Lecture 8: Brownian Motion

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Continuous time

- As we assume continuous trading: need to consider continuous time, instead of discrete time
- **Diffusion models** are a standard way to represent random variables in continuous time
- The ideas are analogous to discrete-time stochastic processes
- The basic building block of a diffusion model is a **Brownian motion** (or **Wiener process**), which is a real-valued continuous-time stochastic process

Brownian motion

- Brownian motion is the **random movement** of microscopic particles suspended in a fluid, caused by constant collisions with the fluid molecules.
- It is a classic example of a continuous random walk.
- **Robert Brown** (1827): A Scottish botanist first observed this phenomenon while studying pollen grains in water under a microscope. He noted their erratic movement but could not explain why it happened
- **Albert Einstein** (1905): Provided a theoretical explanation, proving that Brownian motion was due to the random collisions of molecules in a fluid
- **Norbert Wiener** (1923): Developed the mathematical theory of continuous random walks, leading to the Wiener process, a key part of modern stochastic processes

Applications

- Physics and Chemistry:
  - Modeling Particle Motion: Brownian motion provides a model for understanding the random motion of small particles suspended in a fluid, like pollen grains in water or dust particles in air
  - Diffusion: It's crucial for understanding diffusion processes, where particles move from areas of high concentration to low concentration
  - Micromanipulation of DNA: Brownian motion is used in techniques to manipulate DNA molecules
- Finance:
  - Stock Market Modeling: Geometric Brownian motion, a variation of Brownian motion, is used to model the fluctuations of stock prices and other financial assets
  - Options Pricing: The assumption that asset prices follow Brownian motion is essential to options pricing models
- Biology and Medicine: Movement of bacteria, cellular transport
- Computer Science: Randomized algorithms, Monte Carlo methods

Brownian motion

- A **Brownian motion** is the natural generalization of a **random walk** in discrete time
- Can think of a **random walk** as modelling a person's erratic path when intoxicated in discrete time:

$$z_t - z_{t-1} = \varepsilon_t$$

$$\varepsilon_t \sim N(0,1), \;\; E(\varepsilon_t \varepsilon_s) = 0, \;\; s \neq t$$

- A **Brownian motion** z_t:

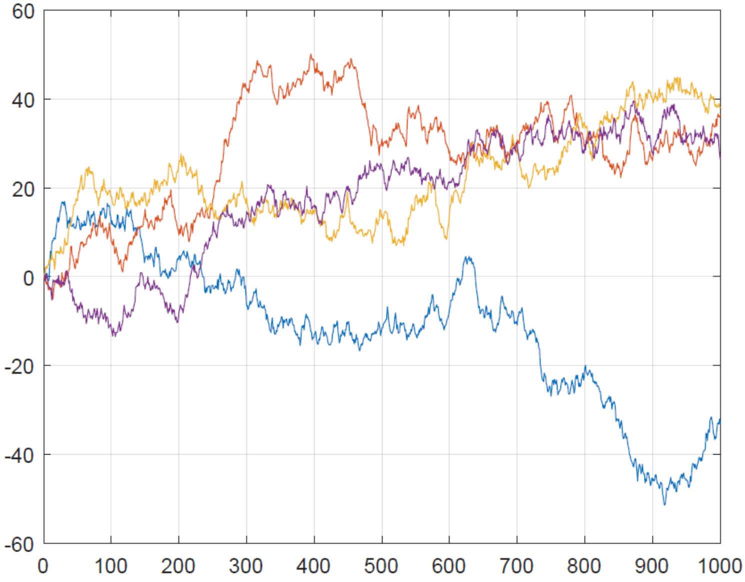
$$z_{t+\Delta} - z_t \sim N(0, \Delta)$$

  i.e. mean zero and variance Δ

As $E(\varepsilon_t \varepsilon_s) = 0$ in discrete time, increments to z for nonoverlapping intervals are also independent

$$cov(z_{t+\Delta} - z_t, z_{s+\Delta} - z_s) = 0$$

Example Brownian motion



Brownian motion

$$dz_t \equiv z_{t+dt} - z_t \sim N(0, dt)$$

- That is, the change in z_t over a small time interval dt, follows a normal distribution with:
  - Mean: 0
  - Variance: dt
  - Independent increments: The increments dz_t over non-overlapping time intervals are independent.

