STATISTICAL METHODS

Master in Industrial Management,

Operations and Sustainability (MIMOS)

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https://doity.com.br/estatistica-aplicada-a-nutricao

https://basiccode.com.br/produto/informatica-basica/

PROGRAM

1. Fundamental Concepts of Statistics
2. Descriptive Data Analysis
3. Introduction to Inferential Analysis
4. Parametric Hypothesis Testing
5. Non-Parametric Hypothesis Testing
6. Linear Regression Analysis

LECTURE 10 HOMEWORK: QUESTIONS AND SOLUTIONS

EXERCISE 7.29

7.29 A car-rental company is interested in the amount of time its vehicles are out of operation for repair work. State all assumptions and find a 90% confidence interval for the mean number of days in a year that all vehicles in the company's fleet are out of operation if a random sample of nine cars showed the following number of days that each had been inoperative:

16 10 21 22 8 17 19 14 19

Newbold et al (2013)

EXERCISE 7.29: SOLUTION

Answer:

Problem data (sample):

16, 10, 21, 22, 8, 17, 19, 14, 19 (n = 9)

Confidence Interval for the Mean (Unknown Variance)

Assumptions:

- The sample is a simple random sample from the company's fleet.
- The distribution of days out of operation is approximately **normal** (required because n is small).
- Observations are independent.

Under these assumptions we use the **t-distribution.**

$$CI_{(1-\alpha)}(\mu) = \left(\bar{x} - t_{1-\frac{\alpha}{2};n-1} \times \frac{s}{\sqrt{n}}, \bar{x} + t_{1-\frac{\alpha}{2};n-1} \times \frac{s}{\sqrt{n}}\right)$$

Step 1 — Sample statistics

Computing the mean and the standard deviation of the sample mean

$$n = 9$$

$$\bar{x} = \frac{16+10+21+22+8+17+19+14+19}{9} = 16.2222 \text{ (days)}$$

(sample mean)

Sample standard deviation (unbiased):

$$s \approx 4.7900 \text{ (days)}$$

Standard error:

$$SE = \frac{s}{\sqrt{n}} = \frac{4.7900}{\sqrt{9}} = \frac{4.7900}{3} \approx 1.5967$$

Degrees of freedom: $df = n - 1 = 8$.

EXERCISE 7.29: SOLUTION

Answer:

1-α = 0.90 ⇔ α = 0.10
1- α/2 = 0.95
$t_{1-\frac{\alpha}{2};n-1}$ = $t_{0.95;8}$ ~1.8595
(see student´s t table)

Step 2 — Critical t value for 90% CI

For a 90% two-sided confidence interval, $\alpha = 0.10$ and $\alpha/2 = 0.05$.

With $df = 8$,

0.95 quantile of a t-distribution with 8 degrees of freedom

$$t_{0.95;8} \approx 1.8595.$$

Margin of error:

$$ME = t_{0.05,8} \times SE \approx 1.8595 \times 1.5967 \approx 2.9691$$

Step 3 — 90% confidence interval

$$\bar{x} \pm ME = 16.2222 \pm 2.9691$$

$$\boxed{90\% \text{ CI for the mean} \approx [13.25,\ 19.19] \text{ days}}$$

$CI_{0.90}(\mu) = (13.25; 19.19)$

EXERCISE 7.41

7.41 It is important for airlines to follow the published scheduled departure times of flights. Suppose that one airline that recently sampled the records of 246 flights originating in Orlando found that 10 flights were delayed for severe weather, 4 flights were delayed for maintenance concerns, and all the other flights were on time.

a. Estimate the percentage of on-time departures using a 98% confidence level.
b. Estimate the percentage of flights delayed for severe weather using a 98% confidence level.

Newbold et al (2013)

EXERCISE 7.41 A): SOLUTION

Answer:

Good — let's do both parts with a 98% confidence level (use $z \approx 2.3263$).

Given $n = 246$.

On-time flights: $x_{\text{on}} = 246 - 10 - 4 = 232$.

Severe-weather delays: $x_{\text{weather}} = 10$.

Confidence Interval for the Population Proportion

$$CI_{(1-\alpha)}(p) = \left(\hat{p} - z_{1-\frac{\alpha}{2}} \times \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + z_{1-\frac{\alpha}{2}} \times \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$$

a) On-time departures

Sample proportion:

$$\hat{p}_{\text{on}} = \frac{232}{246} = 0.943089 \ (\approx 94.31\%).$$

Computing the mean and the standard deviation of the sample proportion

Standard error:

$$SE = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\frac{0.943089(1-0.943089)}{246}} \approx 0.0147708.$$

1-α = 0.98 ⇔ α = 0.02
1- α/2 = 0.95
$z_{1-\frac{\alpha}{2}} = z_{0.99} = 2.326$ (see standard normal table)

98% CI:

$$\hat{p} \pm z\,SE = 0.943089 \pm 2.3263(0.0147708)$$
$$= 0.943089 \pm 0.034362$$
$$\Rightarrow (0.9087,\ 0.9775)$$

In percentages: ≈ **90.87% to 97.75%.**

$CI_{98\%}(p) = (0.9087, 0.9775)$

EXERCISE 7.41 B): SOLUTION

Confidence Interval for the Population Proportion

Answer:

b) Flights delayed for severe weather

Sample proportion:

$$\hat{p}_{\text{weather}} = \frac{10}{246} = 0.0406504 \ (\approx 4.07\%).$$

Standard error:

$$SE = \sqrt{\frac{0.0406504(1 - 0.0406504)}{246}} \approx 0.0125908.$$

98% CI:

$$0.0406504 \pm 2.3263(0.0125908)$$

$$= 0.0406504 \pm 0.0292906$$

$$\Rightarrow (0.01136, \ 0.06994)$$

In percentages: ≈ **1.14% to 6.99%**.

$CI_{98\%}(p) = (0.01136, 0.06994)$

EXERCISE 7.50

7.50 A manufacturer bonds a plastic coating to a metal surface. A random sample of nine observations on the thickness of this coating is taken from a week's output, and the thicknesses (in millimeters) of these observations are as follows:

19.8 21.2 18.6 20.4 21.6 19.8 19.9 20.3 20.8

Assuming normality, find a 90% confidence interval for the population variance.

Newbold et al (2013)

EXERCISE 7.50: SOLUTION

Answer:

Confidence Interval for the Population Variance

$$CI_{(1-\alpha)}(\sigma^2) = \left(\frac{(n-1)s^2}{\chi^2_{1-\frac{\alpha}{2},n-1}}, \frac{(n-1)s^2}{\chi^2_{\frac{\alpha}{2},n-1}} \right)$$

We use the chi-square confidence interval for the population **variance**.

