Decision Making and Optimization

Master in Data Analytics for Business

Lisbon School
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Transportation Problem And Variants

Transportation Problem

Transportation Problem (TP)

Determine the quantities of a commodity to be shipped from a set of distribution centers - `the origins (or sources)` - to a set of receiving centers - `the destinations` - such that the total cost is minimized.

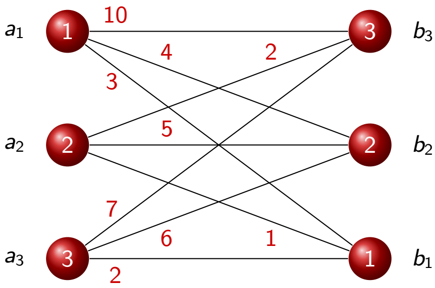
Applications:

- Transportation of products
- Production planning
- Scheduling human resources

Transportation Problem

Data

- m origin points, each with a_i $(i = 1, \ldots, m)$ units of a certain product;
- n destination points, each requiring b_j $(j = 1, \ldots, n)$ units of the same product;
- c_{ij} unit transportation cost between each origin i and destination j.



Determine the way of transporting the product between origins and destinations with minimal cost.

Define the decision variables

x_{ij} as the nº of units transported between source i and destination j.

Assume that, with a_i and b_j non-negative, the TP is balanced

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

that is, the total supply and the total demand are equal,

if $\sum_{i=1}^{m} a_i > \sum_{j=1}^{n} b_j$ a destination is created fictitious;
if $\sum_{i=1}^{m} a_i < \sum_{j=1}^{n} b_j$ an origin is created fictitious;

in both cases the associated transportation costs will be zero.

Transportation model

Considering the decision variables x_{ij} that indicate the quantity transported between origin i and destination j,
the LP formulation of the transportation problem (TP) is:

$$\min \quad \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$$

s. a:

$$\sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1, \ldots, m$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1, \ldots, n$$

$$x_{ij} \geq 0, \quad i = 1, \ldots, m, \; j = 1, \ldots, n$$

Small example

Let us consider a T.P. with 3 origins and 4 destinations, with

$a = [a_i] = [6\ 8\ 10]$,

$b = [b_j] = [4\ 6\ 8\ 6]$

and the unit transportation costs given by $C = [c_{ij}] = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 0 \\ 0 & 2 & 2 & 1 \end{bmatrix}$

Small example

$$\min \; x_{11} + 2x_{12} + 3x_{13} + 4x_{14} + 4x_{21} + 3x_{22} + 2x_{23} + 2x_{32} + 2x_{33} + x_{34}$$

$$\text{s. a:} \quad \begin{aligned} x_{11} + x_{12} + x_{13} + x_{14} &= 6 \\ x_{21} + x_{22} + x_{23} + x_{24} &= 8 \\ x_{31} + x_{32} + x_{33} + x_{34} &= 10 \end{aligned}$$

$$\begin{aligned} x_{11} + x_{21} + x_{31} &= 4 \\ x_{12} + x_{22} + x_{32} &= 6 \\ x_{13} + x_{23} + x_{33} &= 8 \\ x_{14} + x_{24} + x_{34} &= 6 \end{aligned}$$

$$x_{ij} \geq 0, \quad i = 1, \ldots, m, \; j = 1, \ldots, n$$

Transportation Problem: example

MG Auto has three plants in Los Angeles, Detroit, and New Orleans and two major distribution centers in Denver and Miami. The quarterly capacities of the three plants are 1000, 1500, and 1200 cars, and the demands at the two distribution centers for the same period are 2300 and 1400 cars. Mileage between plants and distribution centers is shown in the table at left.

	Distribution Centers		
Plants	Denver	Miami	Capacity
Los Angeles	1027	2342	1000
Detroit	1158	1086	1500
New Orleans	1303	661	1200
Demand	2300	1400	3700

The trucking company in charge of transporting the cars charges 8 cents per mile per car. Formulate as a linear programming problem to find the transportation plan that minimizes the total cost.