Brownian motion

- The variance of a random walk scales with time

$$var\left(z_{t+k} - z_t\right) = var\left(\varepsilon_{t+1} + ... + \varepsilon_{t+k}\right) = kvar\left(z_{t+1} - z_t\right)$$

- And the variance of a Brownian motion scales with time too

$$var(z_{t+k\Delta} - z_t) = kvar(z_{t+\Delta} - z_t)$$

- The standard deviation is the "typical size" of a movement in a normally distributed random variable
- The "typical size" of $z_{t+\Delta} - z_t$ in time interval Δ is $\sqrt[2]{\Delta}$
- This means that $\dfrac{z_{t+\Delta} - z_t}{\Delta}$ has "typical size" $1/\sqrt[2]{\Delta}$
- Thus, the sample path of z_t is continuous but is not differentiable: moves infinitely fast (up and down)

Brownian motion

- **Definition:** Differential dz_t is the forward difference

$$dz_t = \lim_{\Delta \searrow 0} (z_{t+\Delta} - z_t)$$

- Can be represented as an integral

$$z_t = z_0 + \int_0^t dz_t$$

- **Define** dt as the smallest positive real number such that $dt^{\alpha} = 0$ if $\alpha > 1$
- Properties of dz:

$$\begin{aligned} E_t(dz_t) &= 0 \\ E_t(dz_t dt) &= dt E_t(dz_t) = 0, \ dt \text{ is a constant} \end{aligned}$$

Brownian motion

- Properties of dz:

$$\begin{aligned} dt &= var\left(dz_t\right) = E_t\left[z_{t+\Delta} - z_t - E_t\left(z_{t+\Delta} - z_t\right)\right]^2 \\ &= E_t\left(z_{t+\Delta} - z_t\right)^2 - E_t\left[E_t\left(z_{t+\Delta} - z_t\right)\right]^2 \\ &= E_t\left(z_{t+\Delta} - z_t\right)^2 \equiv E_t\left(dz_t^2\right) \end{aligned}$$

  i.e. the expected value of the squared random variable is the same as the variance.

- **Observation**: notation $dz_t^2 \equiv \left(dz_t\right)^2$

Brownian motion

- Additional properties of dz:

$$var(dz_t^2) = E\left(dz_t^4\right) - E^2\left(dz_t^2\right) = 3dt^2 - dt^2 = 0$$

$$\text{fourth central moment of a normal is } 3\sigma^2 \text{ and } dt^2 \text{ is } 0$$

$$E_t\left(dz_t dt\right)^2 = dt^2 E_t\left(dz_t^2\right) = 0$$

$$var\left(dz_t dt\right) = E_t\left(dz_t dt\right)^2 - E^2\left(dz_t dt\right) = 0$$

$$dz_t^2 = dt, \text{ because the variance of } dz_t^2 \text{ is zero and } E_t\left(dz_t^2\right) = dt$$

$$dz_t dt = 0, \text{ because the variance of } dz_t dt \text{ is zero and } E_t\left(dz_t dt\right) = 0$$

Stochastic differential equation (diffusion)

- Can construct more complicated time-series processes by adding drift, $\mu(\cdot)$, and volatility, $\sigma(\cdot)$, terms to dz_t,

$$dx_t = \mu(t, x_t)\, dt + \sigma(t, x_t)\, dz_t$$

as a short-cut to express

$$x_t = x_0 + \int_0^t \mu(t, x_s)\, ds + \int_0^t \sigma(t, x_s)\, dz_s$$

- Some examples:
    - **Random walk with drift**

$$dx_t = \mu dt + \sigma dz_t, \text{continuous time}$$

$$x_{t+1} - x_t = \mu + \sigma \varepsilon_{t+1}, \text{discrete time}$$

-
    - **Geometric Brownian motion with drift**

$$dx_t = x_t \mu dt + x_t \sigma dz_t$$

Diffusion model

- From the standard Brownian motion case, we already know that $dz_t \sim N(0, dt)$. Since multiplying a normal variable by σ scales its mean and variance, we get

$$\sigma dz_t \sim N(0, \sigma^2 dt)$$

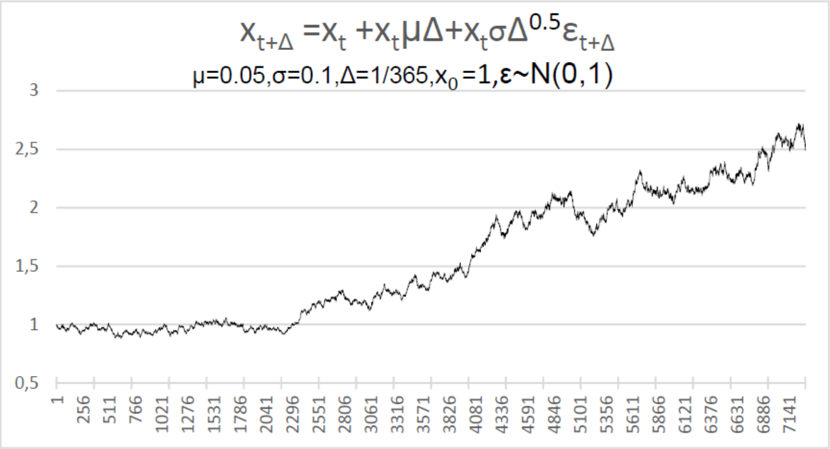
- Adding the drift term μdt gives:

$$\mu dt + \sigma dz_t = dx_t \sim N(\mu dt, \sigma^2 dt)$$

- Any stochastic process (without jumps) can be approximated by a diffusion.