Data: $n = 9$ observations

$$19.8,\ 21.2,\ 18.6,\ 20.4,\ 21.6,\ 19.8,\ 19.9,\ 20.3,\ 20.8.$$

1. Compute sample mean and sample variance (use $n-1$ denominator):

$$\bar{x} = 20.2666667, \qquad s^2 = 0.7875, \qquad s \approx 0.8874.$$

2. For a 90% CI we have $\alpha = 0.10$, so $\alpha/2 = 0.05$. Degrees of freedom $\nu = n - 1 = 8$.

   Chi-square critical values (df = 8):

$$\chi^2_{0.95,8} \approx 15.507313, \qquad \chi^2_{0.05,8} \approx 3.490295.$$

n = 9 (sample size)
s = 0.8874 (sample standard deviation)
α = 0.10 (significance level)
1 – α = 0.90 (confidence level)
$\chi^2_{(0.05,8)}$ = 3.490295
$\chi^2_{(0.95\ 8)}$ = 15.507313 (see chi-square table)

EXERCISE 7.50: SOLUTION

Answer:

3. The $100(1-\alpha)\%$ CI for the variance σ^2 is

$$\left(\frac{\nu s^2}{\chi^2_{0.95,\nu}}, \frac{\nu s^2}{\chi^2_{0.05,\nu}}\right).$$

Numerically,

$$\text{Lower} = \frac{8 \times 0.7875}{15.507313} \approx 0.40626,$$

$$\text{Upper} = \frac{8 \times 0.7875}{3.490295} \approx 1.80501.$$

Final answer (90% CI)

- For the **variance**:

$CI_{90\%}(\sigma^2) = (0.406, 1.805)$

$$\boxed{0.406 \leq \sigma^2 \leq 1.805}$$

- For the **standard deviation** (by taking square roots):

$$\boxed{0.637 \leq \sigma \leq 1.344}$$

(Values rounded to three decimal places.)

$CI_{90\%}(\sigma) = (0.637, 1.344)$

LECTURE 11: PARAMETRIC HYPOTHESIS TESTS

TESTS OF THE MEAN OF A NORMAL DISTRIBUTION (σ^2 UNKNOWN)

1. Hypotheses

- Null hypothesis: $H_0 : \mu = \mu_0$, $H_0: \mu \leq \mu_0$ OR $H_0: \mu \geq \mu_0$
- Alternative hypothesis: $H_1 : \mu \neq \mu_0$ (two-tailed)
  or $H_1 : \mu > \mu_0$ / $H_1 : \mu < \mu_0$ (one-tailed)

2. Test Statistic

$$T = \frac{\bar{X} - \mu_0}{S/\sqrt{n}} \sim t_{n-1}$$

where $\bar{X}$ is the sample mean, S the sample standard deviation, and n the sample size.

Note:
For a mean test, there are three possible types: **two-tailed**, **right-tailed**, and **left-tailed.** They have different hypotheses, the same test statistic, but potentially different decision rules.

3. Decision Rule

- Using critical value(s): Reject H_0 if $T \in \text{Rejection Region}$ (RR)
- Using p-value: Reject H_0 if $p\text{-value} < \alpha$

TESTS OF THE MEAN OF A NORMAL DISTRIBUTION (σ^2 UNKNOWN)

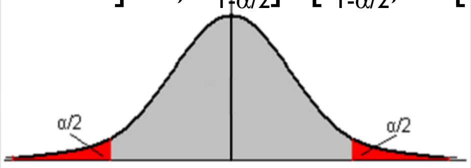
- A parametric hypothesis test for the parameter μ (the population mean) may be:

Two-Tailed Test
$H_0: \mu = \mu_0$
$H_1: \mu \neq \mu_0$

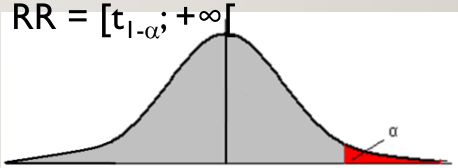
Right-Tailed Test
$H_0: \mu \leq \mu_0$
$H_1: \mu > \mu_0$

Left-Tailed Test
$H_0: \mu \geq \mu_0$
$H_1: \mu < \mu_0$

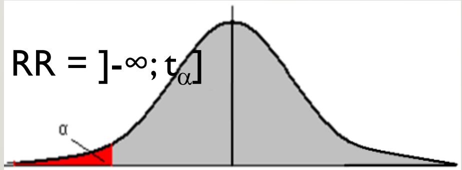
RR = $]-\infty; -t_{1-\alpha/2}] \cup [t_{1-\alpha/2}; +\infty[$



RR = $[t_{1-\alpha}; +\infty[$



RR = $]-\infty; t_{\alpha}]$



where μ_0 is the specific numerical value considered in H_0 and H_1.

Decision rule: Reject H_0 if the test statistic t is in the rejection region (RR), or if p-value < α.

- **Two-tailed:** P-value = 2 · P(T > |t|)
- **Right-tailed:** P-value = P(T > t)
- **Left-tailed:** P-value = P(T < t)

- **Two-tailed test:** $t \in \mathrm{RR} \iff |t| \geq t_{1-\alpha/2;\,n-1}$
- **Right-tailed test:** $t \in \mathrm{RR} \iff t \geq t_{1-\alpha;\,n-1}$
- **Left-tailed test:** $t \in \mathrm{RR} \iff t \leq t_{\alpha;\,n-1}$

EXERCISE 9.25

9.25 A statistics instructor is interested in the ability of students to assess the difficulty of a test they have taken. This test was taken by a large group of students, and the average score was 78.5. A random sample of eight students was asked to predict this average score. Their predictions were as follows:

72 83 78 65 69 77 81 71

Assuming a normal distribution, test the null hypothesis that the population mean prediction would be 78.5. Use a two-sided alternative and a 10% significance level.

Newbold et al (2013)

EXERCISE 9.25: SOLUTION

One-Sample t-Test for the Mean (unknown Variance)

Answer:

Step 1 – Problem Setup

- Sample size: $n = 8$
- Sample mean: $\bar{x} = 74.5$
- Sample standard deviation: $s \approx 6.23$
- Null hypothesis: $H_0 : \mu = 78.5$
- Alternative hypothesis: $H_1 : \mu \neq 78.5$
- Significance level: $\alpha = 0.10$
- Test: **two-tailed t-test** (σ unknown, small sample)

Two-tailed Test

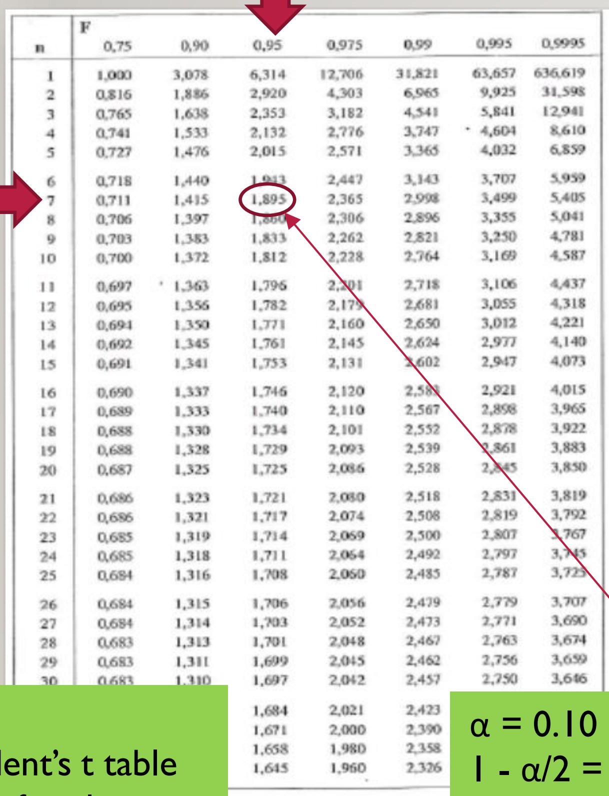
Step 2 – Test Statistic

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{74.5 - 78.5}{6.23/\sqrt{8}} \approx -1.82$$

