1) There are only two firms in a market. The demand curve is $Y = 20 - p$. The firms' cost functions are $c_1(y_1) = 0.25y_1^2$ and $c_2(y_2) = 0.2y_2^2$.
   a) Assume the firms set output simultaneously. Find the reaction functions and draw them in a single diagram with y_1 on one axis and y_2 on the other.
   b) Find the Cournot equilibrium quantities, price, and profits.
   c) Assume the following. Firms adjust output levels, if necessary, every month. Managers in both firms observe the rival's previous output and, being a bit dumb, assume it is never going to change, and set their own current output accordingly. Initially firm 1 is alone in the market, and then firm 2 enters. How do the firms' outputs change over time? Show the pairs of outputs in the first couple of months in the graph you drew in part a).
   d) Now the firms form a cartel to maximise total profit. Find the optimal outputs, price and profits. Compare with the values you obtained in part b). Do they change as you would expect?
   e) What is the marginal cost in each firm in the cartel solution? Are they equal? Why?
   f) What is the cartel's marginal revenue? How does it compare with the firms' marginal costs?
   g) Suppose firm 1 believes it can change its output without firm 2 noticing or reacting. How much would it produce? What would happen to price and the firms' profits?
   h) What would happen if firm 2, rather than firm 1, believed it could change its output without firm 1 reacting? What if both firms behaved in this manner?
   i) Is it plausible that one firm succeeded in changing its output without the other noticing in the present case? In what conditions would it be more plausible?

2) Stores often offer to match or beat the price offered by a rival store (or to pay back costumers the price difference if after the purchase they find the article at a lower price elsewhere). Explain why firms may want to make this offer.

3) In a duopolistic market with demand curve $p = 66 - 2Y$, the firms' cost functions are $c_1(y_1) = 4y_1$ and $c_2(y_2) = y_2^2$.
   a) Find the Cournot equilibrium quantities, price, and profits.
   b) Suppose the demand curve shifts. Would firm 2 always produce less than firm 1 in the Cournot equilibrium regardless of how much and in which direction the demand shifted? Why?
   c) Now the firms join efforts in a cartel. Without calculating the cartel's optimal quantities and price, find the values of the cartel's marginal revenue and the firms' marginal costs at the cartel solution. Could you do this so easily in any type of cost function, for instance those of exercise 1?
   d) Find the cartel's optimal quantities, price, and profits.
   e) Would both firms be happy with the outcome? What arrangements would be necessary for both firms to be willing to cooperate?

4) Two duopolists face the demand curve $Y = 24 - p$ and their cost functions are $c_1(y_1) = 5y_1$ and $c_2(y_2) = 4y_2$.
   a) Is the product homogeneous or differentiated?
   b) The two firms set their quantities simultaneously and independently. Find the equilibrium price, quantities and profit.
   c) Now the firms form a cartel. Find the price, quantities, and profits.
   d) Will both firms be happy with the outcome? What arrangements between the two would be necessary so that both firms agreed to cooperate?

5) A duopoly faces the inverse demand curve $p = 21 - Y$, and both firms have a constant marginal cost equal to 3 and no fixed costs.
   a) The firms simultaneously and independently set their outputs. Can you determine which firm will produce more and which will earn a bigger profit without calculating the equilibrium outputs? Explain.
   b) Find the Cournot equilibrium quantities, price, and profits.
   c) Now the firms join in a cartel. Can you find the cartel's optimal total quantity and price? What about each firm's output? And total profit? Explain.
   d) The firms agree to allocate output so that both will benefit the same from forming the cartel. How much will each produce?

6) In an industry with inverse demand curve $p = 50 - Y$ there are four firms, each of which has a constant marginal cost given by $MC = 10$. If the firms form a profit-maximising cartel and agree to operate subject to the constraint that each firm will produce the same output level, how much does each firm produce?

7) (Adapted from Perloff 15.3.9, p. 547) Consider the Cournot model with n identical firms. The inverse demand curve is $p = a - bY$. Each firm has the same cost function $c(y_i) = Ay_i + 0.5By_i^2$, where $a > A$. In terms of n, what is each firm firm's Nash equilibrium output and price? As n gets very large (approaches infinity) what happens to price and each firm's output and profit? Why?

8) (Adapted from Perloff 15.3.5, p. 547) In a Nash-Cournot model, each of the n firms face constant marginal cost m, the inverse demand curve is $p = a - bY$, and the government assesses a specific tax t per unit.
   a) What is the incidence of this tax on consumers?
   b) How does the incidence vary as the number of firms increase? Does this seem paradoxical result? Explain.