Geometric Brownian motion

Can simulate a diffusion process by approximating it with a small time interval,



Price of stock

- Let P_t be the price of a generic stock at any moment in time that pays dividends at the rate $D_t dt$

The instantaneous return is

$$\frac{dP_t}{P_t} + \frac{D_t}{P_t} dt$$

Let the price be a geometric Brownian motion

$$\frac{dP_t}{P_t} = \mu_p dt + \sigma_p dz_t$$

The risk-free rate can be thought as the return on an asset that does not pay dividend and has the price

$$\frac{dP_t}{P_t} = r_t^f dt$$

Ito's Lemma

- Suppose we have a diffusion representation for one variable, say

$$dx_t = \mu(\cdot)\, dt + \sigma(\cdot)\, dz_t$$

- Define a new variable in terms of the old one,

$$y_t = f(x_t)$$

- What is the diffusion representation for y_t. **Ito's lemma** tells you how to get it
- Use a second-order Taylor expansion, keep terms dz, dt , and $dz^2 = dt$, but terms $dt \times dz$, dt^2, and higher go to zero

Ito's Lemma

- Start with the second order Taylor expansion

$$dy = \frac{df}{dx}dx + \frac{1}{2}\frac{d^2 f}{dx^2}dx^2$$

- Expanding the second term

$$dx^2 = [\mu dt + \sigma dz_t]^2 = \mu^2 dt^2 + \sigma^2 dz_t^2 + 2\mu\sigma dz_t dt = \sigma^2 dt$$

- Substituting for dx^2 and dx

$$\begin{aligned} dy &= \frac{df}{dx}[\mu dt + \sigma dz_t] + \frac{1}{2}\frac{d^2 f}{dx^2}\sigma^2 dt \\ &= \left(\frac{df}{dx}\mu + \frac{1}{2}\frac{d^2 f}{dx^2}\sigma^2\right) dt + \frac{df}{dx}\sigma dz_t \end{aligned}$$

Pricing

- The utility function in continuous time is

$$E_0 \int_0^{\infty} e^{-\delta t} u\left(c_t\right) dt$$

- Let P_t be the price of an asset that pays dividends D_t
- The price must satisfy

$$P_t e^{-\delta t} u'\left(c_t\right) = E_t \int_{s=0}^{\infty} D_{t+s} e^{-\delta(t+s)} u'\left(c_{t+s}\right) ds$$

In discrete time we have:

$$P_t = E_t \sum_{s=0}^{\infty} D_{t+s} \left[\frac{\beta^s u'\left(c_{t+s}\right)}{u'\left(c_t\right)}\right]$$

Pricing

- Define $\Lambda_t \equiv e^{-\delta t} u' \left(c_t \right)$ as the discount factor in continuous time. It follows that

$$P_t \Lambda_t = E_t \int_{s=0}^{\Delta} D_{t+s} \Lambda_{t+s} ds + E_t \int_{s=\Delta}^{\infty} D_{t+s} \Lambda_{t+s} ds$$

or

$$P_t \Lambda_t = E_t \int_{s=0}^{\Delta} D_{t+s} \Lambda_{t+s} ds + E_t \left[P_{t+\Delta} \Lambda_{t+\Delta} \right]$$

- For small Δ the integral above can be approximated by $D_t \Lambda_t \Delta$

$$P_t \Lambda_t \approx D_t \Lambda_t \Delta + E_t \left[P_{t+\Delta} \Lambda_{t+\Delta} \right]$$

or

$$0 \approx D_t \Lambda_t \Delta + E_t \left[P_{t+\Delta} \Lambda_{t+\Delta} - \Lambda_t P_t \right]$$

- For $\Delta \longrightarrow dt$

$$0 = D_t \Lambda_t dt + E_t \left[d \left(\Lambda_t P_t \right) \right]$$

Pricing

- Define the function

$$f\left(\Lambda_t P_t\right) = \Lambda_t P_t$$

where

$$d\Lambda_t = \mu_\Lambda dt + \sigma_\Lambda dz_t \text{ and } dP_t = \mu_P dt + \sigma_P dz_t$$