Degrees of freedom: $df = n - 1 = 7$

CRITICAL VALUE $t_{1-\alpha/2;\, n-1}$: CALCULATION

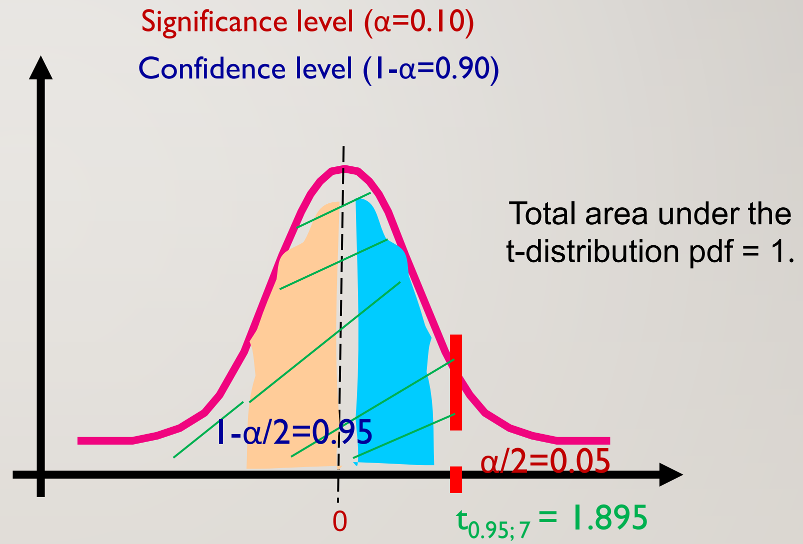
Two-tailed Test → RR =] -∞; $-t_{1-\alpha/2}$]U[$t_{1-\alpha/2}$; +∞[?



n	F 0,75	0,90	0,95	0,975	0,99	0,995	0,9995
1	1,000	3,078	6,314	12,706	31,821	63,657	636,619
2	0,816	1,886	2,920	4,303	6,965	9,925	31,598
3	0,765	1,638	2,353	3,182	4,541	5,841	12,941
4	0,741	1,533	2,132	2,776	3,747	4,604	8,610
5	0,727	1,476	2,015	2,571	3,365	4,032	6,859
6	0,718	1,440	1,943	2,447	3,143	3,707	5,959
7	0,711	1,415	1,895	2,365	2,998	3,499	5,405
8	0,706	1,397	1,860	2,306	2,896	3,355	5,041
9	0,703	1,383	1,833	2,262	2,821	3,250	4,781
10	0,700	1,372	1,812	2,228	2,764	3,169	4,587
11	0,697	1,363	1,796	2,201	2,718	3,106	4,437
12	0,695	1,356	1,782	2,179	2,681	3,055	4,318
13	0,694	1,350	1,771	2,160	2,650	3,012	4,221
14	0,692	1,345	1,761	2,145	2,624	2,977	4,140
15	0,691	1,341	1,753	2,131	2,602	2,947	4,073
16	0,690	1,337	1,746	2,120	2,583	2,921	4,015
17	0,689	1,333	1,740	2,110	2,567	2,898	3,965
18	0,688	1,330	1,734	2,101	2,552	2,878	3,922
19	0,688	1,328	1,729	2,093	2,539	2,861	3,883
20	0,687	1,325	1,725	2,086	2,528	2,845	3,850
21	0,686	1,323	1,721	2,080	2,518	2,831	3,819
22	0,686	1,321	1,717	2,074	2,508	2,819	3,792
23	0,685	1,319	1,714	2,069	2,500	2,807	3,767
24	0,685	1,318	1,711	2,064	2,492	2,797	3,745
25	0,684	1,316	1,708	2,060	2,485	2,787	3,725
26	0,684	1,315	1,706	2,056	2,479	2,779	3,707
27	0,684	1,314	1,703	2,052	2,473	2,771	3,690
28	0,683	1,313	1,701	2,048	2,467	2,763	3,674
29	0,683	1,311	1,699	2,045	2,462	2,756	3,659
30	0,683	1,310	1,697	2,042	2,457	2,750	3,646
			1,684	2,021	2,423		
			1,671	2,000	2,390		
			1,658	1,980	2,358		
			1,645	1,960	2,326		

Note:
The Student's t table reports left-tail probabilities: P(T ≤ t).

α = 0.10
1 - α/2 = 0.95
$t_{1-\frac{\alpha}{2};n-1}$ = $t_{0.95;7}$ = 1.895



RR =] -∞; -1.895]U[1.895; +∞[

P-VALUE FOR A STUDENT´S T-STATISTIC: CALCULATION

Two-tailed Test → P-value = $2 \times P(T \geq |t_0|)$

n	F 0,75	0,90	0,95	0,975	0,99	0,995	0,9995
1	1,000	3,078	6,314	12,706	31,821	63,657	636,619
2	0,816	1,886	2,920	4,303	6,965	9,925	31,598
3	0,765	1,638	2,353	3,182	4,541	5,841	12,941
4	0,741	1,533	2,132	2,776	3,747	4,604	8,610
5	0,727	1,476	2,015	2,571	3,365	4,032	6,859
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7	0,711	1,415	1,895	2,365	2,998	3,499	5,405
8	0,706	1,397	1,860	2,306	2,896	3,355	5,041
9	0,703	1,383	1,833	2,262	2,821	3,250	4,781
10	0,700	1,372	1,812	2,228	2,764	3,169	4,587
11	0,697	1,363	1,796	2,201	2,718	3,106	4,437
12	0,695	1,356	1,782	2,179	2,681	3,055	4,318
13	0,694	1,350	1,771	2,160	2,650	3,012	4,221
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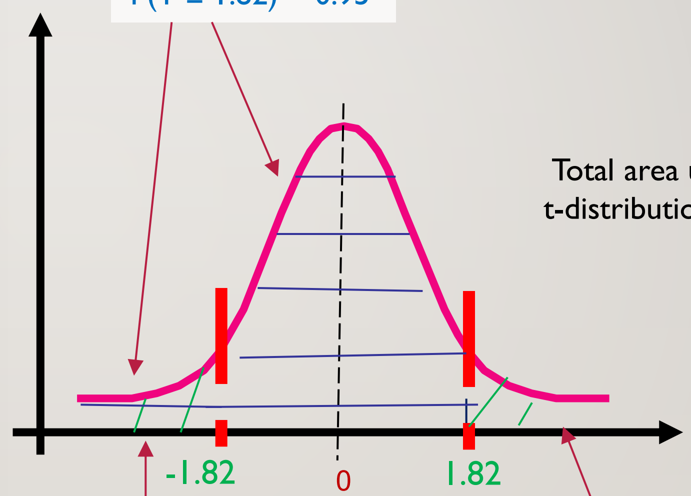
19	0,688	1,328	1,729	2,093	2,539	2,861	3,883
20	0,687	1,325	1,725	2,086	2,528	2,845	3,850
21	0,686	1,323	1,721	2,080	2,518	2,831	3,819
			1,717	2,074	2,508	2,819	3,792
			1,714	2,069	2,500	2,807	3,767
			1,711	2,064	2,492	2,797	3,745
			1,708	2,060	2,485	2,787	3,725
			1,706	2,056	2,479	2,779	3,707
			1,703	2,052	2,473	2,771	3,690
			1,701	2,048	2,467	2,763	3,674
			1,699	2,045	2,462	2,756	3,659
			1,697	2,042	2,457	2,750	3,646
40	0,681	1,303	1,684	2,021	2,423	2,704	3,551

Note:
The Student's t table reports left-tail probabilities: $P(T \leq t)$.

