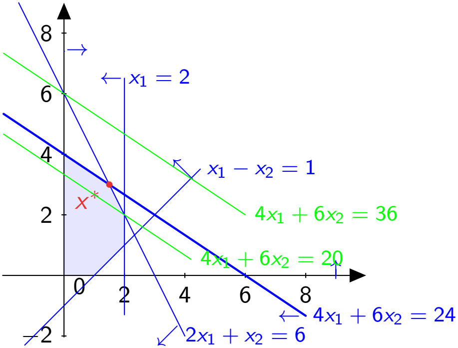
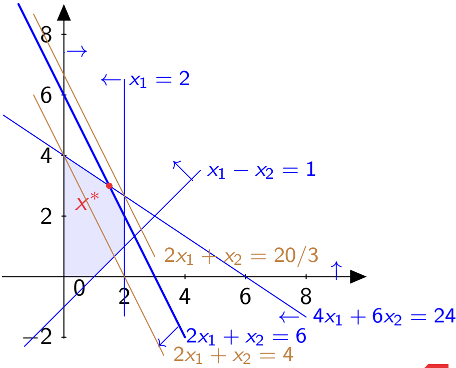




Changes at the RHS

The Sensitivity Report: see allowable increase and allowable decrease columns for the constraints

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B2	value x1	1,5	0	4	6	0,666666667
C2	value x2	3	0	5	1	3

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
D6	c1	24	0,75	24	12	4
D7	c2	6	0,5	6	0,666666667	2
D8	c3	-1,5	0	1	1E+30	2,5
D9	c4	1,5	0	2	1E+30	0,5

$b_1 = 24$ and can decrease 4 and increase 12, i.e. $-4 \leq \Delta b_1 \leq 12$

$b_2 = 6$ and can decrease 2 and increase $2/3$, i.e. $-2 \leq \Delta b_2 \leq 2/3$

Changes at the RHS

$b_1 = 24$ and can decrease 4 and increase 12, $-4 \leq \Delta b_1 \leq 12$ thus

$$20 = 24 - 4 \leq b_1 \leq 24 + 12 = 36$$

shadow prices maintain their value
minor change in the solution value $\hat{z}^* = z^* + \Delta b_1 y_1^*$

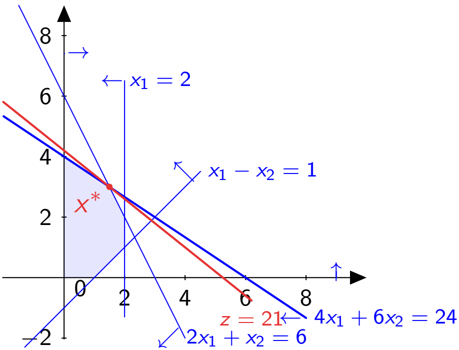
$b_2 = 6$ and can decrease 2 and increase $2/3$, $-2 \leq \Delta b_2 \leq 2/3$ thus

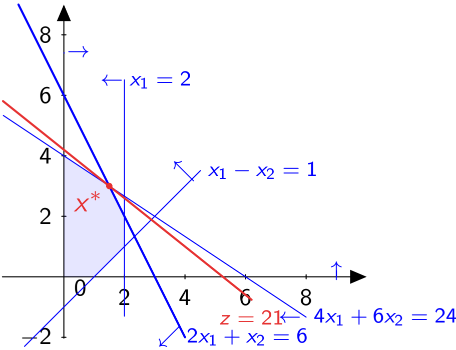
$$4 = 6 - 2 \leq b_2 \leq 6 + 2/3 = 20/3$$

shadow prices maintain their value
minor change in the solution value $\hat{z}^* = z^* + \Delta b_2 y_2^*$

Changes to the objective function coefficients

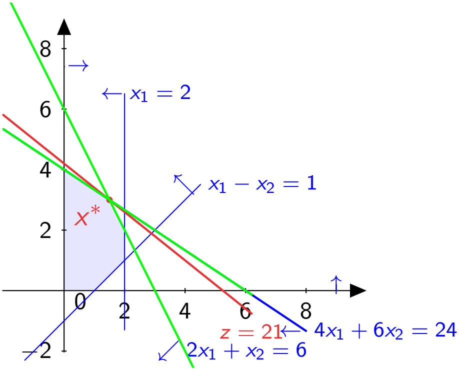
$$z = c_1 x_1 + c_2 x_2 \iff x_2 = \frac{1}{c_2} z - \frac{c_1}{c_2} x_1 \implies \nabla z = (c_1, c_2) \text{ and slope } = -\frac{c_1}{c_2}$$