Taylor expansion of $d\left(\Lambda_t P_t\right)$

$$\begin{aligned} d\left(\Lambda_t P_t\right) &= \frac{\partial f}{\partial \Lambda_t} d\Lambda_t + \frac{\partial f}{\partial P_t} dP_t + \frac{1}{2}\frac{\partial^2 f}{\partial \Lambda_t^2}\left(d\Lambda_t\right)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial P_t^2}\left(dP_t\right)^2 + \\ &\quad \frac{1}{2}\frac{\partial^2 f}{\partial P_t \partial \Lambda_t} dP_t d\Lambda_t + \frac{1}{2}\frac{\partial^2 f}{\partial \Lambda_t \partial P_t} d\Lambda_t dP_t \\ &\quad + \text{higher order terms} \end{aligned}$$

Since higher order terms $= 0$, and replacing the derivatives $\frac{\partial^2 f}{\partial \Lambda_t^2} = \frac{\partial^2 f}{\partial P_t^2} = 0$

$$d\left(\Lambda_t P_t\right) = \Lambda_t dP_t + P_t d\Lambda_t + d\Lambda_t dP_t$$

Pricing

- Replacing $d\Lambda_t P_t$ in the pricing equation

$$0 = D_t \Lambda_t dt + E_t \left[d \left(\Lambda_t P_t \right) \right]$$

and dividing by $\Lambda_t P_t$ get

$$0 = \frac{D_t}{P_t} dt + E_t \left[\frac{dP_t}{P_t} + \frac{d\Lambda_t}{\Lambda_t} + \frac{d\Lambda_t}{\Lambda_t} \frac{dP_t}{P_t} \right]$$

or

$$\frac{D_t}{P_t} dt + E_t \left[\frac{dP_t}{P_t} \right] = -E_t \left[\frac{d\Lambda_t}{\Lambda_t} + \frac{d\Lambda_t}{\Lambda_t} \frac{dP_t}{P_t} \right]$$

For the risk free rate:

$$D_t = 0, \frac{dP_t}{P_t} = r_t^f dt$$

implying

$$\frac{d\Lambda_t}{\Lambda_t} \frac{dP_t}{P_t} = 0, \text{ and } r_t^f dt = -E_t \left[\frac{d\Lambda_t}{\Lambda_t} \right]$$

Pricing

- Replacing

$$r_t^f dt = -E_t\left[\frac{d\Lambda_t}{\Lambda_t}\right]$$

in

$$\frac{D_t}{P_t}dt + E_t\left[\frac{dP_t}{P_t}\right] = -E_t\left[\frac{d\Lambda_t}{\Lambda_t} + \frac{d\Lambda_t}{\Lambda_t}\frac{dP_t}{P_t}\right]$$

- get:

$$\frac{D_t}{P_t}dt + E_t\left[\frac{dP_t}{P_t}\right] = r_t^f dt - E_t\left[\frac{d\Lambda_t}{\Lambda_t}\frac{dP_t}{P_t}\right]$$

which is the equivalent in discrete time to

$$E_t R_{t+1} = R_{t+1}^f - R_{t+1}^f cov_t\left(m_{t+1}, R_{t+1}\right)$$

Black-Scholes formula

- The Black–Scholes formula provides the price of an option
- We are going to use the discount factor approach to derive the formula
- The risk free bond price follows the process:

$$\frac{dB_t}{B_t} = rdt$$

  where r is the riskless rate
- The stochastic discount factor follows the process:

$$\frac{d\Lambda_t}{\Lambda_t} = -rdt - \frac{\mu - r}{\sigma} dz_t$$

where $\frac{\mu - r}{\sigma}$ is the Sharpe ratio

- **Recall** that $\frac{d\Lambda_t}{\Lambda_t}$ is a discount factor if it can price the bond and the stock

Black-Scholes formula

- Let S_t be the price of a stock that pays no dividends (alternatively can think that the dividend is already included in the drift: μ_S)
- We established that $\frac{d\Lambda_t}{\Lambda_t}$ must satisfy the condition

$$E_t\left[\frac{dS_t}{S_t}\right] = -E_t\left[\frac{d\Lambda_t}{\Lambda_t} + \frac{d\Lambda_t}{\Lambda_t}\frac{dS_t}{S_t}\right]$$

- Thus, for $\frac{d(\Lambda_t)}{\Lambda_t}$ to be a stochastic discount factor must satisfy

$$-rdt = E_t\left[\frac{d\Lambda_t}{\Lambda_t}\right]$$

$$E_t\left[\frac{dS_t}{S_t}\right] - rdt = -E_t\left[\frac{d\left(\Lambda_t\right)}{\Lambda_t}\frac{dS_t}{S_t}\right]$$