TP example

Transportation cost per car (rounded to the nearest $):

	Distribution Centers	
Plants	Denver	Miami
Los Angeles	82	187
Detroit	92	86
New Orleans	104	52

Decision variables:

x_{ij} as the number of cars transported between plant $i = 1, 2, 3$ and distribution center $j = 1, 2$, with $i = \begin{cases} 1 \rightarrow & \text{Los Angeles} \\ 2 \rightarrow & \text{Detroit} \\ 3 \rightarrow & \text{New Orleans} \end{cases}, \quad j = \begin{cases} 1 \rightarrow & \text{Denver} \\ 2 \rightarrow & \text{Miami} \end{cases}.$

TP example

The data of this example can be summarized in the following table

	Distribution Centers		
Plants	Denver	Miami	Capacity
Los Angeles	82	187	1000
Detroit	92	86	1500
New Orleans	104	52	1200
Demand	2300	1400	3700

that shows transportation cost per car (rounded to the nearest $), the capacity and the demands.

the model is

$$
\begin{aligned}
\min \quad & 82x_{11} + 187x_{12} + 92x_{21} + \\
& + 86x_{22} + 104x_{31} + 52x_{32} \\
\text{s. t.:} \quad & \\
& x_{11} + x_{12} = 1000 \\
& x_{21} + x_{22} = 1500 \\
& x_{31} + x_{32} = 1200 \\
& \\
& x_{11} + x_{21} + x_{31} = 2300 \\
& x_{12} + x_{22} + x_{32} = 1400 \\
& \\
& x_{ij} \geq 0, \quad i = 1, 2, 3, \; j = 1, 2
\end{aligned}
$$

Transportation Model Properties

- The transportation problem has at least one feasible solution which is

$$x_{ij} = \frac{a_i b_j}{\sum a_i} = \frac{a_i b_j}{\sum b_j}, \quad \forall i, j$$

- The values of the variables satisfy

$$0 \le x_{ij} \le \min\{a_i, b_j\}, \quad \forall i, j$$

- From the two previous items it follows that the T.P. always has an optimal solution
- When supplies a_i ($\forall i$) and demands b_j ($\forall j$) are integer values, then any feasible basic solution has integer values, so is the optimal solution

Solving the problem with Excel Solver

All data of the T.P. are easily represented by the next table with m rows and n columns in addition to the a_i column and the b_j line

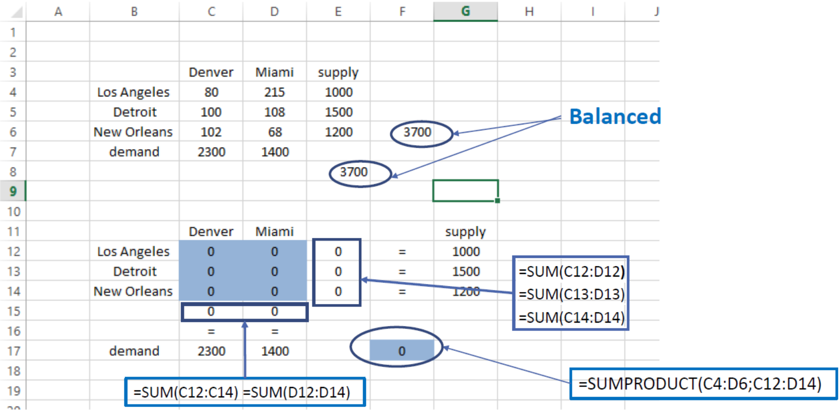
c_{11}	c_{12}	$\cdots$	c_{1j}	$\cdots$	c_{1n}	a_1
c_{21}	c_{22}	$\cdots$	c_{2j}	$\cdots$	c_{2n}	a_2
$\vdots$			$\vdots$		$\vdots$	$\vdots$
c_{i1}	c_{i2}	$\cdots$	c_{ij}	$\cdots$	c_{in}	a_i
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
c_{m1}	c_{m2}	$\cdots$	c_{mj}	$\cdots$	c_{mn}	a_m
b_1	b_2	$\cdots$	b_j	$\cdots$	b_n	

Specific Cases

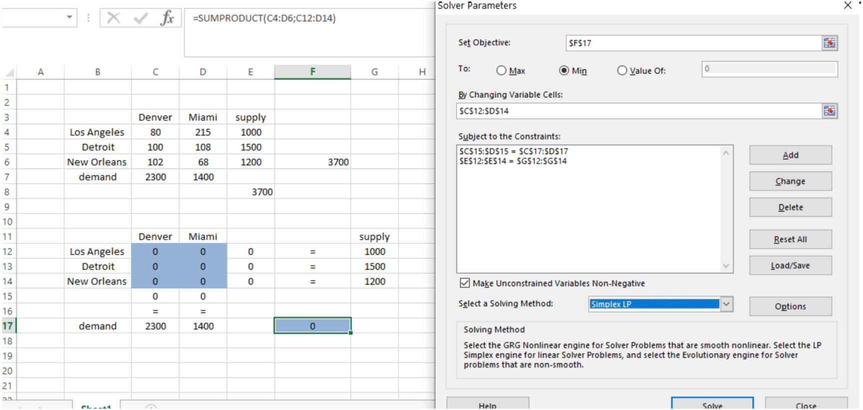
Problems that have the same structure of parameters but:

