Universidade de Lisboa
Instituto Superior de Economia e Gestão
Msc in Economics, Mathematical Finance,
Monetary and Financial Economics

<u>Mathematical Economics</u> – 1st Semester - 2024/2025

Resit Assessment - 09th of January 2025

Duration: $(120 + \varepsilon)$ minutes, $|\varepsilon| \leq 30$

Version A

Name: ....

Student ID #: ....

Part I

- Complete the following sentences in order to obtain true propositions. The items are independent from each other.
- There is no need to justify your answers.

(a) *(4)* The (maximal) domain of $f : D_f \to \mathbb{R}$ defined by

$$f(x, y) = \frac{1}{y-1} + \ln(x^2 y)$$

is the set

$D_f =$ ....

(simplify the expression)

(b) *(5)* Consider the set

$$A = \{(x, y) \in \mathbb{R}^2 : x^2 + (y + 2)^2 < 4\}.$$

The border (frontier) of A is analytically defined by

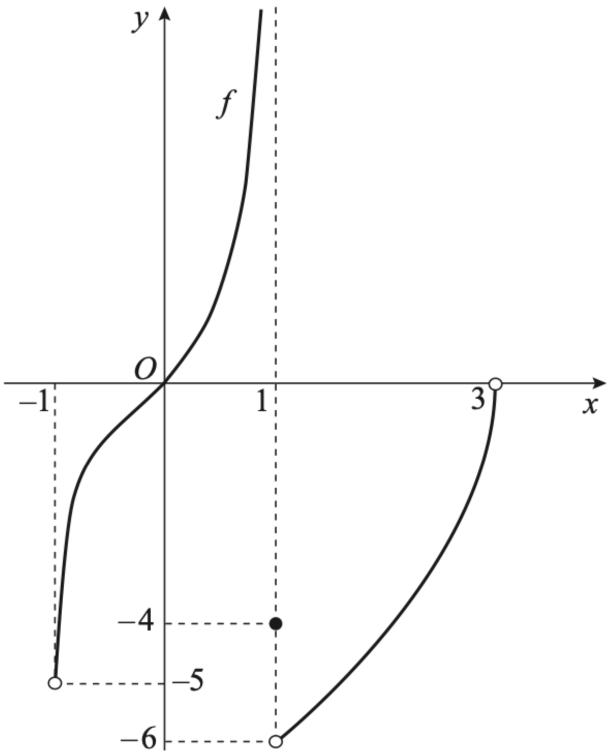
$\partial A =$ ....

Since $cl(A) \neq A$, then A is not .... and, consequently, A is not compact.

(c) *(5)* The graph of $f :]-1, 3[\to \mathbb{R}$ is below. We know that

$$\lim_{x \to 1^-} f(x) = +\infty, \qquad f(1) = -4$$

and $(x_n)_{n\in\mathbb{N}}$ is a sequence in $]-1, 0[$ such that $\lim_{n\in\mathbb{N}} x_n = -1$.



Then

$$\lim_{n\to+\infty} f(x_n) = \text{..........}$$

(d) *(5)* With respect to the map $f : \mathbb{R}^2 \to \mathbb{R}$, one knows that

$$\nabla f(x, y) = (4 - 2x\sin(x^2 + y); -\sin(x^2 + y)).$$

If $f(x, y)$ does not have constant terms, then

$$f(x, y) = \text{..}$$

(e) *(4)* The sets $I_1 = \{(x, y) \in \mathbb{R}^2 : x^2+y^2 < 1\}$ and $I_2 = \{(x, y) \in \mathbb{R}^2 : 2 \leq x^2+y^2 \leq 3\}$ are non-empty and disjoint. However, the Hyperplane Separation Theorem **cannot** be applied because the set .... is not ....

