Mathematical Economics - Resit Assessment 09/01/2025

Part I

a) $D_f = \{(x,y) \in \mathbb{R}^2 \; y-1 \neq 0 \wedge x^2 y > 0\}$

$= \{(x,y) \in \mathbb{R}^2 : y \neq 1 \wedge x \neq 0 \wedge y > 0\}$

b) $\partial A = \{ (x,y) \in \mathbb{R}^2 : x^2 + (y+2)^2 = 4 \}$

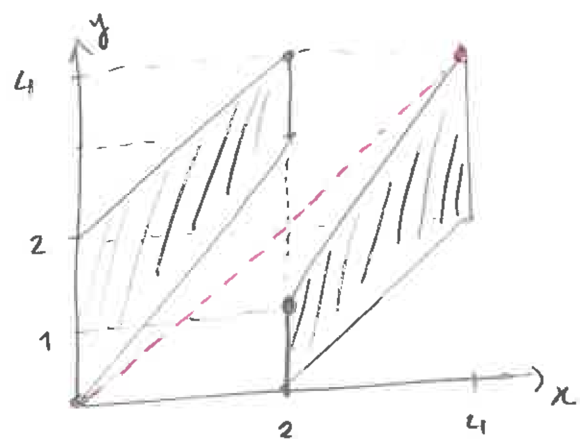
closed

c) $\lim\limits_{n \to +\infty} f(x_n) = -5$

d) $f(x,y) = 4x + \cos(x^2 + y)$

e) I_2 is not convex

f)



- closed graph
- For $x = 2$, $H(x)$ is not convex
- $\{0, 4\}$

g)

$$\nabla f(x,y) = (2-2x\,;\,-1-2y)$$

$$H_f(x,y) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$

$f(1,-1/2)$ is a global maximum

h) $$\mathcal{L}(x,y,z,\lambda) = x-2y+2z-\lambda(x^2+y^2+z^2-1)$$

i) $$\mathcal{L}(x,y,z,\lambda) = x^2-y-\lambda(x^2+y^2-1)$$

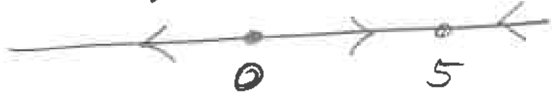
$$\begin{cases} 2x-2\lambda x=0 \\ -1-2\lambda y=0 \\ \lambda(x^2+y^2-1)=0 \\ x^2+y^2 \leq 1 \end{cases}$$

j) $p' = 10p - 2p^2$

Equilibria: $\{0, 5\}$ (zeros of $10p-2p^2$)

Phase portrait

unnecessary because $p \geq 0$



increasing, 5 ; $5/2$

k)

$y(x) = (1+x)\, e^{-2x}$ $\qquad$ $y'(x) = e^{-2x} - 2(1+x)\, e^{-2x}$

$\Downarrow$

-2 is a double zero of the charact polynomial

$p(\lambda) = (\lambda+2)^2 = \lambda^2 + 4\lambda + 4$

$$\begin{cases} y'' + 4y' + 4y = 0 \\ y(0) = 1 \\ y'(0) = 1 - 2 = -1 \end{cases}$$

l)

$$\begin{cases} \dot{x} = -x + x y^{25} \\ \dot{y} = 3y + y x^{25} \end{cases} \qquad \begin{cases} \dot{x} = -x \\ \dot{y} = 3y \end{cases}$$

$(0,0)$ is unstable

Hartman-Grobman.

m) v_2, $\det(A) = 0$.

n) $\cdot$ $H(t,x,u,p) = -\dfrac{x^2}{2} - \dfrac{u^2}{2} + p\cdot(u)$

$\cdot$ $\dfrac{\partial H}{\partial u}(t,x,u,p) = 0 \iff -u + p = 0 \iff \boxed{p = u}$

$\cdot$ $\begin{cases} \dot{x} = \dfrac{\partial H}{\partial p} \\ \dot{p} = -\dfrac{\partial H}{\partial x} \end{cases} \iff \begin{cases} \dot{x} = u \\ \dot{p} = +x \end{cases} \iff \boxed{\begin{cases} \dot{x} = p \\ \dot{p} = +x \end{cases}}$

Part II

1 a) $f(\Delta_2) \subset \Delta_2$

$(x, y, z) \in \Delta_2$

$$f(x, y, z) = \underbrace{\left(\tfrac{1}{2}y + z ; x, \tfrac{1}{2}y\right)}_{\text{summing all coordinates we get } 1.}$$

summing all coordinates we get 1.

