Recap: Consumer Theory

Sources: Microeconomic Analysis 3rd Ed (Varian); Notes on Microeconomic Theory (Miller)

Preferences and assumptions

Consumption bundle $x \in \mathbb{R}^k_+$. **Consumption set** X.

Preference relations:

$$x \succ y, \; x \succeq y, \; x \sim y$$

Standard assumptions:

- Completeness: Any two bundles can be compared.
- Transitivity: Rankings are consistent.
- Continuity: No jumps in preferences.
- Monotonicity: More of every good is better.
- Convexity: Averages are preferred to extremes.

Utility and indifference curves

- Under completeness, transitivity, continuity. There exists a continuous utility function $u(x)$ that represents preferences.
- Monotonicity and convexity guarantee that the SOCs of constrained maximization are met.
- Only the *ordering* of $u(x)$ matters. Any increasing transformation $g(u)$ represents the same preferences $\Rightarrow$ this is why we can take *ln* tranformations.
- Indifference curve at level $\bar{u}$:

  $$I(\bar{u}) = \{x : u(x) = \bar{u}\}$$

  Indifference curve for e.g., $u(x) = x_1^{1/2} x_2^{1/2}$, fix $\bar{u} = u$ to obtain: $x_2 = \frac{u^2}{x_1}$
- Indifference curves are downward sloping, cannot cross. Under convexity they are convex to the origin.

Marginal utility and MRS

For two goods x_1, x_2:

$$MU_i = \frac{\partial u(x)}{\partial x_i}, \quad i = 1, 2$$

Marginal rate of substitution of x_1 for x_2:

$$MRS = -\frac{MU_1}{MU_2}$$

- MRS measures how many units of good 2 the consumer is willing to give up for one extra unit of good 1, keeping utility constant.
- With convex preferences: Value of MRS falls as x_1 increases (indifference curves becomes flatter).

Budget constraint

Income m, prices p_1, p_2. Budget line:

$$p_1 x_1 + p_2 x_2 = m$$

Solve for x_2 (isobudget line):

$$x_2 = \frac{m}{p_2} - \frac{p_1}{p_2} x_1$$

- Vertical intercept: m/p_2. Horizontal intercept: m/p_1
- Slope: $-p_1/p_2$. Economic rate of substitution along the budget line.
- A change in m shifts the line in parallel. A change in prices rotates it.

Utility maximization and optimality condition

Utility maximization problem (UMP):

$$\max_x u(x) \quad \text{s.t. } px = m$$

Lagrangian for two goods:

$$\mathcal{L} = u(x_1, x_2) - \lambda(p_1 x_1 + p_2 x_2 - m)$$

First order conditions for an interior optimum:

$$\frac{\partial u}{\partial x_1} = \lambda p_1, \quad \frac{\partial u}{\partial x_2} = \lambda p_2, \quad p_1 x_1 + p_2 x_2 = m$$

Divide the first two FOCs:

$$\frac{p_1}{p_2} = \frac{\partial u/\partial x_1}{\partial u/\partial x_2} \quad \Leftrightarrow \quad ERS = MRS$$

At the optimum, the slope of the indifference curve equals the slope of the budget line.

Marshallian Demand, Hicksian Demand

Marshallian demand $x(p, m)$:

- Solution x^* of the UMP.

$$\max_x u(x) \quad \text{s.t. } px = m$$

- Gives optimal bundle as function of prices and income.
- Captures both substitution and income effects.

Indirect utility:

$$v(p, m) = u(x(p, m))$$

Hicksian demand $h(p, u)$:

- Solution of the expenditure minimization problem (EMP)

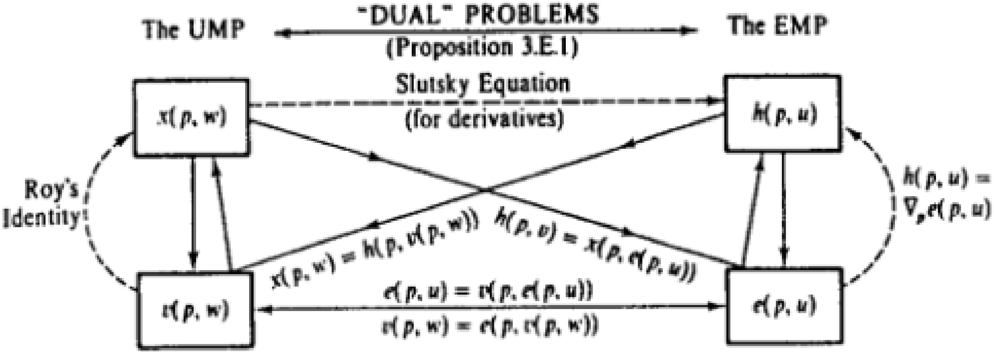
$$\min_x px \quad \text{s.t. } u(x) = u$$

- Keeps utility fixed. Compensated demand.