9) Firm 1 and firm 2 produce a homogeneous good, have positive marginal costs, and are maximising joint profit.
   a) What happens to the cartel's revenue and to each of the firms' revenue when firm 1, in breach of the

agreement, slightly increases its output (i.e what is $\partial(r_1 + r_2)/\partial y_1$, $\partial r_1/\partial y_1$ and $\partial r_2/\partial y_1$). Show mathematically and explain in words.

b) Using the results you found in part a) show that a cartel member has an incentive to increase output, in breach of the agreement, if it believes the other members will not retaliate. (This generalises the results you found in the particular case of exercise 1, parts g) and h).)

10) Two duopolists face the inverse demand curve $p = 24 - Q$, and their cost curves are $C_1(Q_1) = 6Q_1$ and $C_2(Q_2) = 0.5Q_2^2$.
a) Find the Cournot equilibrium outputs, price and profits.
b) Now firm 1 sets its output first, and firm 2 sets its own knowing the firm 1's decision. What are the equilibrium quantities, price, and profits?
c) What if firm 2 sets its quantity first?
d) Will the leader in the Stackelberg model always (i.e. with any demand curve and cost curves) earn a higher profit than it would in the Cournot equilibrium? Explain.
e) Explain in words why the Stackelberg leader produces more than it would in the Cournot equilibrium, and why the follower produces less.
f) The firms now consider forming a cartel. Without calculations can you tell whether it would be in the interest of a Stackelberg leader to enter into such a deal?
g) Suppose the firms form the cartel. Find the optimal quantities, price, and profit.
h) Would some redistribution of profits be necessary?
i) Without calculations, how would the firms' output levels change if demand expanded? What if demand contracted?

11) In a duopoly the inverse demand curve is é $p = 66 - 2Q$, and the cost curves are $C_1(Q_1) = 4Q_1$ e $C_2(Q_2) = Q_2^2$.
a) Find the Cournot equilibrium quantities, price, and profit.
b) Find the Stackelberg equilibrium where firm 1 is the leader.
c) Find the Stackelberg equilibrium where firm 2 is the leader.
d) Find the cartel's optimal solution.
e) Find the Bertrand equilibrium quantities, price, and profits. Does the firms with the largest output earn the bigger profit? Why or why not?

12) In a duopoly the inverse demand curve is é $p = 33 - Q$, and the cost curves are $C_1(Q_1) = 6Q_1$ e $C_2(Q_2) = 3Q_2$.
a) Find the Cournot equilibrium quantities, price, and profit.
b) Find the Stackelberg equilibrium where firm 1 is the leader.
c) Find the Stackelberg equilibrium where firm 2 is the leader.
d) Find the cartel's optimal solution.
e) Find the Bertrand equilibrium quantities, price, and profits.

13) (Perloff 14.4.2, p. 548) Determine the Stackelberg equilibrium with one leader firm and two follower firms if the market demand curve is linear and each firm faces a constant marginal cost m and no fixed costs. (Hint: to facilitate comparison with the answer provided below make the demand curve $p = a - bQ$ and let the leader be firm 1.)

14) Two duopolists face the demand curve $Q = 30 - p$, and both firms have constant marginal cost equal to €20.01 and no fixed costs. The good is homogeneous, and firms set prices simultaneously and independently.
a) Find the equilibrium prices, quantities, and profits.
b) How does the equilibrium change if firm 1 manages to lower its marginal cost to €17?
c) Suppose firm 1 manages to lower its marginal cost to €6. How much profit will the firm earn if it charges a price of €20? And if it charges a price of €19? Find the market equilibrium.

Multiple-Choice Questions

1) A two-firm cartel faces the demand curve $p = 10 - 0.5y$. The firms marginal costs are $MC_1(y_1) = y_1$ and $MC_2(y_2)=4$. What quantities maximise the cartel's profit?
a) $y_1 = 4$ and $y_2 = 4$.
b) $y_1 = 4$ and $y_2 = 2$.
c) $y_1 = 3$ and $y_2 = 3$.
d) The information is not enough to answer.

2) In a Bertrand model with two firms, both firms have constant marginal costs, but firm A's marginal cost is lower than B's. How much is the equilibrium price?
a) It equals firm A's marginal cost.
b) It is the same as firm A would charge if it were a monopoly.
c) It is the same as if firm A were the leader in a Stackelberg model.
d) None of the other options is correct.

3) In a Stackelberg model which firm maximises its profit given the other's output?
a) Both.
b) None.
c) The follower only.
d) The leader only.