Exercise: Check that these 2 conditions are satisfied. Remember $E_t\left(dz_t\right) = 0$, $dz_t^2 = dt$, $dz_t dt = 0$ and $dt^\alpha = 0$, if $\alpha > 1$

- To find the value of

$$
\begin{aligned}
C_0\Lambda_0 &= E_0\Lambda_T \max(S_T - X, 0) \\
&= \int_0^\infty \Lambda_T \max(S_T - X, 0) df\left(\Lambda_T, S_T\right)
\end{aligned}
$$

- we need to find the values Λ_T and S_T
- we need the solution of the stochastic differential equation for Λ_t and S_t:

A little Math

$$
\begin{aligned}
d\ln S_t &= \frac{1}{S_t}dS_t - \frac{1}{2}\frac{1}{S_t^2}dS_t^2 \\
&= \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dz_t
\end{aligned}
$$

Black-Scholes formula

- Integrating

$$d \ln S_t = \left(\mu - \frac{1}{2}\sigma^2 \right) dt + \sigma dz_t$$

from 0 to T gives

$$\int_0^T d \ln S_t = \left(\mu - \frac{1}{2}\sigma^2 \right) \int_0^T dt + \sigma \int_0^T dz_t$$

$$\ln S_T = \ln S_0 + \left(\mu - \frac{1}{2}\sigma^2 \right) T + \sigma \left(z_T - z_0 \right)$$

where $z_T - z_0$ is a normally distributed random variable with **mean** zero and **variance** T.

- Thus, $\ln S_T$ is conditionally (on the information at date 0) normal with **mean** $\ln S_0 + \left(\mu - \frac{1}{2}\sigma^2 \right) T$ and **variance** $\sigma^2 T$.

Black-Scholes formula

- The solutions can be written as

$$\ln S_T = \ln S_0 + \left(\mu - \frac{1}{2}\sigma^2\right) T + \sigma \sqrt[2]{T} \varepsilon$$

$$\ln \Lambda_T = \ln \Lambda_0 - \left(r + \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2\right) T - \frac{\mu - r}{\sigma} \sqrt[2]{T} \varepsilon$$

where

$$\varepsilon = \frac{z_T - z_0}{\sqrt[2]{T}} \sim N(0, 1)$$

Recall

$$\frac{d\Lambda_t}{\Lambda_t} = -rdt - \frac{\mu - r}{\sigma} dz_t$$

Black-Scholes formula

- Now we can do the integral:

$$
\begin{aligned}
C_0 &= \int_0^\infty \frac{\Lambda_T}{\Lambda_0} \max(S_T - X, 0) df\left(\Lambda_T, S_T\right) \\
&= \int_{S_T = X}^\infty \frac{\Lambda_T}{\Lambda_0} (S_T - X) df\left(\Lambda_T, S_T\right) \\
&= \int_{S_T = X}^\infty \frac{\Lambda_T\left(\varepsilon\right)}{\Lambda_0} (S_T\left(\varepsilon\right) - X) f(\varepsilon) d\varepsilon
\end{aligned}
$$

  where f is the density of ε

- We know the joint distribution of the terminal stock price S_T and discount factor Λ_T on the right hand side, so we have all the information we need to calculate this integral.

Black-Scholes formula

Start by breaking up the integral into two terms

$$C_0 = \int_{S_T = X}^{\infty} \frac{\Lambda_T(\varepsilon)}{\Lambda_0} S_T(\varepsilon) f(\varepsilon) d\varepsilon - X \int_{S_T = X}^{\infty} \frac{\Lambda_T(\varepsilon)}{\Lambda_0} f(\varepsilon) d\varepsilon$$

use

$$\frac{S_T}{S_0} = e^{\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}\varepsilon}$$

$$\frac{\Lambda_T}{\Lambda_0} = e^{-\left(r + \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2\right)T - \frac{\mu - r}{\sigma}\sqrt{T}\varepsilon}$$

$$\begin{aligned} C_0 &= S_0 \int_X^{\infty} e^{-\left(r + \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2\right)T - \frac{\mu - r}{\sigma}\sqrt{T}\varepsilon} e^{\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}\varepsilon} f(\varepsilon) d\varepsilon \\ &\quad - X \int_X^{\infty} e^{-\left(r + \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2\right)T - \frac{\mu - r}{\sigma}\sqrt{T}\varepsilon} f(\varepsilon) d\varepsilon \end{aligned}$$

