Two-Sided Matching

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Exercise

There are 3 firms f_1, f_2, f_3 and 4 students i_1, i_2, i_3, i_4. The preferences are as follows:

R_{i_1} : f_1, f_2, f_3
R_{i_2} : f_1, f_2, f_3
R_{i_3} : f_2, f_3, f_1
R_{i_4} : f_3, f_2, f_1
Rf_1 : i_4, i_3, i_2, i_1
Rf_2 : i_3, i_2, i_1, i_4
Rf_3 : i_1, i_2, i_3, i_4

1. Compute μ^F and μ^I.
2. Are μ^F and μ^I Pareto efficient? What if only the welfare of workers is considered?
3. Can you think of a sufficient condition for $\mu^F = \mu^I$?

A college admissions problem (Gale and Shapley, 1962) is a four-tuple (C, I, q, R) where C is a finite set of colleges, I is finite a set of students, $q = (q_c)_{c \in C}$ is a vector of college capacities, and $R = (P_l)_{l \in C \cup I}$ is a list of preferences.

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Here R_i denotes the preferences of student i over $C \cup \{\emptyset\}$, R_c denotes the preferences of college c over 2^I , and P_c, P_i denote strict preferences derived from R_c, R_i.

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Formally, college preferences R_c are responsive iff

- for any $J \subset I$ with $|J| < q_c$ and any $i \in I \setminus J$, $(J \cup \{i\}) P_c J$ if and only if $\{i\} P_c \emptyset$,

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- for any $J \subset I$ with $|J| < q_c$ and any $i, j \in I \setminus J$, $(J \cup \{i\}) P_c (J \cup \{j\})$ if and only if $\{i\} P_c \{j\}$.

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A matching for college admissions is a correspondence $\mu : C \cup I \to 2^{C \cup I}$ such that:

- $\mu(c) \subseteq I$ such that $|\mu(c)| \leq q_c$ for all $c \in C$,
- $\mu(i) \subseteq C$ such that $|\mu(i)| \leq 1$ for all $i \in I$, and
- $i \in \mu(c)$ if and only if $\mu(i) = \{c\}$ for all $c \in C$ and $i \in I$.

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A matching μ is blocked by a pair $(c, i) \in C \times I$ if

- $c P_i \mu(i)$, and
- (a) either there exists $j \in \mu(c)$ such that $\{i\} P_c \{j\}$, or (b) $|\mu(c)| < q_c$ and $\{i\} P_c \emptyset$.

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  Observe that this version of blocking by a pair is plausible only under responsiveness. A matching is stable if it is not blocked by any agent or pair.

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College-Proposing Deferred Acceptance Algorithm

Step 1 Each college c proposes to its top q_c acceptable students (and if it has less acceptable choices than q_c, then it proposes to all its acceptable students). Each student rejects any unacceptable proposals and, if more than one acceptable proposal is received, she "holds" the most-preferred and rejects the rest.
In general, at

Step k Any college c who was rejected at step $k-1$ by any student proposes to its most-preferred q_c acceptable students who have not yet rejected it (and if among the remaining students there are fewer than q_c acceptable students, then it proposes to all). Each students "holds" her most-preferred acceptable offer to date and rejects the rest. The algorithm terminates when there are no more rejections. Each student is matched with the college she has been holding in the last step.

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Many results for marriage problems extend to college admissions problems.

The following "trick" is very useful to extend some of these. Given a college admissions problem (C, I, q, R), construct a related marriage problem as follows:

First "Divide" each college c_l into q_{c_l} separate pieces $c_l^1, ..., c_l^{q_{c_l}}$ where each piece has a capacity of one; and let each piece have the same preferences over I as college c has. (Since college preferences are responsive, R_c is consistent with a unique ranking of students.)

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C^* : The resulting set of college "pieces" (or seats).

Second For any student i, extend her preferences to C^* by replacing each college c_l in her original preferences R_i with the block $c_l^1, ..., c_l^{q_c}$ in that order.

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So, in the related marriage problem, each seat of a college c is an individual unit that has the same preferences with college c, and students rank seats at different colleges as they rank the colleges whereas they rank seats at the same college based on the index of the seat.

Given a matching for a college admissions problem, it is straightforward to define a corresponding matching for its related marriage problem: Given any college *c*, assign the students who were assigned to c in the original problem one at a time to pieces of c starting with lower index pieces.