(f) *(11)* The graph of the correspondence $H : [0,4] \rightrightarrows [0,4]$

$$H(x) = \begin{cases} \left[\frac{3}{2}x, x+2\right] & x < 2 \\ [0,1] \cup [3,4], & x = 2 \\ \left[x-2, \frac{3}{2}x - 2\right], & x > 2 \end{cases}$$

is

Since the .... property is valid, then H is upper hemicontinuous. The following hypothesis of the Kakutani fixed point Theorem is failing: (*apply to the correspondence under consideration*):

....

The set of fixed points of H is explicitly given by ....

(g) *(5)* With respect to the map $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = 2x - y - x^2 - y^2$, one knows that $\nabla f(1,-1/2) = (0,0)$. Since

$$H_f(x,y) = \begin{pmatrix} \cdots\cdots & 0 \\ \cdots\cdots & \cdots\cdots \end{pmatrix},$$

then f is stricly concave in $\mathbb{R}^2$ and thus $f(1,-1/2)$ is a .... maximum of f.

(h) *(4)* The minimisers and maximisers of the map $f : \mathbb{R}^3 \to \mathbb{R}$ given by

$$f(x,y,z) = x - 2y + 2z$$

restricted to the set

$$M = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

are the critical points of the map:

$$\mathcal{L}(x,y,z,\lambda) = \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

(i) *(4)* Consider the following **optimization** problem

$$\text{minimize} \quad x^2 - y, \qquad \text{subject to} \quad x^2 + y^2 \le 1.$$

The Karush-Kuhn-Tucker conditions are:

$$\begin{cases} 2x - 2\lambda x = 0 \\ \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = 0 \\ \\ \lambda(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots) = 0 \\ \\ x^2 + y^2 \le 1 \end{cases}$$

(j) *(9)* A given population of size p depends on the time $t \ge 0$ and follows the logistic law:

$$p' = 10p - 2p^2.$$

The equilibria of the differential equation are .... and its phase portrait is:

If $p(0) = 1$, then:

- the solution of the previous differential equation is monotonic ....
- $\lim\limits_{t \to +\infty} p(t) = \ldots\ldots.$
- the graph of p has an inflexion point at $p = \ldots..$

(k) *(8)* The map $y(x) = (1 + x)e^{-2x}$, $x \in \mathbb{R}$, is a solution of the IVP

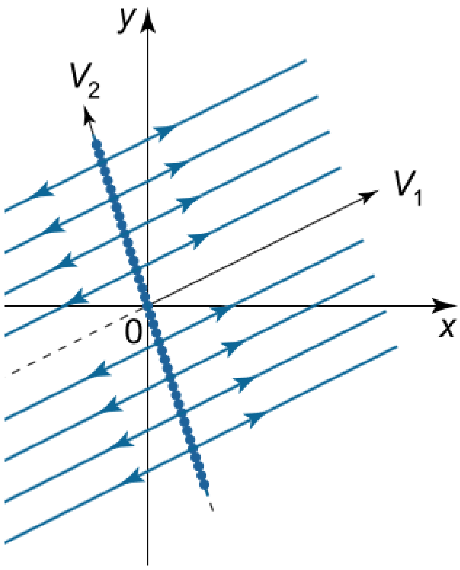
$$\begin{cases} y'' + \ldots\ldots y' + \ldots\ldots y = 0 \\ y(0) = \ldots.. \\ y'(0) = \ldots.. \end{cases} .$$

(l) *(8)* Assuming that x and y depend on t, the linearisation of

$$(*) \begin{cases} \dot{x} = -x + xy^{25} \\ \dot{y} = \ldots. - yx^{25} \end{cases} \quad \text{around the equilibrium } (0,0) \quad \text{is} \quad (**) \begin{cases} \dot{x} = \ldots\ldots\ldots\ldots \\ \dot{y} = 3y \end{cases}$$

With respect to the Lyapunov's **stability**, we may conclude that (0,0) is ....
Furthermore, ...-.........................Theorem says that there exists a small neighbourhood of $(0, 0)$ where the dynamics of (*) and (**) are "qualitatively" the same (topologically conjugated).