4) The follower in a Stackelberg model:
a) Produces the output level that suits the other firm the best.
b) Produces where its marginal cost equals the price.
c) Maximises its profit given the other firm's output.
d) None of the other options is correct.

5) In a Stackelberg model the leader produces
   a) Twice as much as the follower.
   b) More than the follower, not necessarily twice as much.
   c) Less than the follower.
   d) None of the other options is correct.

6) Two firms in Bertrand model have identical marginal cost curves. The equilibrium price and quantity will be the same as in:
   a) The Cournot model.
   b) A cartel.
   c) The Stackelberg model.
   d) Perfect competition.

7) Cartel agreements tend to be violated because each firm can increase its own profits by:
   a) Producing more than the quantity that maximises the cartel's joint profit.
   b) Producing less than the quantity that maximises the cartel's joint profit.
   c) Charging a higher price than that that maximises the cartel's joint profit.
   d) None of the other options is correct.

8) Which of the following is true in a Cournot model?
   a) The firms set output levels.
   b) The firms decide simultaneously.
   c) The firms do not cooperate.
   d) All are true.

9) Two firms in a Stackelberg model have identical cost functions. Which firm will produce the most?
   a) The leader.
   b) The follower.
   c) Both will produce the same.
   d) The information is not enough to answer.

10) In a Bertrand model two firms have constant marginal costs equal to 30 and face the demand curve $p = 90 - y$. How much is the price and quantities sold?
   a) $p = 60$, $y_1 = 20$, $y_2 = 10$.
   b) $p = 30$, $y_1 = 30$, $y_2 = 30$.
   c) $p = 30$, $y_1 = 60$, $y_2 = 0$.
   d) The information is not enough to answer.

Answers

1.a) $f_1(y_2) = 8 - 0.4y_2$, $f_2(y_1) = (100 - 5y_1)/12$. See similar diagram in Varian.

1.b) $y_1 = 5.6$, $y_2 = 6$, $P = 8.4$, $\pi_1 = 39.2$, $\pi_2 = 43.2$.

1.c) Use the best response functions. If firm 1 is alone, $y_2 = 0$, and $f_1(0) = 8$; firm 2 observes $y_1 = 8$, and produces $f_2(8) = 5$, then firm 1 adjust to $f_1(5) = 6$, and so on. The values converge to the Cournot equilibrium quantities. See similar diagram in Varian.

1.d) $y_1 = 4$, $y_2 = 5$, $p = 11$, $\pi_1 = 40$, $\pi_2 = 50$. As you should expect, total output falls, price and total profit rises.

1.e) $MC_1(4) = MC_2(5) = 2$. The profit-maximising solution requires that all firms (actually producing) have the same marginal cost. Otherwise the same total output could be produced at a lower total cost by transferring production from the high marginal-cost firms to the low marginal-cost firms.

1.f) $MR = 20 - 2Q$. With $Y = 9$, $MR = 2$, the same as marginal cost. If $MR > MC$ the cartel could increase profit by producing more; if $MR < MC$ it could increase profit producing less. Note that the cartel's marginal revenue is the change in total revenue, sum of the firms' revenues, when one more unit is sold. It depends on the total output, not on an individual firm's output.

1.g) $f_1(5) = 6$, $p = 9$, $\pi_1 = 45$, $\pi_2 = 40$. Firm 1 increases its profit, firm 2 sees its profit fall because of lower price; total profit falls.

1.h) $f_1(4) = 6.667$, $P = 9.333$, $\pi_1 = 33.333$, $\pi_2 = 53.333$. Probably they would go back to the Cournot equilibrium.

1.i) With only two firms and no uncertainty regarding demand, if one firm produced its cartel quota and saw the price fall it would know the other had increased output. If demand fluctuates unpredictably the firm might not know whether the other had increased production or demand had contracted. If there were non-cartel members in the market prices could also drop if they, rather than one firm in the cartel increased output.

2.a) See Varian, sections 27.1 and 27.11.

3.a) $y_1 = 12$, $y_2 = 7$, $p = 28$, $\pi_1 = 288$, $\pi_2 = 147$.

3.b) No. If demand contracts sufficiently firm 2 will produce more than firm 1. Can you explain why?

3.c) They are all 4. We could do this because firm 1's marginal cost is 4 no matter how much it produces. Of course if it did not produce at all (if demand contract sufficiently) we could no longer find the cartel's marginal revenue and firm 2's marginal cost in this way.

3.d) $y_1 = 13.5$, $y_2 = 2$, $p = 35$, without profit redistribution it is $\pi_1 = 418.5$, $\pi_2 = 66$.