P-value = $2 \times P(T \geq |t_0|)$ ⇔
P-value = $2 \times P(T \geq 1.82)$ ⇔
P-value ~ $2 \times P(T \geq 1.895)$ ⇔
P-value = $2 \times [1 - P(T < 1.895)] \sim 2 \times [1 - 0.95] = 0.1$

The value of the test statistic is $t_0 = -1.82$

$P(T \leq 1.82) \sim 0.95$



Total area under the t-distribution pdf = 1.

$P(T \leq -1.82) = P(T \geq 1.82) \sim 1 - 0.95 = 0.05$

P-value = sum of the two green areas = 0.05 + 0.05 = 0.1

EXERCISE 9.25: SOLUTION

Answer:

Step 3 – Critical Value

RR =] -∞; -1.895]U[1.895; +∞[

- Two-tailed test, α = 0.10 → $t_{0.05,7} \approx \pm 1.895$

Decision rule: Reject H_0 if $|t| > 1.895$

Note:
The rejection region and the p-value were calculated in the two previous slides, respectively.

Step 4 – p-value

- Using t-distribution with 7 df:

P-value = 0.1
Approximate value (from the Student's t-distribution table)

$$p\text{-value} = 2 \cdot P(T > |t|) \approx 2 \cdot P(T > 1.82) \approx 0.11$$

Exact value

Step 5 – Conclusion

- Calculated t = -1.82 → $|t| < 1.895$
- p-value ≈ 0.11 > 0.10

Decision: Do not reject H_0

Decision rule: Reject H_0 if the test statistic t is in the rejection region (RR), or if p-value < α.

Note:
In the following slides, we will examine **both the left-tailed and right-tailed tests** to compare the results.

Interpretation: There is **not enough evidence** at the 10% significance level to conclude that the mean predicted score differs from 78.5.

EXERCISE 9.25: LEFT-TAILED TEST

One-Sample t-Test for the Mean (unknown Variance)

Answer:

Step 1 – Problem Setup

Left-tailed Test

- Sample size: $n = 8$
- Sample mean: $\bar{x} = 74.5$
- Sample standard deviation: $s \approx 6.23$
- Degrees of freedom: $df = 7$
- Test statistic: $t \approx -1.82$

Step 2 – Left-tailed test

- Null hypothesis: $H_0 : \mu \geq 78.5$
- Alternative hypothesis: $H_1 : \mu < 78.5$
- Significance level: $\alpha = 0.10$

Critical value: $t_{0.10,7} \approx -1.415$

RR =] -∞; -1.415]

Decision rule: Reject H_0 if $t < -1.415$

p-value: $P(T < -1.82) \approx 0.058$

Exact value

Note:
The rejection region and the p-value will be computed in the next two slides, respectively.

Conclusion:

- P-value ~ 0.058 < 0.1 → reject H_0
- t = -1.82 < -1.415 → **reject** H_0
- There is **enough evidence** at the 10% significance level to conclude that the mean predicted score is **less than 78.5**.

Note:
The value of the test statistic is the same for both two-tailed and one-tailed tests: $t = -1.82$.

CRITICAL VALUE $t_{\alpha;\, n-1}$: CALCULATION

Left-tailed Test → RR =] -∞; t_α]

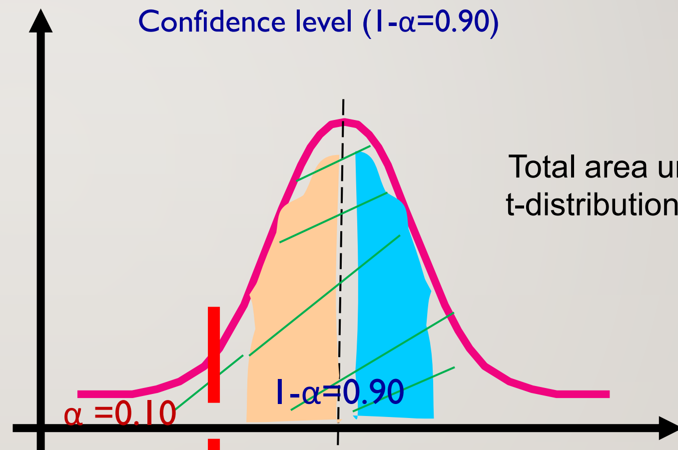
?

n	0,75	0,90	0,95	0,975	0,99	0,995	0,9995
1	1,000	3,078	6,314	12,706	31,821	63,657	636,619
2	0,816	1,886	2,920	4,303	6,965	9,925	31,598
3	0,765	1,638	2,353	3,182	4,541	5,841	12,941
4	0,741	1,533	2,132	2,776	3,747	4,604	8,610
5	0,727	1,476	2,015	2,571	3,365	4,032	6,859
6	0,718	1,440	1,943	2,447	3,143	3,707	5,959
7	0,711	1,415	1,895	2,365	2,998	3,499	5,405
8	0,706	1,397	1,860	2,306	2,896	3,355	5,041
9	0,703	1,383	1,833	2,262	2,821	3,250	4,781
10	0,700	1,372	1,812	2,228	2,764	3,169	4,587
11	0,697	1,363	1,796	2,201	2,718	3,106	4,437
12	0,695	1,356	1,782	2,179	2,681	3,055	4,318
13	0,694	1,350	1,771	2,160	2,650	3,012	4,221
14	0,692	1,345	1,761	2,145	2,624	2,977	4,140
15	0,691	1,341	1,753	2,131	2,602	2,947	4,073
16	0,690	1,337	1,746	2,120	2,583	2,921	4,015
17	0,689	1,333	1,740	2,110	2,567	2,898	3,965
18	0,688	1,330	1,734	2,101	2,552	2,878	3,922
19	0,688	1,328	1,729	2,093	2,539	2,861	3,883
20	0,687	1,325	1,725	2,086	2,528	2,845	3,850
21	0,686	1,323	1,721	2,080	2,518	2,831	3,819
22	0,686	1,321	1,717	2,074	2,508	2,819	3,792
23	0,685	1,319	1,714	2,069	2,500	2,807	3,767
24	0,685	1,318	1,711	2,064	2,492	2,797	3,745
25	0,684	1,316	1,708	2,060	2,485	2,787	3,725
26	0,684	1,315	1,706	2,056	2,479	2,779	3,707
27	0,684	1,314	1,703	2,052	2,473	2,771	3,690
28	0,683	1,313	1,701	2,048	2,467	2,763	3,674
29	0,683	1,311	1,699	2,045	2,462	2,756	3,659
30	0,683	1,310	1,697	2,042	2,457	2,750	3,646
[illegible]	[illegible]	[illegible]	1,684	2,021	2,423	2,704	3,551
[illegible]	[illegible]	[illegible]	1,671	2,000	2,390	2,660	3,460
[illegible]	[illegible]	[illegible]	1,658	1,980	2,358	2,617	3,373
[illegible]	[illegible]	[illegible]	1,645	1,960	2,326	2,576	3,291

Note:
The Student's t table reports left-tail probabilities: P(T ≤ t).

Significance level (α=0.10)

Confidence level (1-α=0.90)



Total area under the t-distribution pdf = 1.