- or

$$
\begin{aligned}
C_0 &= S_0 \int_X^{\infty} e^{\left(\mu - r - \frac{1}{2}\left(\sigma^2 + \left(\frac{\mu - r}{\sigma}\right)^2\right)\right)T + \left(\sigma - \frac{\mu - r}{\sigma}\right)\sqrt{T}\varepsilon} f(\varepsilon) d\varepsilon \\
&\quad - X \int_X^{\infty} e^{-\left(r + \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2\right)T - \frac{\mu - r}{\sigma}\sqrt{T}\varepsilon} f(\varepsilon) d\varepsilon
\end{aligned}
$$

Now we replace the formula for $f(\varepsilon)$

$$
f(\varepsilon) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\varepsilon^2}
$$

$$
\begin{aligned}
C_0 &= \frac{S_0}{\sqrt{2\pi}} \int_X^{\infty} e^{\left[\mu - r - \frac{1}{2}\left(\sigma^2 + \left(\frac{\mu - r}{\sigma}\right)^2\right)\right]T + \left(\sigma - \frac{\mu - r}{\sigma}\right)\sqrt{T}\varepsilon - \frac{1}{2}\varepsilon^2} d\varepsilon \\
&\quad - \frac{X}{\sqrt{2\pi}} \int_X^{\infty} e^{-\left(r + \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2\right)T - \frac{\mu - r}{\sigma}\sqrt{T}\varepsilon - \frac{1}{2}\varepsilon^2} d\varepsilon
\end{aligned}
$$

Black-Scholes formula

- or

$$C_0 = \frac{S_0}{\sqrt{2\pi}} \int_X^{\infty} e^{-\frac{1}{2}\left(\varepsilon - \left(\sigma - \frac{\mu - r}{\sigma}\right)\sqrt{T}\right)^2} d\varepsilon$$
$$- \frac{X}{\sqrt{2\pi}} e^{-rT} \int_X^{\infty} e^{-\frac{1}{2}\left(\varepsilon + \frac{\mu - r}{\sigma}\sqrt{T}\right)^2} d\varepsilon$$

- Notice that the integrals have the form of a normal distribution with nonzero mean and variance 1.
- **Recall:** $x \sim N\left(\widetilde{\mu}, \widetilde{\sigma}^2\right)$ if

$$f(x) = \frac{1}{\sqrt{2\pi}\widetilde{\sigma}} e^{-\frac{1}{2}\frac{(x-\widetilde{\mu})^2}{\widetilde{\sigma}^2}}$$

Black-Scholes formula

- The lower bound X can be expressed in terms of ε

$$\ln X = \ln S_T = \ln S_0 + \left(\mu - \frac{1}{2}\sigma^2\right) T + \sigma\sqrt{T}\varepsilon$$

implies

$$\varepsilon = \frac{\ln X - \ln S_0 - \left(\mu - \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}}$$

- The integrals can be expressed using the cumulative standard normal, Φ

$$\Phi(a - \mu) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-\frac{(x-\mu)^2}{2}} dx$$

- where $\Phi(\cdot)$ is the area under the left tail of the standard normal distribution.

Black-Scholes formula

- because Φ is symmetric around zero

$$\Phi\left(a-\mu\right)=1-\Phi\left(\mu-a\right)$$

$$\Phi\left(\mu-a\right)=\frac{1}{\sqrt{2\pi}}\int_{a}^{\infty}e^{-\frac{(x-\mu)^{2}}{2}}dx$$

Black-Scholes formula

- Substituting in

$$
\begin{aligned}
C_0 &= \frac{S_0}{\sqrt{2\pi}} \int_X^{\infty} e^{-\frac{1}{2}\left(\varepsilon - \left(\sigma - \frac{\mu - r}{\sigma}\right)\sqrt{T}\right)^2} d\varepsilon \\
&\quad - \frac{X}{\sqrt{2\pi}} e^{-rT} \int_X^{\infty} e^{-\frac{1}{2}\left(\varepsilon + \frac{\mu - r}{\sigma}\sqrt{T}\right)^2} d\varepsilon
\end{aligned}
$$

$$
\begin{aligned}
C_0 &= S_0 \Phi \left(- \frac{\ln X - \ln S_0 - \left(\mu - \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}} + \left(\sigma - \frac{\mu - r}{\sigma}\right)\sqrt{T} \right) \\
&\quad - X e^{-rT} \Phi \left(- \frac{\ln X - \ln S_0 - \left(\mu - \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}} - \frac{\mu - r}{\sigma}\sqrt{T} \right)
\end{aligned}
$$

- Simplifying, we get the Black-Scholes formula

$$
C_0 = S_0 \Phi \left(\frac{\ln \frac{S_0}{X} + \left(r + \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}} \right) - X e^{-rT} \Phi \left(\frac{\ln \frac{S_0}{X} + \left(r - \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}} \right)
$$

