Lisbon University
Lisbon School of Economics and Management
Ms in Economics, Mathematical Finance and
Monetary and Financial Economics

Mathematical Economics – 1st Semester - 2025/2026

More exercises - Group III

1. The initial value problem $\begin{cases} y' = \sqrt{y-1} \\ y(0) = 1 \end{cases}$ admits two different solutions.

   (a) Find explicitly the two different solutions.

   (b) Explain why this does not contradict the Existence and Uniqueness Theorem for ordinary differential equations.

2. Assume that $a > b > 0$ and the Initial value problem (IVP) where p is a function of t

$$p' = ap - bp^2, \quad p(0) = p_0 > 0$$

   (a) With respect to the differential equation above, what is the name of a/b? (This IVP is called the logistic law).
   (b) Write the unique solution of the differential equation, say $p(t)$.
   (c) Write the domain of p when $p_0 < a/b$.
   (d) Write the domain of p when $p_0 > a/b$.
   (e) Compute $\lim_{t \to +\infty} p(t)$.

3. Consider the differential equation (y is a function of x):

$$y'' - 4y' + 4y = e^{2x}$$

   (a) Show that $y(x) = \frac{1}{2}x^2e^{2x}$ is a solution of the differential equation.
   (b) Write the general solution of the differential equation.

4. Consider the following differential equation where y is a function of x:

$$y'' + 3y' = 0$$

   (a) Find the solutions of the differential equations whose graph in the plane (y, y') lie in a line.
   (b) Find the solution of the differential equation such that $y(0) = 1$, $y'(0) = -3$. Represent it in the plane (t, y)

5. Consider the following differential equation (y is a function of t)

$$y'' + 4y' + 5y = 1.$$

   (a) Find a particular solution of the differential equation.
   (b) Find the general solution of the differential equation.
   (c) If $y(0) = 6/5$ and $y'(0) = -2$, trace the graph of the unique solution of the differential equation.

6. Consider the following differential equation and write its general solution.

   (a)
$$\begin{cases} \dot{x} = x + 2y \\ \dot{y} = x + y \end{cases}$$

   (b)
$$\begin{cases} \dot{x} = -x \\ \dot{y} = x - y \end{cases}$$

   (c)
$$\begin{cases} \dot{x} = 2x + y \\ \dot{y} = -2x + 4y \end{cases}$$

7. Find the linearisation of the following systems around their equilibria. Classify the equilibria according to their Lyapunov stability.

   (a)
$$\begin{cases} \dot{x} = x + x^2 + xy^2 \\ \dot{y} = y + y^3 \end{cases}$$

   (b)
$$\begin{cases} \dot{x} = e^{x+y} - y \\ \dot{y} = -x + xy \end{cases}$$

   (c)
$$\begin{cases} \dot{x} = y \\ \dot{y} = -\sin x - y \end{cases}$$

8. Consider the following differential equation:

$$\begin{cases} \dot{x} = 1 - 6y + x^2 \\ \dot{y} = 1 - 2y - x^2 \end{cases}$$

   (a) Find the equilibria and determine their Lyapunov stability.
   (b) Draw the null-clines.

9. Consider the following differential equation in $\mathbb{R}^3$:

$$\begin{cases} \dot{x} = -3x - 4y + x^2yz \\ \dot{y} = 4x - 3y - xy^2z \\ \dot{z} = z + xyz^2 \end{cases} \tag{1}$$

(a) Show that $(0, 0, 0)$ is an **unstable** equilibrium (according to Lyapunov).

(b) Draw the phase portrait associated to (1) in a small neighbourhood of $(0, 0, 0)$.

(c) In the phase portrait of (b), locate the **local stable** and the **local unstable** manifolds of $(0, 0, 0)$.

(d) Suppose that $p \in W^u_{loc}(0, 0, 0)$. What is the α-limit set of p?

(e) Suppose that $p \in W^s_{loc}(0, 0, 0)$. What is the ω-limit set of p?