$\therefore f(x, y, z) \in \Delta_2$

b) $\left\{ \begin{array}{l} \cdot \ \Delta_2 \text{ is closed} \\ \cdot \ \Delta_2 \text{ is bounded } (\Delta_2 \subset [0,1]^3), \\ \Longrightarrow \ \Delta_2 \text{ is compact} \end{array} \right.$

- f is continuous (all components are polynomial)
- Δ_2 is convex (triangle)

$f(x, y, z) = (x, y, z) \iff \begin{cases} \frac{1}{2}y + z = x \\ x = y \\ \frac{1}{2}y = z \end{cases}$

$\iff \begin{cases} z = \frac{1}{2}x \\ x = y \\ \frac{1}{2}x = z \end{cases}$

General point $(a, a, \frac{1}{2}a)$ $a \in \mathbb{R}$

$\longrightarrow$

Since $(a, a, \frac{1}{2}a) \in \Delta_2$, then: $a + a + \frac{1}{2}a = 1$

$\Leftrightarrow 2a + 2a + a = 2$

$\Leftrightarrow a = 2/5$

$Fix(f) = \{(\frac{2}{5}, \frac{2}{5}, \frac{1}{5})\}$

2) $f(x,y) = y^2 - 2y - x^2y + 2x^2$

$\nabla f(x,y) = (-2xy + 4x ; 2y - 2 - x^2)$

$\nabla f(x,y) = \bar{0} \Leftrightarrow \begin{cases} -2xy + 4x = 0 \\ 2y - 2 - x^2 = 0 \end{cases} \Leftrightarrow \begin{cases} x(-2y+4) = 0 \\ \_\_\_\_\_\_ \end{cases}$

$\Leftrightarrow \begin{cases} x = 0 \\ y = 1 \end{cases} \vee \begin{cases} y = 2 \\ x = \pm\sqrt{2} \end{cases}$

Critical points: $(0,1) ; (\pm\sqrt{2}, 2)$

$Hf(x,y) = \begin{pmatrix} -2y+4 & -2x \\ -2x & 2 \end{pmatrix}$

$Hf(0,1) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ P.D $\Longrightarrow$ $(0,1)$ is a local minimizer

$$H_f(\pm\sqrt{2}, 2) = \begin{pmatrix} 0 & \mp 2\sqrt{2} \\ \mp 2\sqrt{2} & 2 \end{pmatrix}$$

$\Delta_1 = 0$

$\Delta_2 = -8 < 0$

$\Downarrow$

$H_f(\pm\sqrt{2}, 2)$ UND

$(\pm\sqrt{2}, 2)$ saddles $\Longleftarrow$

b) $f(x,0) = -x^2 \cdot (-2) = 2x^2$

$\lim\limits_{x \to +\infty} f(x,0) = +\infty \implies f$ is unbounded.

3)

a) $y_1(x) = 2 \implies y_1'(x) = 0$

$0 \cdot 2 = x \cdot 2^2 - 4x$

$0 = 0$ // $\therefore y_1(x) = 2$ is a const solution.

$y_2(x) = -2 \implies$ same procedure as before.

b) Separable

$$y' y = x y^2 - 4x \Leftrightarrow$$

$y^2-4\neq 0$ $\Leftrightarrow \dfrac{y y'}{y^2-4} = x \Leftrightarrow$

$y^2-4>0.$ $\Leftrightarrow \dfrac{1}{2} \ln |y^2-4| = \dfrac{x^2}{2} + C \;, C\in\mathbb{R}$

$$\Leftrightarrow \ln|y^2-4| = x^2 + 2C \;, C\in\mathbb{R}$$

$$\Rightarrow |y^2-4| = e^{x^2+2C}$$

$$\Leftrightarrow y^2 = 4 \pm K e^{x^2}$$

$$\Leftrightarrow y = \sqrt{4 + K e^{x^2}} \;, C\in\mathbb{R}$$

$$y(1) = \sqrt{3} \Leftrightarrow \sqrt{3} = \sqrt{4 + K e} \Leftrightarrow K = -\frac{1}{e}$$

Solution: $y(x) = \sqrt{4 - e^{x^2-1}}$

$$4 - e^{x^2-1} > 0 \Leftrightarrow e^{x^2-1} < 4 \Leftrightarrow x^2 - 1 < \ln 4$$

$$\Leftrightarrow x^2 < \ln 4 + 1$$

$$-\sqrt{\ln 4 + 1} < x < \sqrt{\ln 4 + 1}$$

Domain: $\{ x \in \mathbb{R} : x \in \left] -\sqrt{\ln 4 + 1}, \sqrt{\ln 4+1} \right[\}$