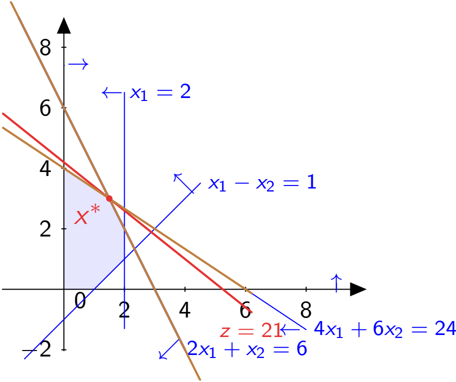




Changes to the objective function coefficients

$$z = c_1x_1 + c_2x_2 \iff x_2 = \frac{1}{c_2}z - \frac{c_1}{c_2}x_1 \implies \nabla z = (c_1, c_2) \text{ and slope } = -\frac{c_1}{c_2}$$





Changes to the objective function coefficients

The Sensitivity Report: see allowable increase and allowable decrease columns for the variables

Variable Cells

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D8	c3	-1,5	0	1	1E+30	2,5
D9	c4	1,5	0	2	1E+30	0,5

$c_1 = 4$ and can decrease $2/3$ and increase 6, i.e. $-2/3 \leq \Delta c_1 \leq 6$

$c_2 = 5$ and can decrease 3 and increase 1, i.e. $-3 \leq \Delta b_2 \leq 1$

Changes to the objective function coefficients

$c_1 = 4$ and can decrease $2/3$ and increase 6, $-2/3 \leq \Delta c_1 \leq 6$ thus

$$10/3 = 4 - 2/3 \leq c_1 \leq 4 + 6 = 10$$

primal solution maintain their value
minor change in the solution value $\hat{z}^* = z^* + \Delta c_1 x_1^*$

$c_2 = 5$ and can decrease 3 and increase 1, $-3 \leq \Delta c_2 \leq 1$ thus

$$2 = 5 - 3 \leq c_2 \leq 5 + 1 = 6$$

primal solution maintain their value
minor change in the solution value $\hat{z}^* = z^* + \Delta c_2 x_2^*$

Add a new constraint

- If the new constraint satisfies the optimal solution, nothing changes, the current solution is the optimal solution.

- Otherwise the problem must be solved again with the new constraint.

Add a new constraint to the RM problem

- The paint produced needs to be stored during five days and the capacity of the warehouse is 40 tons of paint:

$$x_1 + x_2 \leq 8$$

$1.5 + 3 = 4.5 \leq 8$ the optimal solution satisfies this additional constraint

- The paint produced needs to be stored during five days and the capacity of the warehouse is 20 tons of paint.

$$x_1 + x_2 \leq 4$$

$1.5 + 3 = 4.5 > 4$ the optimal solution does not satisfies this additional constraint, solve the problem with the new constraint to obtain the new optimal solution

Add a new variable

the addition of a new activity is considered

using the new activity coefficients, write the corresponding dual constraint

- If the new constraint satisfies the dual optimal solution, nothing changes, the current solution is the optimal solution.

- Otherwise the problem must be solved again with the new activity.

Add a new variable to the RM problem

Suppose that RM is considering to introduce in the production plan a new product:

A new special paint is to be produced which will give a profit of 5000 per ton and will use 5 tons of RM1 and 4 tons of RM2 per ton produced. Should the production plan be changed to introduce this new product?

the associated dual constraint is

$$5y_1 + 4y_2 \geq 5$$

as the dual optimal solution is $y^* = (0.75, 0.5, 0, 0, 0, 0)$

we have $5 \times 0.75 + 4 \times 0.5 = 3.75 + 2 = 5.75 > 5$!!!

This new product should not be produced.
Do not change the production plan!