Black-Scholes formula

- We repeat the formula again here:

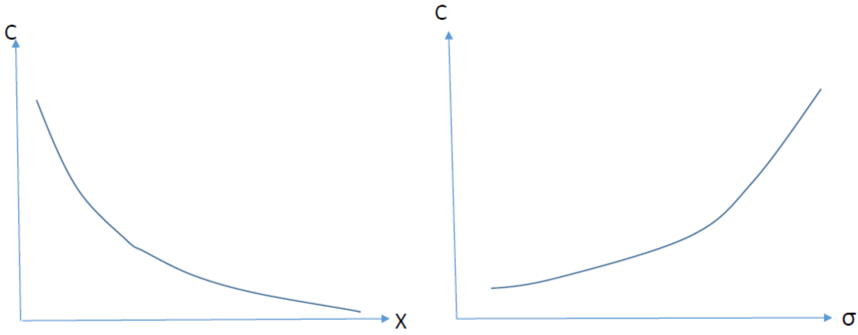
$$C_0 = S_0 \Phi \left(\frac{\ln \frac{S_0}{X} + \left(r + \frac{1}{2}\sigma^2 \right) T}{\sigma \sqrt{T}} \right) - Xe^{-rT} \Phi \left(\frac{\ln \frac{S_0}{X} + \left(r - \frac{1}{2}\sigma^2 \right) T}{\sigma \sqrt{T}} \right)$$

- The price is a function:
  - S_0 (stock price)
  - r (risk free rate)
  - X (strike price)
  - T (time to expiration date)
  - σ (volatility of the underlying stock)

Black-Scholes formula

$$C_0 = S_0 \Phi \left(\frac{\ln \frac{S_0}{X} + \left(r + \frac{1}{2}\sigma^2\right) T}{\sigma \sqrt{T}} \right) - X e^{-rT} \Phi \left(\frac{\ln \frac{S_0}{X} + \left(r - \frac{1}{2}\sigma^2\right) T}{\sigma \sqrt{T}} \right)$$

- This formula is useful to assess how the price of the option changes when the variables in the r.h.s. of the equation change



Black-Scholes formula

- The price is a monotonic increasing function of the σ
- This formula is often used to solve for σ (once C_0 is known). The σ is the **implied volatility**
- Typically options are quoted in units of sigma

Exercise:

Determine the price of an European call option with $S_0 = 50$ euros, $r = 4\%$, $X = 48$ euros, $T = 60$ days and $\sigma = 30\%$. What is the price of an European put option on the same stock, with the same exercise price and time to maturity?

$$\frac{\ln \frac{S_0}{X} + \left(r + \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}} = \frac{\ln \frac{50}{48} + \left(0.04 + \frac{1}{2}\left(0.3\right)^2\right) \frac{60}{365}}{0.3\sqrt{\frac{60}{365}}} = 0.450\,49$$

$$\frac{\ln \frac{S_0}{X} + \left(r - \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}} = \frac{\ln \frac{50}{48} + \left(0.04 - \frac{1}{2}\left(0.3\right)^2\right) \frac{60}{365}}{0.3\sqrt{\frac{60}{365}}} = 0.328\,86$$

$$\Phi\left(0.450\,49\right) = 0.67382$$

Black-Scholes formula

In Excel the command to get the cumulative normal is "=NORM.S.DIST(0,45049;TRUE)"

$$\Phi\left(0.328\,86\right) = 0.62886$$

$$C_0 = 50\left(0.67382\right) - 48e^{-0.04\frac{60}{365}}\left(0.62886\right) = 3.7035$$