4) a) $\begin{cases} \dot{x} = 3x \\ \dot{y} = -2x + 3y \end{cases} \Leftrightarrow \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \underbrace{\begin{pmatrix} 3 & 0 \\ -2 & 3 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$

eigenvalues of A: $3_{(2)}$

E_3 $\begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} x = 0 \\ y \in \mathbb{R} \end{cases}$

$E_3 = \langle (0,1) \rangle$

$$\begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

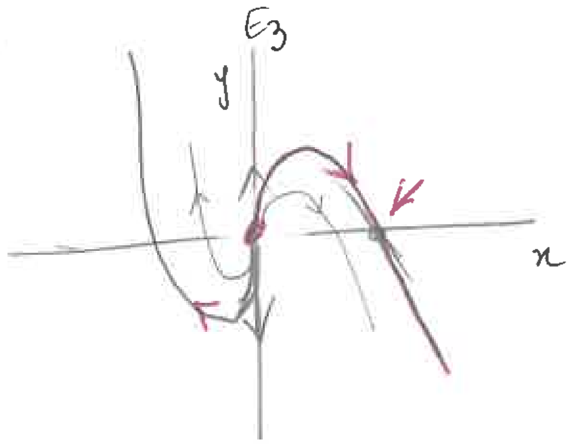
Solution

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{3t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2 e^{3t} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$$

$$= c_1 e^{3t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2 e^{3t} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -2t \end{pmatrix} \right)$$

$$= \begin{pmatrix} c_2 e^{3t} \\ c_1 e^{3t} - 2c_2 t e^{3t} \end{pmatrix} \qquad t \in \mathbb{R}$$

b)



(1,0)

$\dot{x} = 3$

$\dot{y} = -2 + 0 = -2$

5

a)

$$\max \int_0^T \underbrace{\left(x - \frac{1}{2}\dot{x}\right)^2}_{F(t,x,\dot{x})} dt \qquad F(t,x,\dot{x}) = x^2 - x\dot{x} + \frac{1}{4}\dot{x}^2$$

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}} = 0 \iff 2x - \dot{x} - \frac{d}{dt}\left(-x + \frac{1}{2}\dot{x}\right) = 0$$

$$\iff 2x - \cancel{\dot{x}} + \cancel{\dot{x}} - \frac{1}{2}\ddot{x} = 0$$

$$\iff \boxed{4x = \ddot{x}}$$

Transv. condition

$$\frac{\partial F}{\partial \dot{x}}(2, x(2), \dot{x}(2)) = 0 \iff -x(2) + \frac{1}{2}\dot{x}(2) = 0$$

$$\iff \boxed{\dot{x}(2) = 2x(2)}$$

~~T=2~~.

b) $\ddot{x} = 4x \Leftrightarrow \ddot{x} - 4x = 0$

$\Leftrightarrow x(t) = C_1 e^{2t} + C_2 e^{-2t}, \quad C_1, C_2 \in \mathbb{R}$

$$\begin{cases} x(0) = 1 \\ \frac{\partial f}{\partial \dot{x}}(2, x(2), \dot{x}(2)) = 0 \end{cases} \Leftrightarrow \begin{cases} C_1 + C_2 = 1 \\ \dot{x}(2) - 2x(2) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} C_1 + C_2 = 1 \\ \cancel{2C_1 e^4} - 2C_2 e^{-4} - \cancel{2C_1 e^4} - 2C_2 e^{-4} = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} C_1 + C_2 = 1 \\ -4C_2 e^{-4} = 0 \end{cases} \Leftrightarrow \begin{cases} C_1 = 1 \\ C_2 = 0 \end{cases}$$

$$x(t) = e^{2t}, \quad t \in \mathbb{R}$$

$\nabla F(x, \dot{x}) = (2x - \dot{x} \,;\, -x + \frac{1}{2}\dot{x})$

$HF(x, \dot{x}) = \begin{pmatrix} 2 & -1 \\ -1 & 1/2 \end{pmatrix}$

eigenvalues of $HF(x, \dot{x})$: $0, 5/2$

Eigenvalues:

$$P(\lambda) = (2-\lambda)\left(\tfrac{1}{2}-\lambda\right) - 1$$

$$= -2\lambda - \tfrac{1}{2}\lambda + \lambda^2 = \lambda^2 - \tfrac{5}{2}\lambda = \lambda\left(\lambda - \tfrac{5}{2}\right)$$

$HF(x, \dot{x})$ is convex (not strictly)

$$\Downarrow$$

$\boxed{x(t) = e^{2t},\ t \in \mathbb{R}}$ is the solution.