- total supply $>$ total demand: origin constraints type $\leq$
  Opt. Sol. : part of the supply is not transported.
- total supply $<$ total demand: destination constraints type $\leq$
  Opt. Sol. : part of the demand is not satisfied.
- Destination requiring demand between a minimum and a maximum value:
  2 constraints at the destination: "$\leq$ maximum demand" and "$\geq$ minimum demand".
- Origin producing supply between a minimum and a maximum value:
  2 constraints at the origin: "$\leq$ maximum supply" and "$\geq$ minimum supply".
- Infeasible link: corresponding variable is set to zero.
- Maximization problem: in solver/excel choose OF: Max.

Solving using the Solver of the Excel



Solving using the Solver of the Excel



TP variants

Problems that have the same structure of parameters but differ from the TP:

- total supply $>$ total demand: origin constraints $\leq$
  Opt. Sol. : Part of the supply is not transported
- total supply $<$ total demand: destination constraints $\leq$
  Opt. Sol. : Part of the demand is not satisfied
- Destination requiring demand between a minimum and a maximum value: 2 constraints at the destination: "$\leq$ maximum demand" and "$\geq$ minimum demand"
- Origin producing supply between a minimum and a maximum value: 2 constraints at the origin: "$\leq$ maximum supply" and "$\geq$ minimum supply"
- Infeasible link: corresponding variable is set to zero or assign a huge cost (in a minimization problem)
- Maximization problem: in solver/excel choose OF as Max.

Assignment Problem

Assignment Model

Given

- n individuals,
- n tasks,
- and being c_{ij} the cost of assigning the individual i to task j.

The goal is to assign each individual to one and only one task in such a way that the total cost of performing the tasks is minimum.

Assignment Model

Considering the binary variables x_{ij} that indicate whether the individual i is assigned to task j, $x_{ij} = 1$, or not $x_{ij} = 0$,
the LP model of the assignment problem (AP) is:

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \\
\text{s. to:} \quad & \\
& \sum_{j=1}^{n} x_{ij} = 1, \quad i = 1, \ldots, n \\
& \sum_{i=1}^{n} x_{ij} = 1, \quad j = 1, \ldots, n \\
& x_{ij} \in \{0, 1\}, \quad i = 1, \ldots, n, \; j = 1, \ldots, n
\end{aligned}
$$

Example

Let's consider a Factory with 3 sections (assembly (A), painting (P) and packaging (K))
and 3 candidates (C1, C2, C3),

the allocation costs are given by

	A	P	K
$C1$	4	5	3
$C2$	1	4	2
$C3$	3	1	5

$$
\begin{aligned}
\min \quad & \\
& 4x_{11} + 5x_{12} + 3x_{13} + x_{21} + 4x_{22} + 2x_{23} + 3x_{31} + x_{32} + 5x_{33} \\
\text{s. to:} \quad & x_{11} + x_{12} + x_{13} = 1 \\
& x_{21} + x_{22} + x_{23} = 1 \\
& x_{31} + x_{32} + x_{33} = 1 \\
& \\
& x_{11} + x_{21} + x_{31} = 1 \\
& x_{12} + x_{22} + x_{32} = 1 \\
& x_{13} + x_{23} + x_{33} = 1 \\
& \\
& x_{ij} \in \{0, 1\}, \quad i = 1, \dots, 3,\ j = 1, \dots, 3
\end{aligned}
$$

Properties and Applications

Properties

- Is a particular case of T.P. in which $m = n$ and $a_i = b_j = 1$, as such any feasible basic solution has integer values.
- Due to its special structure, constraints $x_{ij} \in \{0, 1\}$ can be replaced by constraints $x_{ij} \geq 0, \ \forall i, j$.
- Several variants can also be considered.

Applications

- Assign people to tasks;
- Production planning (operations to machines; products to plants)

Exercise

A department has opened three vacancies for translators:
Vacancy 1: Portuguese/French;
Vacancy 2: Portuguese/German;
Vacancy 3: Portuguese/Greek.
Four candidates applied and in the selection tests they achieved the following grades (in scale from a minimum of zero to a maximum of ten):

Candidate	Portuguese/French	Portuguese/German	Portuguese/Greek
A	8.5	7.0	6.0
B	7.5	8.0	6.5
C	6.0	7.5	8.5
D	7.0	6.5	8.0

Determine the assignment that provides the best service quality.