(m) *(6)* The phase portrait of $\dot{X} = AX$, where $A \in M_{2\times 2}(\mathbb{R})$ and $X = (x, y) \in \mathbb{R}^2$, is given by:



where $v_1 \neq (0,0)$ and $v_2 \neq (0,0)$ are different eigenvectors associated to A. We may conclude that the eigenvalue associated to the vector .... is 0 and det(A).......0.

(n) *(8)* Consider the following problem of optimal control where $x : [0, 10] \to \mathbb{R}$ is the *state*, $u : [0, 10] \to \mathbb{R}$ is the *control* and t is the independent variable:

$$\max_{u(t)\in\mathbb{R}} \int_0^{10} -\frac{x(t)^2}{2} - \frac{u(t)^2}{2}\, dt, \quad x'(t) = u(t), \quad x(0) = 1 \quad \text{and} \quad x(10) \text{ free.}$$

Then the Hamiltonian is given by (*specify the formulas to the case under consideration*):

$$H(t, x, u, p) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

The Pontryagin maximum principle says that the optimal control $u^\star$ should satisfy the equality .... The Hamiltonian differential equations are given by:

$$\begin{cases} \dot{x} = \ldots\ldots\ldots\ldots\ldots\ldots \\ \\ \dot{p} = \ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

(the right hand side of the system should depend only on the state and the co-state)

Part II

- Give your answers in exact form. For example, $\frac{\pi}{3}$ is an exact number while 1.047 is a decimal approximation for the same number.
- In order to receive credit, you must show all of your work. If you do not indicate the way in which you solve a problem, you may get little or no credit for it, even if your answer is correct.

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1. Consider the following matrix

$$A = \begin{pmatrix} 0 & 1/2 & 1 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}$$

and define the map $f : \Delta^2 \to \Delta^2$ as $f(v) = Av$ where

$$\Delta^2 = \{(x, y, z) \in \mathbb{R}^3 \colon x + y + z = 1,\ x, y, z \geq 0\}.$$

(a) Show that $f(\Delta^2) \subset \Delta^2$.

(b) Show that the hypotheses of the Brouwer fixed point theorem may be applied to f and **compute** the fixed point(s) of f explicitly.
Note: *You can use without proof that Δ^2 is closed.*

2. Consider the map $f : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$f(x, y) = (y - x^2)(y - 2)$$

(a) Identify and classify the critical points of f .

(b) Compute $f(x, 0)$ and conclude that f is unbounded.

3. Consider the following differential equation (y is a function of x):

$$y'y = xy^2 - 4x.$$

(a) Show that the constant maps $y_1(x) = 2$ and $y_2(x) = -2$ are particular solutions of the differential equation.

(b) Write explicitly the solution of the IVP:

$$\begin{cases} y'y = xy^2 - 4x \\ y(1) = \sqrt{3} \end{cases}$$

and identify its maximal domain.

4. Consider the linear system of ODEs in $\mathbb{R}^2$ given by (x and y depend on t):

$$\begin{cases} \dot{x} = 3x \\ \dot{y} = -2x + 3y \end{cases}$$

(a) Write the general form of the solution.

(b) Locate, in the phase portrait, the **unique** solution such that $x(0) = 1, y(0) = 0$.

5. Consider the following Problem on *Calculus of Variations*, where $x : [0, 2] \to \mathbb{R}$ is a smooth function on t:

$$\min_{x(t)\in\mathbb{R}} \int_0^2 \left[x(t) - \frac{1}{2}\dot{x}(t) \right]^2 dt, \quad \text{with} \quad x(0) = 1 \quad \text{and} \quad x(2) \in \mathbb{R}.$$

(a) Write the Euler-Lagrange equation and the transversality condition applied to the case under consideration.

(b) Find the solution of the problem.

DO NOT DO THIS! :)

Credits:

I	II.1(a)	II.1(b)	II.2(a)	II.2(b)	II.3(a)	II.3(b)	II.4(a)	II.4(b)	II.5(a)	II.5(b)
85	10	15	15	5	5	15	15	10	10	15