Lemma (Roth and Sotomayor 1989)

A matching of a college admissions problem is stable if and only if the corresponding matching of its related marriage problem is stable.

This lemma can be used to extend the following results for marriage problems to college admissions:

1. There exists a student-optimal stable matching μ^I that every student likes at least as well as any other stable matching. Furthermore, the outcome of the student-proposing deferred acceptance algorithm yields the student-optimal stable matching.

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4. The set of students filled and the set of positions filled is the same at each stable matching.
5. The join as well as the meet of two stable matchings is each a stable matching.
6. There is no individually rational matching μ' where $\mu'(i) P_i \mu^I(i)$ for all $i \in I$.

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Theorem (Theorem 1 in Roth 1986)

Any college that does not fill all its positions at some stable matching is assigned precisely the same set of students at every stable matching.

Theorem (Theorem 4 in Roth and Sotomayor 1989)

Let μ and μ' be two stable matchings. For any college c,

- *either $\{i\}P_c\{j\}$ for all $i \in \mu(c) \setminus \mu'(c)$ and $j \in \mu'(c) \setminus \mu(c)$, or*

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There is an implicit competition between different seats of a college in the sense that a college may lose a student in one of its seats in order to get a more-preferred student for another seat.

Example

There are 2 colleges c_1, c_2 with $q_{c_1} = 2, q_{c_2} = 1$, and 2 students i_1, i_2.
The preferences are as follows:

$R_{i_1} : \{c_1\}\{c_2\}\emptyset$

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Here both colleges strictly prefer μ to μ^C where:

$\mu = \{(c_1, i_2), (c_2, i_1)\}$

$\mu^C = \{(c_1, i_1), (c_2, i_2)\}$

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Theorem (Theorem 3 in Roth 1982b)

There exists no mechanism that is stable and strategy-proof.

Theorem (Proposition 1 in Alcalde and Barbera 1994)

There exists no mechanism that is Pareto efficient, individually rational, and strategy-proof.

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For colleges, however, the situation is different. The following example makes this point.

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If c_1 submits the manipulated preferences R'_{c_1} where only student i_2 is acceptable, for problem (R_{-c_1}, R'_{c_1}) the only stable matching is $\mu^C = \{(c_1, i_2), (c_2, i_1)\}$.

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Hence college c_1 benefits by manipulating its preferences under any stable mechanism (including the college-optimal stable mechanism).

Theorem (Theorem 4 in Roth 1985)

There exists no stable mechanism where truth-telling is a weakly dominant strategy for all colleges.

Capacity Manipulation

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A college c manipulates mechanism ϕ via capacities at problem (R, q) if

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A mechanism is immune to manipulation via capacities if it can never be manipulated via capacities.

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Let $q'_{c_1} = 1$ be a potential capacity manipulation by college c_1. Then:

$$\phi^C[R, q] = \{(c_1, i_1), (c_2, i_2)\},$$
$$\phi^C[R, q'_{c_1}, q_{c_2}] = \{(c_1, i_2), (c_2, i_1)\}.$$

Hence college c_1 benefits by reducing the number of its positions under ϕ^C.

Theorem (Theorem 1 in Sonmez 1997)

Suppose there are at least 2 colleges and 3 students. Then there exists no stable mechanism that is immune to manipulation via capacities.

If, however, colleges prefer larger groups of students to smaller groups of students, then a positive result is obtained. College preferences are strongly monotonic if

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Theorem (Theorem 5 in Konishi and Unver 2006)

Suppose college preferences are strongly monotonic. Then the student-optimal stable mechanism is immune to manipulation via capacities.

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This modeling choice is quite realistic for a vast majority of practical applications, although there are exceptions. For example in New York City High School Match, some schools (potentially) strategically rank students in preference order as part of the admissions process.

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In a school choice problem, schools are simply objects to be allocated and only student welfare matters.

Consequently, stability, a key notion in two-sided matching, does not imply efficiency. Stability is no longer incompatible with strategy-proofness.