To compute the put price must use the put-call parity formula

$$C_0 - P_0 = S_0 - \frac{X}{R^f}$$

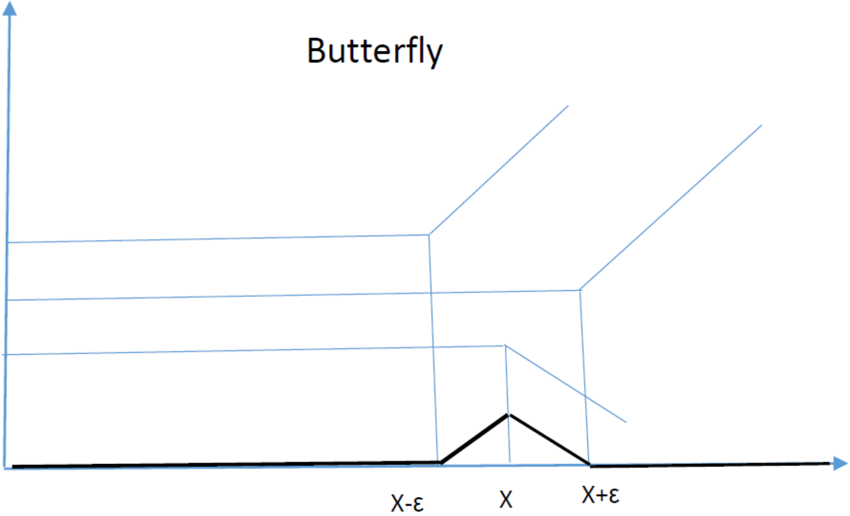
$$P_0 = C_0 + \frac{X}{R^f} - S_0$$

$$P_0 = 3.7035 + 48e^{-0.04\frac{60}{365}} - 50 = 1.3889$$

Spanning

- Given contingent prices can get discount factors, contingent claims and risk neutral probabilities

Proposition: The second derivative of the call option price with respect to the exercise price gives a stochastic discount factor.
Proof: We can construct a contingent claim. Consider the strategy of buying 2 call options, one with strike price $X - \varepsilon$ and another with strike price $X + \varepsilon$, and selling 2 call options with strike price X. The payoff of that portfolio (known as butterfly) is


Butterfly
X-ε
X
X+ε

Spanning

As $\varepsilon \searrow 0$ we are creating a contingent claim.
The payoff of the contingent claim is the area of the triangle ε^2.
The cost of this portfolio is

$$C(X-\varepsilon) - 2C(X) + C(X+\varepsilon)$$

But this is $\varepsilon^2 \frac{\partial^2 C}{\partial X^2}$. Recall that $f''(x) = \lim_{\varepsilon \longrightarrow 0} \frac{f'(x+\varepsilon) - f'(x)}{\varepsilon}$ and $f'(x) = \lim_{\varepsilon \longrightarrow 0} \frac{f(x)-f(x-\varepsilon)}{\varepsilon}$. Thus, $f''(x) = \lim_{\varepsilon \longrightarrow 0} \frac{\frac{f(x+\varepsilon)-f(x)}{\varepsilon} - \frac{f(x)-f(x-\varepsilon)}{\varepsilon}}{\varepsilon}$.

Spanning

Thus, if we buy $\frac{1}{\varepsilon^2}$ of the butterfly we get a payoff of 1 if the $S_T = X$ and a payoff zero for any other value of S_T.
Conclusion: The price of this contingent claim is $\frac{\partial^2 C}{\partial X^2}$.

- Once we have contingent claims we can price any payoff that is a function of S_T, $x(S_T)$
- The price of a portfolio with payments $x(S_T)$ is

$$P = \int_{S_T} \frac{\partial^2 C}{\partial X^2}(X = S_T)\, x(S_T)\, dS_T$$

Spanning

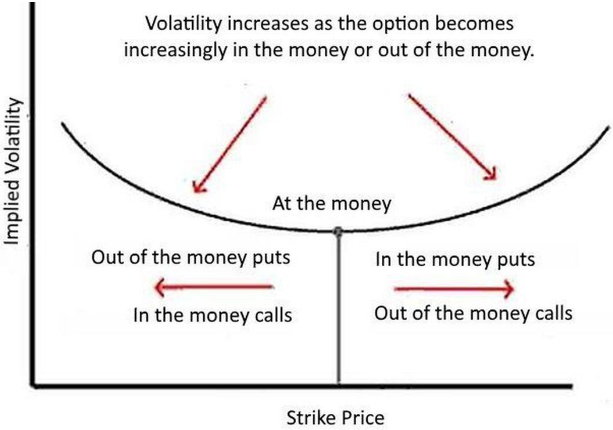
- Discount factor $m_{S_T} = \frac{\frac{\partial^2 C}{\partial X^2}(X = S_T)}{f(S_T)}$, where $f(S_T)$ is the probability of S_T
- Risk neutral probabilities $p_{S_T} = (1+r)^T \frac{\partial^2 C}{\partial X^2}(X = S_T)$

$$P = \frac{E^p(x(S_T))}{(1+r)^T}$$

Data

- Are actual prices equal to the ones predicted by the Black-Scholes formula?
- When options with the same maturity T, same S, but different X, are graphed for implied volatility the tendency is for that graph to show a **smile**.
- The smile shows that the options that are furthest in or out-of-the-money have the highest implied volatility.
- Options with the lowest implied volatility have strike prices at or near-the-money.
- **But** the Black-Scholes model predicts that the implied volatility curve is flat when plotted against varying strike prices**!**

Data



- This means that calls near-the-money have a lower price than the others
- Solution: Consider that the underlying asset price follows a distribution with fatter tails, or that the volatility is a stochastic