3.e) Firm 2 would not cooperate in the cartel unless it earned at least as much profit as in the alternative situation. Hence firm 1 would have to transfer at least 81 of its profits to firm 2. Firm 1 could transfer up to 130.5 without being worse off. The transfer would plausibly be somewhere between 81 and 130.5.

4.a) Homogeneous: there is a single price.

4.b) $y_1 = 6$, $y_2 = 7$, $p = 11$, $\pi_1 = 36$, $\pi_2 = 49$.

4.c) $y_1 = 10$, $y_2 = 0$, $p = 14$, without profit redistribution profits are $\pi_1 = 0$, $\pi_2 = 100$.

4.d) Firm 2 would have to transfer anything between 36 and 51 to firm 1 for both firm to be willing to cooperate.

5.a) The firms are identical: they face the same demand curve and have the same costs. So they will produce the same.

5.b) $y_1 = y_2 = 6$, $p = 9$, $\pi_1 = \pi_2 = 36$.

5.c) $y_1 + y_2 = 9$, $p = 11$, $\pi_1 + \pi_2 = 81$. Individual productions cannot be determined. Both firms have the same constant marginal cost, therefore total cost, and hence total profit, is the same no matter how you split total output between firms.

5.d) $y_1 = y_2 = 4.5$.

6) Each firm produces $y_i = 5$.

7) Write the profit function for firm i and differentiate it with respect to y_i to find the first order condition for profit maximisation for firm i. Then note that, as all firms are identical, each firm will in equilibrium produce the same: $y_1 = y_2 = \ldots = y_n$. From that you get: $y_i=(a - A)/[b(n + 1) + B]$, $p = (Ba + ba + nbA)/[b(n + 1) + B]$. As n tends to infinity y_i tends to zero and p tend to A, which is the limit of the marginal cost when y_i tends to zero.

8.a) From the previous exercise you get (note that you can just substitute m for A and zero for B) $y_i=(a - m)/[b(n + 1)$. All firms produce the same, so $Y = ny_i = (a - m)n/[b(n+1)]$. The the price is $p = a - (a - m)n/(n+1)$. The tax simply increases marginal cost from m to $m + t$. So the effect of the tax on price is $dp/dt = n/(n+1)$ (per euro of tax).

8.b) The incidence of tax on consumers is the higher the more firms there are in the market ($n/(n+1)$ approaches 1 when n tends to infinity). As the number of firms increases the market tends to perfect competition, i.e., the price converges towards the marginal cost. Why does the tax incidence on consumers increase as the market becomes closer to perfect competition?

9.a) $r_i = p(y_1,y_2)y_i$, $i = 1, 2$. $\partial r_2/\partial y_1 = y_2\partial p(y_1,y_2)/\partial y_1 < 0$ because $\partial p(y_1,y_2)/\partial y_1 < 0$. That is, when firm 1 increases output, the price falls, and firm 2's revenue falls as well (because its output remained the same). For firm 1, $MR_1 = \partial r_1/\partial y_1 = y_1\partial p(y_1,y_2)/\partial y_1 + p(y_1,y_2)$; in general this can be negative or positive (the price falls but the firm is selling more, so revenue may rise or fall). But at the profit-maximising quantities the cartel's marginal revenue equals marginal cost, and is therefore positive. As firm 2's revenue falls when firm 1 increases output, firm 1's marginal revenue must be positive and larger than the marginal cost: the cartel's marginal revenue is $\partial(r_1 + r_2)/\partial y_i$. At the joint-maximising quantities $\partial(r_1 + r_2)/\partial y_1 = MC \Leftrightarrow MR_1 = MC - \partial r_2/\partial y_1 > MC > 0$ (because $\partial r_2/\partial y_1 = < 0$).

9.b) It follows immediately from the previous result: at the cartel's profit-maximising quantities firm 1's marginal revenue exceeds its marginal cost, so it will increase profit if it increases output and the other firm keeps its output constant.

10.a) $y_1 = 6$, $y_2 = 6$, $p = 12$, $\pi_1 = 36$, $\pi_1 = 54$.

10.b) $y_1 = 7.5$, $y_2 = 5.5$, $p = 11$, $\pi_1 = 37.5$, $\pi_1 = 45.375$.

10.c) $y_1 = 5.25$, $y_2 = 7.5$, $p = 11.25$, $\pi_1 = 27.5625$, $\pi_1 = 56.25$.

10.d) We can guarantee it will never make a lower profit: the leader could choose to produce its Cournot equilibrium output, the follower uses its best-response function, so it would produce its Cournot equilibrium output too, and the leader (and the follower) would earn the same profit as in the Cournot equilibrium. So it would never settle for less, and typically (except in the case of funnily-shaped marginal curves, say if the leader's marginal cost curve becomes vertical at its Cournot equilibrium output) it will be able to do better.