A school choice problem (Abdulkadiroglu and Sonmez, 2003) is a five-tuple (I, S, q, R_S, P_I) where:

- I is a finite set of students,
- S is a finite set of schools,
- $q = (q_s)_{s \in S}$ is a capacity profile for schools where q_s is the number of available seats at school $s \in S$,
- $R_S = (R_s)_{s \in S}$ is a profile of weak priority relations for schools where R_s is a complete, reflexive and transitive binary relation over $I \cup \emptyset$ for school $s \in S$, and
- $P_I = (P_i)_{i \in I}$ is a profile of strict preference relations for students where P_i is a complete, irreflexive, and transitive binary relation over $S \cup \{\emptyset\}$ for student $i \in I$.

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For $i \in I$, let R_i be the symmetric extension of P_i . That is, for all $s, j \in S \cup \{\emptyset\}$, if sR_ij then sP_ij or $s = j$.

A matching of students to schools is a function $\mu : I \cup S \to 2^{I \cup S}$ such that

1. $\mu(i) \subset S$ with $|\mu(i)| \leq 1$ for all $i \in I$; and
2. $\mu(s) \subset I$ with $|\mu(s)| \leq q_s$ for all $s \in S$; and
3. $s \in \mu(i)$ if and only if $i \in \mu(s)$ for all $i \in I$ and $s \in S$.

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A stable matching is student-optimal if it is not Pareto dominated by another stable matching.

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The so-called Boston mechanism, (Abdulkadiroglu and Sonmez, 2003) and was in use in Boston until 2005, has been adopted widely by school districts in the U.S..

The Boston Mechanism

1. For each school, a priority ordering is exogenously determined. (In the case of Boston, priorities depend on home address, whether student has a sibling already attending a school, and a lottery number to break ties).
2. Each student submits a preference ranking of the schools.
3. Student assignment based on preferences and priorities:

Round 1 For each school, consider the students who have listed it as their first choice and assign seats of the school to these students one at a time following their priority order until either there are no seats left or there is no student left who has listed it as her first choice. In general at:

Round k Consider the remaining students. For each school with available seats, consider the students who have listed it as their kth choice and assign the remaining seats to these students one at a time following their priority order until either there are no seats left or there is no student left who has listed it as her kth choice.

At each round, every assignment is final and the algorithm terminates when no more students are assigned.

Example

There are three students $\{i_1, i_2, i_3\}$ and three schools $\{s_1, s_2, s_3\}$ each with one seat. Preferences and school priorities are as follows:

$R_{i_1} : \{s_2\}\{s_1\}\{s_3\}$

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If i_2 reports her preferences as $s_2 P'_{i_2} s_1 P'_{i_2} s_3$ instead, the Boston mechanism produces : $\mu = \{(i_1, s_3), (i_2, s_2), (i_3, s_1)\}$ and student i_2 benefits from submitting a false preference list.

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Another one inspired by Gale's top trading cycles algorithm, which produces an efficient matching - Top Trading Cycles mechanism (TTC).

Theorem (Theorems 1, 2 in Gale and Shapley 1962)

Given (P_I, P_S)*, SOSM produces a matching that is stable at* (P_I, P_S)*, which is also at least as good for every student as any other stable matching at* (P_I, P_S)*.*

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Theorem (Theorem 9 in Dubins and Freedman 1981, Theorem 5 Roth 1982)

Given fixed priorities P_S, SOSM is strategy-proof.

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Abdulkadiroglu and Sonmez (2003) proposed the Top Trading Cycles, a Pareto efficient mechanism.

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At each round there exists at least one cycle where a cycle of students and schools $(i_1, s_1, ..., i_K, s_K)$ is such that each element of the sequence points to the next whereas the last element s_K points to the first element i_1.

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Although TTC is Pareto efficient and SOSM is not, the two are not Pareto ranked in general, as can be seen in the next example.

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The outcomes of the mechanisms are $\mu^{SOSM} = \{(i_1, s_2), (i_2, s_3), (i_3, s_1)\}$ and $\mu^{TTC} = \{(i_1, s_2), (i_2, s_1), (i_3, s_3)\}$.

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Neither matching dominates the other.

Despite the lack of a Pareto ranking between SOSM and TTC, there exists a clear-cut comparison between dominant strategy equilibria of SOSM and Nash of the Boston mechanism.

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Theorem (Theorem 1 in Ergin and Sonmez 2006)

Given strict school priorities, the set of Nash equilibrium outcomes of the Boston mechanism is equal to the set of stable matchings under true preferences. Therefore, the dominant strategy equilibrium of SOSM weakly Pareto dominates every Nash equilibrium outcome of the Boston mechanism.