$t_{0.10;7}$ = -$t_{0.90;7}$ = -1.415

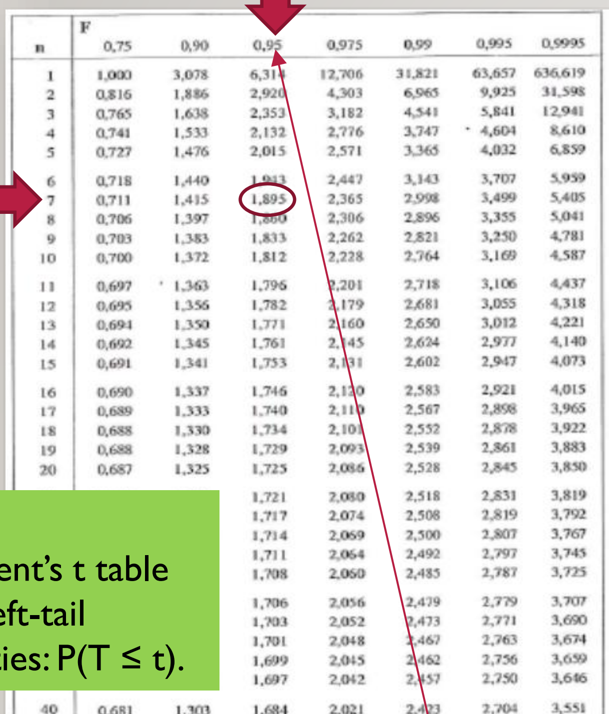
RR =] -∞; -1.415]

α = 0.10
1 - α = 0.90
$t_{\alpha;n-1}$ = -$t_{1-\alpha;n-1}$ = -$t_{0.90;7}$ = -1.415

P-VALUE FOR A STUDENT´S T-STATISTIC: CALCULATION

Left-tailed Test → P-value = $P(T \leq t_0)$

The value of the test statistic is $t_0 = -1.82$

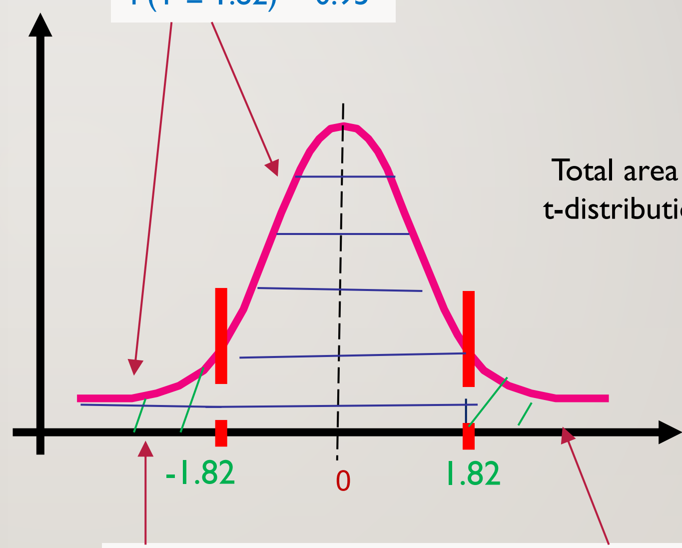


n	F 0,75	0,90	0,95	0,975	0,99	0,995	0,9995
1	1,000	3,078	6,314	12,706	31,821	63,657	636,619
2	0,816	1,886	2,920	4,303	6,965	9,925	31,598
3	0,765	1,638	2,353	3,182	4,541	5,841	12,941
4	0,741	1,533	2,132	2,776	3,747	4,604	8,610
5	0,727	1,476	2,015	2,571	3,365	4,032	6,859
6	0,718	1,440	1,943	2,447	3,143	3,707	5,959
7	0,711	1,415	1,895	2,365	2,998	3,499	5,405
8	0,706	1,397	1,860	2,306	2,896	3,355	5,041
9	0,703	1,383	1,833	2,262	2,821	3,250	4,781
10	0,700	1,372	1,812	2,228	2,764	3,169	4,587
11	0,697	1,363	1,796	2,201	2,718	3,106	4,437
12	0,695	1,356	1,782	2,179	2,681	3,055	4,318
13	0,694	1,350	1,771	2,160	2,650	3,012	4,221
14	0,692	1,345	1,761	2,145	2,624	2,977	4,140
15	0,691	1,341	1,753	2,131	2,602	2,947	4,073
16	0,690	1,337	1,746	2,120	2,583	2,921	4,015
17	0,689	1,333	1,740	2,110	2,567	2,898	3,965
18	0,688	1,330	1,734	2,101	2,552	2,878	3,922
19	0,688	1,328	1,729	2,093	2,539	2,861	3,883
20	0,687	1,325	1,725	2,086	2,528	2,845	3,850
			1,721	2,080	2,518	2,831	3,819
			1,717	2,074	2,508	2,819	3,792
			1,714	2,069	2,500	2,807	3,767
			1,711	2,064	2,492	2,797	3,745
			1,708	2,060	2,485	2,787	3,725
			1,706	2,056	2,479	2,779	3,707
			1,703	2,052	2,473	2,771	3,690
			1,701	2,048	2,467	2,763	3,674
			1,699	2,045	2,462	2,756	3,659
			1,697	2,042	2,457	2,750	3,646
40	0,681	1,303	1,684	2,021	2,423	2,704	3,551

Note:
The Student's t table reports left-tail probabilities: $P(T \leq t)$.

P-value = $P(T \leq t_0)$ ⇔
P-value = $P(T \leq -1.82)$ ⇔ P-value = $P(T \geq 1.82)$ ⇔
P-value ~ $P(T \geq 1.895)$ ⇔
P-value = $1-P(T < 1.895)) \sim (1-0.95) = 0.05$

$P(T \leq 1.82) \sim 0.95$



Total area under the t-distribution pdf = 1.

$P(T \leq -1.82) = P(T \geq 1.82) \sim 1-0.95= 0.05$

P-value = one green area = 0.05

EXERCISE 9.25: RIGHT-TAILED TEST

Answer:

One-Sample t-Test for the Mean (unknown Variance)

Step 1 – Problem Setup

Right-tailed Test

- Sample size: $n = 8$
- Sample mean: $\bar{x} = 74.5$
- Sample standard deviation: $s \approx 6.23$
- Degrees of freedom: $df = 7$
- Test statistic: $t \approx -1.82$

Step 2 - Right-tailed test

- Null hypothesis: $H_0 : \mu \leq 78.5$
- Alternative hypothesis: $H_1 : \mu > 78.5$
- Significance level: $\alpha = 0.10$

Critical value: $t_{0.90;7} \approx 1.415$

RR = [1.415; +∞[

Decision rule: Reject H_0 if $t > 1.415$

p-value: $P(T > -1.82) = 1 - P(T < -1.82) \approx 0.942$

Exact value

Note:
The rejection region and the p-value will be computed in the next two slides, respectively.