10.e) The Cournot equilibrium quantity is the firm's optimal quantity assuming the rival will not change its own output, meaning that its marginal revenue equals its marginal cost. But the Stackelberg leader, by producing a little above the Cournot equilibrium, leads the follower to produce less. So the price does not fall as much as it would in the Cournot equilibrium, which means the leader's marginal revenue will be a little higher, so exceeding the marginal cost, making it profitable to expand output.

10.f) It would (again except in case of funny curves). The cartel could choose to keep the production plans of the Stackelberg equilibrium, guaranteeing the same profits. Typically it can do better, generating some extra profit that can be split between the firms, making it beneficial for them to form the cartel.

10.g) $y_1 = 3$, $y_2 = 6$, $p = 15$, $\pi_1 = 27$, $\pi_1 = 72$.

10.h) Yes, from firm 2 to firm 1, otherwise it would make less profit than in the previous situation.

10.i) Both firms are producing with a marginal cost of 6, the constant marginal cost of firm 1. Firm 1 can expand production as much as it likes at the same marginal cost, whereas firm 2 would incur higher marginal cost if it expanded production. So firm 2 would keep producing 6, and any extra output would come from firm 1. If demanded contracted, firm 1 only would lower output.

11.a) $y_1 = 12$, $y_2 = 7$, $p = 28$, $\pi_1 = 288$, $\pi_2 = 147$.

11.b) $y_1 = 15$, $y_2 = 6$, $p = 26.25$, $\pi_1 = 247.53125$, $\pi_2 = 153.125$.

11.c) $y_1 = 11.125$, $y_2 = 8.75$, $p = 24$, $\pi_1 = 300$, $\pi_2 = 108$.

11.d) $y_1 = 13.5$, $y_2 = 2$, $p = 35$, $\pi_1 = 418{,}5$, $\pi_2 = 66$.

11.e) $y_1 = 29$, $y_2 = 2$, $p = 4$, $\pi_1 = 0$, $\pi_2 = 4$. No. Firm 1 has constant marginal cost, so average cost equals marginal cost and price, whereas firm 2 produces its first units at a very low cost.

12.a) $y_1 = 8$, $y_2 = 11$, $p = 14$, $\pi_1 = 64$, $\pi_2 = 121$.

12.b) $y_1 = 12$, $y_2 = 9$, $p = 12$, $\pi_1 = 72$, $\pi_2 = 81$.

12.c) $y_1 = 5.25$, $y_2 = 16.5$, $p = 11.25$, $\pi_1 = 27.563$, $\pi_2 = 136.125$.

12.d) $y_1 = 0$, $y_2 = 15$, $p = 18$, $\pi_1 = 0$, $\pi_2 = 225$. Some redistribution of profit would be necessary, so that firm 1 would end up with at least as much profit as in the alternative situation

12.e) Firm 2 will charges a price just below 6, firm 1's marginal cost, and firm 1 shuts down. Therefore $y_1 = 0$ and $\pi_1 = 0$. Suppose there can be no fractions of cents,

so $p = 5.99$, i. e. the highest price below 6. $y_2 = 27.01$ and $\pi_2 = 80.7599$.

13) First solve for the intersection of the followers' reaction functions, with y_1 taken as a parameter to get $y_2 = y_3 = (a - m - by_1)/(3b)$. Substitute these expressions into the leader's profit function and maximise it to get $y_1 = (a - m)/(2b)$. Substitute this into the previous expressions to get $y_2 = y_3 = (a - m)/(6b)$.

14.a) $p = 20.01$, $Y = 9.99$, by convention assumed split equally between the firms. $\pi_1 = \pi_2 = 0$.

14.b) Firm 1 charges a price of $p = 20$ and produces 10 with a profit of 30. Firm cannot compete with a price of 20 and produces nothing.

14.c) If $p = 20$, $y_1 = 10$ and $\pi_1 = 140$. If $p = 19$, $\pi_1 = 143$. Firm 1 maximises profit with $p = 18$, $y_1 = 12$ and $\pi_1 = 144$. Even though it could rise the price up to 20 without risking competition from firm 2, it would gain nothing from doing so.

Answers to multiple-choice questions

1b 2d 3c 4c 5d 6d 7a 8d 9a 10b.

Exercises adapted from Perloff can be found in:

Perloff, Jeffrey M., *Microeconomics with Calculus*, 3rd edition, Pearson 2014.