Expenditure function:

$$e(p, u) = p\,h(p, u)$$

UMP and EMP



The UMP
"DUAL" PROBLEMS
(Proposition 3.E.1)
The EMP
Slutsky Equation
(for derivatives)
x(p, w)
h(p, u)
Roy's Identity
h(p, u) =
∇ₚe(p, u)
x(p, w) = h(p, v(p, w))
h(p, u) = x(p, e(p, u))
e(p, u) = v(p, e(p, u))
v(p, w) = e(p, v(p, w))
v(p, w)
e(p, u)

The Slutsky Equation: Motivation

- Marshallian demand $x(p, m)$ gives uncompensated demand, or the total effect of a price change.
- Hicksian demand $h(p, u)$ gives compensated demand, since obtained by fixing utility level u, or the substitution effect of a price change.
- Goal: link the two so that we can do welfare analysis and decompose price effects.
- At the Marshallian optimum:

$$u(x(p, m)) = v(p, m), \qquad e(p, v(p, m)) = m.$$

- This identity allows us to express Hicksian demand as:

$$h(p, u) = x(p, e(p, u)),$$

which leads directly to the Slutsky equation.

Slutsky equation, substitution and income effects

Slutsky equation for good *i*:

$$\underbrace{\frac{\partial x_i(\mathbf{p},m)}{\partial p_i}}_{\text{TE}} = \underbrace{\frac{\partial h_i(\mathbf{p},u)}{\partial p_i}}_{\text{SE}} - \underbrace{\frac{\partial x_i(\mathbf{p},m)}{\partial m} x_i(\mathbf{p},m)}_{\text{IE}} .$$

- Substitution effect (SE): $\frac{\partial h_i}{\partial p_i} \leq 0$. Always for convex preferences.

Intuition

Convexity makes the indifference curve flatten as x_1 increases, so if the price of x_1 rises, the only way to restore tangency is to move to a point with *less* x_1 and *more* x_2.

- Income effect (IE): $-\frac{\partial x_i}{\partial m} x_i$. Positive for normal goods, negative for inferior goods.

Law of demand:

- Ordinary good. TE < 0.
- Giffen good. TE > 0. Must be strongly inferior.

Uncompensated vs. Compensated Price Effects

$$\frac{\partial x_i(p,m)}{\partial p_i} = \frac{\partial h_i(p,u)}{\partial p_i} - \frac{\partial x_i(p,m)}{\partial m} x_i(p,m).$$

- We are interested in explaining an uncompensated change in demand in terms of the compensated change and the income effect.

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- If the price of bananas were to go up, and my wealth *were* adjusted so that I could achieve the same amount of utility before and after the change, I would consume fewer bananas ($\frac{\partial h_i(p,u}{\partial p_i} < 0 \iff SE < 0$).

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- **Intuition**: Compensation raises income. To undo this and get the real change in demand, impose the opposite (a wealth decrease).

Comparative statics and types of goods

Price offer curve. Locus of optimal *bundles* as a price changes, income fixed.

Marshallian demand curve. Relationship between price of a *good* and optimal quantity.

Types of goods:

- Ordinary good. Higher price implies lower demand (downward sloping).
- Giffen good. Higher price implies higher demand (upward sloping).

Income expansion path. Locus of optimal *bundles* as income changes, prices fixed.

Engel curve. Relationship between income and optimal quantity of each *good*.

- Normal good. Demand rises with income (upward sloping).
- Inferior good. Demand falls with income (downward sloping).

Quasilinear utility and zero income effect

Quasilinear utility, linear in good 2:

$$u(x_1, x_2) = \varphi(x_1) + x_2$$

Properties:

- Indifference curves are parallel: they are vertical shifts, the slope does not depend on u.
- This implies that income effect for x_1 is zero.

$$\frac{\partial x_1(p, m)}{\partial m} = 0$$

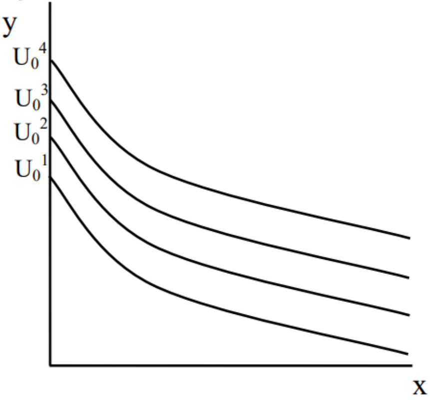
- Thus, Marshallian and Hicksian demand for x_1 coincide.
- All price effects for x_1 are pure substitution effects.