Conclusion:

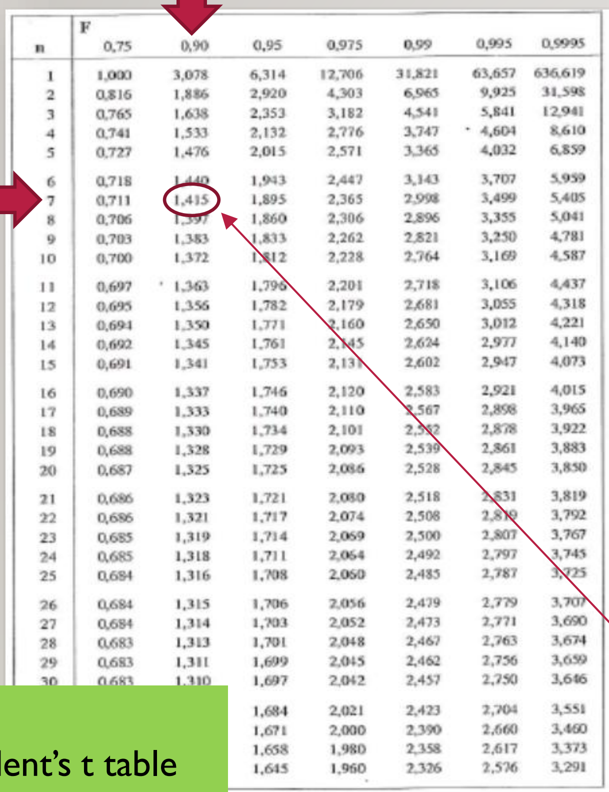
P-value ~ 0.942 > 0.1 → do not reject H_0

- t = -1.82 < 1.415 → **do not reject** H_0
- There is **not enough evidence** at the 10% significance level to conclude that the mean predicted score is **greater than 78.5**.

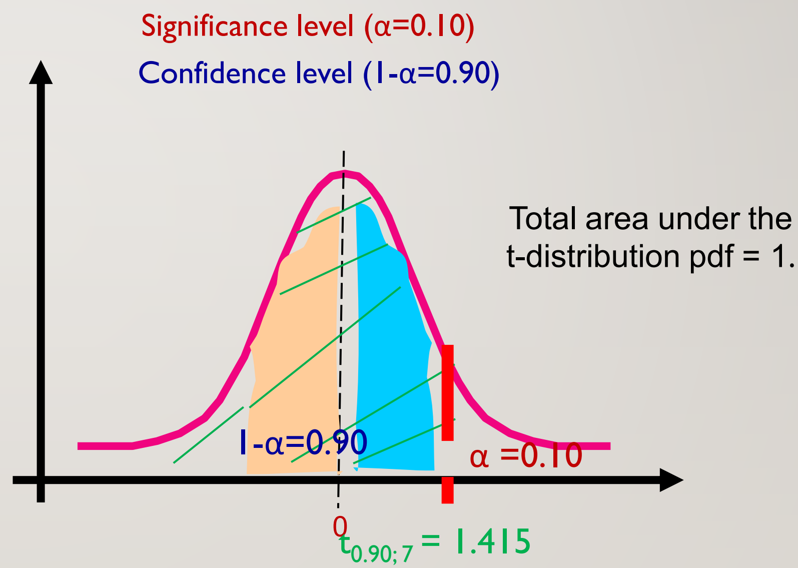
Note:
The value of the test statistic is the same for both two-tailed and one-tailed tests: $t = -1.82$.

CRITICAL VALUE $t_{1-\alpha;\, n-1}$: CALCULATION

Right-tailed Test → RR = $[t_{1-\alpha}; +\infty[$?



n	F 0,75	0,90	0,95	0,975	0,99	0,995	0,9995
1	1,000	3,078	6,314	12,706	31,821	63,657	636,619
2	0,816	1,886	2,920	4,303	6,965	9,925	31,598
3	0,765	1,638	2,353	3,182	4,541	5,841	12,941
4	0,741	1,533	2,132	2,776	3,747	4,604	8,610
5	0,727	1,476	2,015	2,571	3,365	4,032	6,859
6	0,718	1,440	1,943	2,447	3,143	3,707	5,959
7	0,711	1,415	1,895	2,365	2,998	3,499	5,405
8	0,706	1,397	1,860	2,306	2,896	3,355	5,041
9	0,703	1,383	1,833	2,262	2,821	3,250	4,781
10	0,700	1,372	1,812	2,228	2,764	3,169	4,587
11	0,697	1,363	1,796	2,201	2,718	3,106	4,437
12	0,695	1,356	1,782	2,179	2,681	3,055	4,318
13	0,694	1,350	1,771	2,160	2,650	3,012	4,221
14	0,692	1,345	1,761	2,145	2,624	2,977	4,140
15	0,691	1,341	1,753	2,131	2,602	2,947	4,073
16	0,690	1,337	1,746	2,120	2,583	2,921	4,015
17	0,689	1,333	1,740	2,110	2,567	2,898	3,965
18	0,688	1,330	1,734	2,101	2,552	2,878	3,922
19	0,688	1,328	1,729	2,093	2,539	2,861	3,883
20	0,687	1,325	1,725	2,086	2,528	2,845	3,850
21	0,686	1,323	1,721	2,080	2,518	2,831	3,819
22	0,686	1,321	1,717	2,074	2,508	2,819	3,792
23	0,685	1,319	1,714	2,069	2,500	2,807	3,767
24	0,685	1,318	1,711	2,064	2,492	2,797	3,745
25	0,684	1,316	1,708	2,060	2,485	2,787	3,725
26	0,684	1,315	1,706	2,056	2,479	2,779	3,707
27	0,684	1,314	1,703	2,052	2,473	2,771	3,690
28	0,683	1,313	1,701	2,048	2,467	2,763	3,674
29	0,683	1,311	1,699	2,045	2,462	2,756	3,659
30	0,683	1,310	1,697	2,042	2,457	2,750	3,646
[illegible]	[illegible]	[illegible]	1,684	2,021	2,423	2,704	3,551
[illegible]	[illegible]	[illegible]	1,671	2,000	2,390	2,660	3,460
[illegible]	[illegible]	[illegible]	1,658	1,980	2,358	2,617	3,373
[illegible]	[illegible]	[illegible]	1,645	1,960	2,326	2,576	3,291



RR = [1.415; +∞[

Note:
The Student's t table reports left-tail probabilities: P(T ≤ t).

α = 0.10
1 - α = 0.90
$t_{1-\alpha;n-1} = t_{0.90;7} = 1.415$

P-VALUE FOR A STUDENT´S T-STATISTIC: CALCULATION

Right-tailed Test → P-value = P(T ≥ t_0)

n	F 0,75	0,90	0,95	0,975	0,99	0,995	0,9995
1	1,000	3,078	6,314	12,706	31,821	63,657	636,619
2	0,816	1,886	2,920	4,303	6,965	9,925	31,598
3	0,765	1,638	2,353	3,182	4,541	5,841	12,941
4	0,741	1,533	2,132	2,776	3,747	4,604	8,610
5	0,727	1,476	2,015	2,571	3,365	4,032	6,859
6	0,718	1,440	1,943	2,447	3,143	3,707	5,959
7	0,711	1,415	1,895	2,365	2,998	3,499	5,405
8	0,706	1,397	1,860	2,306	2,896	3,355	5,041
9	0,703	1,383	1,833	2,262	2,821	3,250	4,781
10	0,700	1,372	1,812	2,228	2,764	3,169	4,587
11	0,697	1,363	1,796	2,201	2,718	3,106	4,437
12	0,695	1,356	1,782	2,179	2,681	3,055	4,318
13	0,694	1,350	1,771	2,160	2,650	3,012	4,221
14	0,692	1,345	1,761	2,145	2,624	2,977	4,140
15	0,691	1,341	1,753	2,131	2,602	2,947	4,073
16	0,690	1,337	1,746	2,120	2,583	2,921	4,015
17	0,689	1,333	1,740	2,110	2,567	2,898	3,965
18	0,688	1,330	1,734	2,101	2,552	2,878	3,922
19	0,688	1,328	1,729	2,093	2,539	2,861	3,883
20	0,687	1,325	1,725	2,086	2,528	2,845	3,850
				2,080	2,518	2,831	3,819
				2,074	2,508	2,819	3,792
				2,069	2,500	2,807	3,767
				2,064	2,492	2,797	3,745
				2,060	2,485	2,787	3,725
				2,056	2,479	2,779	3,707
				2,052	2,473	2,771	3,690
				2,048	2,467	2,763	3,674
				2,045	2,462	2,756	3,659
				2,042	2,457	2,750	3,646
40	0,681	1,303	1,684	2,021	2,423	2,704	3,551
							3,460
							3,373
							3,291

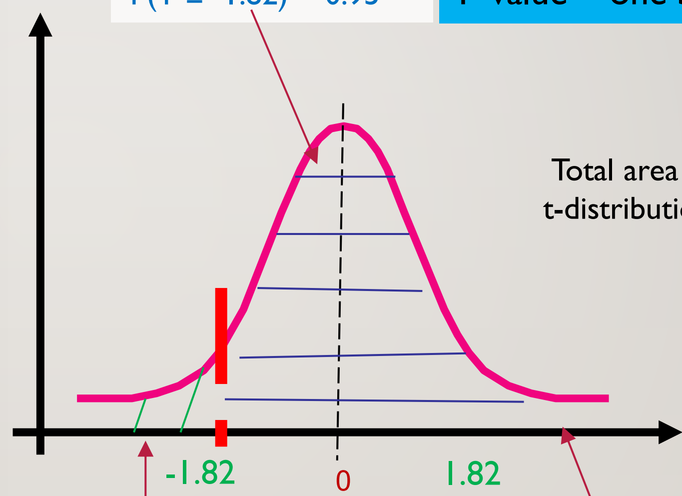
Note:
The Student's t table reports left-tail probabilities: P(T ≤ t).

P-value = P(T ≥ t_0) ⇔
P-value = P(T ≥ -1.82) ⇔ P(T ≤ 1.82)
P-value ~ P(T ≤ 1.895) = 0.95

The value of the test statistic is t_0 = -1.82

P(T ≥ -1.82) ~ 0.95

P-value = one blue area = 0.95



Total area under the t-distribution pdf = 1.

P(T ≤- 1.82) = P(T ≥ 1.82) ~ 1-0.95= 0.05

LECTURE 11: NON-PARAMETRIC HYPOTHESIS TESTS

CHI-SQUARE GOODNESS-OF-FIT TEST

Step 1: Hypotheses

- **Null hypothesis** (H_0): The observed frequencies follow the expected distribution.
- **Alternative hypothesis** (H_1): The observed frequencies do not follow the expected distribution.

Note:
The Chi-square goodness-of-fit test is used when we have a **categorical variable** (nominal or ordinal) and we want to test whether the **observed frequencies follow a specified theoretical distribution**.

Step 2: Test Statistic

$$Q = \sum_{i=1}^{K} \frac{(O_i - E_i)^2}{E_i} \sim \chi^2_{K-1}$$

- O_i = observed frequency
- $E_i = np_i$ = expected frequency
- n = total sample size
- p_i = theoretical probability for class i
- K = number of classes

Note:
The test statistic follows a Chi-square distribution with $K - 1$degrees of freedom.

Note:
Parametric tests rely on assumptions about the population distribution (typically normality), while non-parametric **tests do not require such assumptions**. Chi-square tests are non-parametric.

CHI-SQUARE GOODNESS-OF-FIT TEST

Step 3: Rejection Region

- Degrees of freedom: $df = K - 1$ (or $K - 1 - p$ if parameters are estimated)
- Significance level: α
- **Reject H_0 if**

$$Q \geq \chi^2_{1-\alpha,\, K-1}$$

RR = $[\chi^2_{1-\alpha;k-1}; +\infty[$

α = significance level
1 – α = confidence level
$\chi^2_{1-\alpha;k-1}$ = represents the 1 – α quantile of the Chi-square distribution with k-1degrees of freedom (see chi-square table)

Step 4: p-value

$$p\text{-value} = P(Q \geq q_0)$$

- Where q_0 is the observed value of the test statistic.
- The p-value indicates the probability of observing a test statistic as extreme as q_0, assuming H_0 is true.

Step 5: Decision

- **If q_0 falls in the rejection region** or **p-value < α** → reject H_0
- Otherwise → fail to reject H_0

EXERCISE 14.1

14.1 A random sample of 150 residents in one community was asked to indicate their first preference for one of three television stations that air the 5 p.m. news. The results obtained are shown in the following table. Test the null hypothesis that for this population their first preferences are evenly distributed over the three stations.

Station	A	B	C
Number of first preferences	47	42	61

Newbold et al (2013)

EXERCISE 14.1: SOLUTION

Answer:

Data:

Station	A	B	C
Observed O_i	47	42	61

- Sample size: $n = 150$

Goodness-of-Fit Test

Step 1: Hypotheses + Observed vs. Expected Frequencies

$$H_0 : p_A = p_B = p_C = \frac{1}{3} \quad vs. \quad H_a : \text{not all } p_i \text{ are equal}$$

Station	Observed O_i	Expected $E_i = 150/3$
A	47	50
B	42	50
C	61	50

$$E_i = n \cdot p_i = 150 \cdot \frac{1}{3} = 50$$

EXERCISE 14.1: SOLUTION

Answer:

Step 2: Test statistic (Chi-square)

$$q = \sum \frac{(O_i - E_i)^2}{E_i} = \frac{(47-50)^2}{50} + \frac{(42-50)^2}{50} + \frac{(61-50)^2}{50}$$

Compute each term:

$$\frac{(47-50)^2}{50} = \frac{(-3)^2}{50} = \frac{9}{50} = 0.18$$

$$\frac{(42-50)^2}{50} = \frac{(-8)^2}{50} = \frac{64}{50} = 1.28$$

$$\frac{(61-50)^2}{50} = \frac{(11)^2}{50} = \frac{121}{50} = 2.42$$

$$q = 0.18 + 1.28 + 2.42 = 3.88$$

The value of the test statistic is q = 3.88.

Step 3: Degrees of freedom

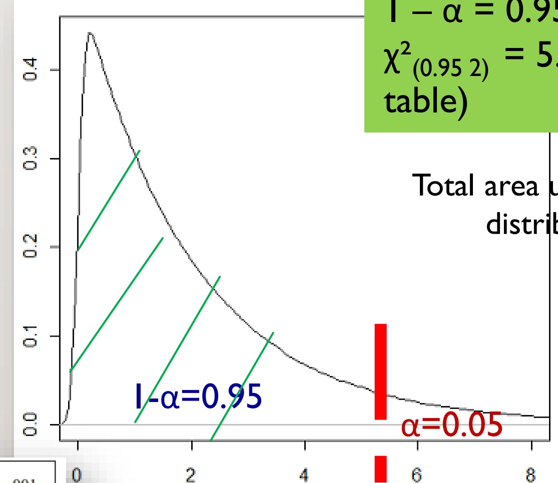
$$df = k - 1 = 3 - 1 = 2$$

CRITICAL VALUE $\chi^2_{1-\alpha;k-1}$: CALCULATION

Confidence Level (1-α = 0.95)

Significance Level (α=0.05)

α = 0.05 (significance level)
1 – α = 0.95 (confidence level)
$\chi^2_{(0.95\ 2)}$ = 5.991 (see chi-square table)



Total area under the chi-square distribution pdf = 1.