This case is useful for linking welfare changes to areas under a single demand curve.

Quasilinear utility graph*

$$U_0 = f(x) + y$$
$$y = U_0 - f(x)$$

So, the indifference curves are all the same shape, except they are vertically shifted up and down by the value of U_0:



*You can replace $y = x_2$ for consistency in notation with the rest of the notes

Welfare Measures

Consider a price increase of good 1 from p_1^0 to p_1^1. Note that p_2 does not change: $p_2 = p_2^0 = p_2^1$.

- **Compensating variation (CV)**:

$$CV = e(p^1, v^0) - e(p^0, v^0) = e(p^1, v^0) - m$$

  Income change at new prices that restores old utility. Area left of $h_1(p, v^0)$.

- **Equivalent variation (EV)**:

$$EV = e(p^1, v^1) - e(p^0, v^1) = m - e(p^0, v^1)$$

  Income change at old prices that gives new utility. Area left of $h_1(p, v^1)$

- **Change in consumer surplus**:

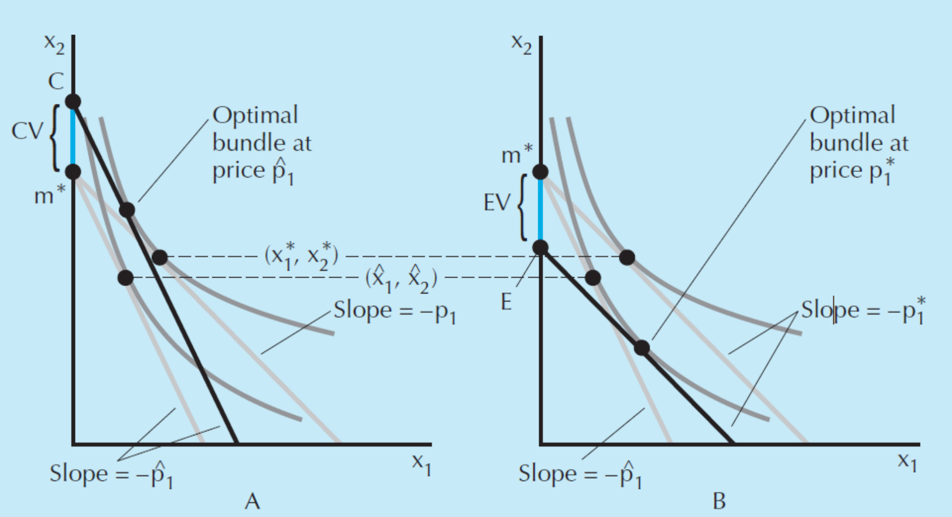
$$\Delta CS = \int_{p_1^0}^{p_1^1} x_1(p, m)\, dp_1$$

  Area to the left of the Marshallian demand curve.

For a normal good (and a price increase): $CV > \Delta CS > EV$. For $IE = 0$ (e.g., quasilinear utility): $CV = EV = \Delta CS$

CV and EV

Consider again a price increase of good 1 from p_1^0 to p_1^1. CV and EV help us answer: How much does this price change hurt the consumer?



Why $CV \geq EV$ for a Normal Good (Price Increase)

- Consider an increase in the price of good x_1.
- To restore the consumer to the *original* indifference curve (CV), we must give more money than what would be required to place them on the *new* indifference curve at old prices (EV).
- Intuition:
  - After the price increase, both goods are effectively more expensive.
  - The marginal utility of income is lower at the new price vector.
  - Therefore, more cash is required to bring the consumer back to their original utility.
- Thus for a normal good and a price increase:

$$CV \geq EV.$$

Which Measure Is Better: EV or CV?

- Both EV and CV give dollar measures of welfare changes from price shifts.
- EV has an important advantage: it is comparable across different price changes.
- Example: with initial prices p^0 and two alternatives p^a and p^b:
  - $EV(p^0, p^a, w)$ and $EV(p^0, p^b, w)$ are both valued at prices p^0 and thus can be meaningfully compared.
  - $CV(p^0, p^a, w)$ is measured at prices p^a, and $CV(p^0, p^b, w)$ at prices p^b: these are *not* directly comparable.
- Policy implication: When comparing the welfare impact of taxing different goods, we need a common reference price. This points to using EV.