$\chi^2_{0.95;2}$ = 5.991

RR = [5.991; +∞[

Given a significance level of α = 0.05, we have 1 − α = 0.95.

$\chi^2_{0.95;2}$ = 5.991 (see chi-square table)

$\chi^2_{n.\varepsilon} : P(X > \chi^2_{n.\varepsilon}) = \varepsilon$

n \ ε	.995	.990	.975	.950	.900	.750	.500	.250	.100	.050	.025	.010	.005	.001
1	.000	.000	.001	.004	.016	.102	.455	1.323	2.706	3.841	5.024	6.635	7.879	10.827
2	.010	.020	.051	.103	.211	.575	1.386	2.773	4.605	5.991	7.378	9.210	10.597	13.815
3	.072	.115	.216	.352	.584	1.213	2.366	4.108	6.251	7.815	9.348	11.345	12.838	16.266
4	.207	.297	.484	.711	1.064	1.923	3.357	5.385	7.779	9.488	11.143	13.277	14.860	18.466
5	.412	.554	.831	1.145	1.610	2.675	4.351	6.626	9.236	11.070	12.832	15.086	16.750	20.515
6	.676	.872	1.237	1.635	2.204	3.455	5.348	7.841	10.645	12.592	14.449	16.812	18.548	22.457
7	.989	1.239	1.690	2.167	2.833	4.255	6.346	9.037	12.017	14.067	16.013	18.475	20.278	24.321
8	1.344	1.647	2.180	2.733	3.490	5.071	7.344	10.219	13.362	15.507	17.535	20.090	21.955	26.124
9	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	8.343	11.389	14.684	16.919	19.023	21.666	23.589	27.877
10	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	9.342	12.549	15.987	18.307	20.483	23.209	25.188	29.588

Note:
The chi-square table reports right-tail probabilities: P(Q ≥ q).

P-VALUE FOR A CHI-SQUARE STATISTIC: CALCULATION

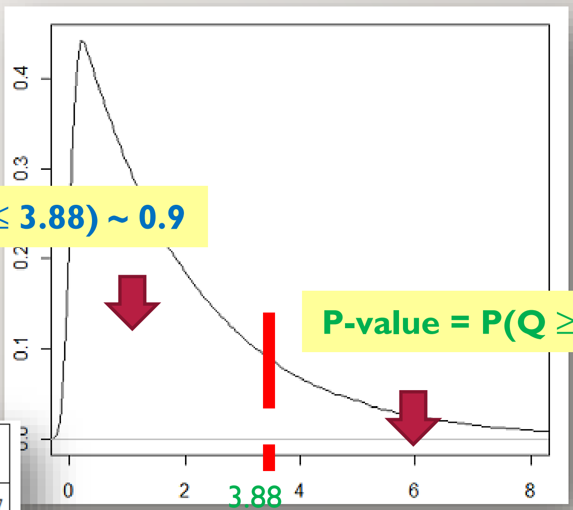
The value of the test statistic is $q_0 = 3.88$

Total area under the chi-square distribution pdf = 1.

P-value = P(Q ≥ 3.88) ~ P(Q ≥ 4.605) = 0.1

P(Q ≤ 3.88) ~ 0.9

P-value = P(Q ≥ 3.88) ~ 0.1



$\chi^2_{n.\varepsilon} : P(X > \chi^2_{n.\varepsilon}) = \varepsilon$

n \ ε	.995	.990	.975	.950	.900	.750	.500	.250	.100	.050	.025	.010	.005	.001
1	.000	.000	.001	.004	.016	.102	.455	1.323	2.706	3.841	5.024	6.635	7.879	10.827
2	.010	.020	.051	.103	.211	.575	1.386	2.773	4.605	5.991	7.378	9.210	10.597	13.815
3	.072	.115	.216	.352	.584	1.213	2.366	4.108	6.251	7.815	9.348	11.345	12.838	16.266
4	.207	.297	.484	.711	1.064	1.923	3.357	5.385	7.779	9.488	11.143	13.277	14.860	18.466
5	.412	.554	.831	1.145	1.610	2.675	4.351	6.626	9.236	11.070	12.832	15.086	16.750	20.515
6	.676	.872	1.237	1.635	2.204	3.455	5.348	7.841	10.645	12.592	14.449	16.812	18.548	22.457
7	.989	1.239	1.690	2.167	2.833	4.255	6.346	9.037	12.017	14.067	16.013	18.475	20.278	24.321
8	1.344	1.647	2.180	2.733	3.490	5.071	7.344	10.219	13.362	15.507	17.535	20.090	21.955	26.124
9	1.735	2.088	2.700	3.325	4.168	5.899	8.343	11.389	14.684	16.919	19.023	21.666	23.589	27.877
10	2.156	2.5[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]9	15.987	18.307	20.483	23.209	25.188	29.588

Note:
The chi-square table reports right-tail probabilities: P(Q ≥ q).

Decision rule based on the p-value:
P-value = P(Q ≥ q) < α ⇒ Reject H_0 for α

EXERCISE 14.1: SOLUTION

Answer:

Note:
The rejection region and the p-value were calculated in the two previous slides, respectively.

Step 4: Rejection region

- Significance level: $\alpha = 0.05$
- Critical value: $\chi^2_{0.95;2} \approx 5.991$
- **Reject** H_0 **if** q > 5.991

α = 0.05 (significance level)
1 – α = 0.95 (confidence level)
$\chi^2_{(0.95\ 2)}$ = 5.991 (see chi-square table)

RR = [5.991; +∞[

Step 5: P-value

$$p = P(\text{Q} > 3.88) \approx 0.144$$

Exact value

P-value = P(Q > 3.88) ~ 0.1
Approximate value (from the chi-table)

Step 6: Conclusion

- $\chi^2 = 3.88 < 5.991 \Rightarrow$ not in rejection region
- $p = 0.144 > 0.05$

Decision: Fail to reject H_0

Interpretation (slide-ready): There is **no statistically significant evidence** that first preferences are not evenly distributed among the three TV stations.

HOMEWORK OF LECTURE 11: QUESTIONS

EXERCISE 9.12

9.12 A company that receives shipments of batteries tests a random sample of nine of them before agreeing to take a shipment. The company is concerned that the true mean lifetime for all batteries in the shipment should be at least 50 hours. From past experience it is safe to conclude that the population distribution

Newbold et al (2013)

EXERCISE 9.26

9.26 An IT consultancy in Singapore that offers telephony solutions to small businesses claims that its new call-handling software will enable clients to increase successful inbound calls by an average of 75 calls per week. For a random sample of 25 small-business users of this software, the average increase in successful inbound calls was 70.2 and the sample standard deviation was 8.4 calls. Test, at the 5% level, the null hypothesis that the population mean increase is at least 75 calls. Assume a normal distribution.

Newbold et al (2013)

EXERCISE 14.2

14.2 A 2008 survey investigated favorite water sports in Australia, and it found out that 45% of the interviewees voted for surfing, 40% voted for scuba diving, and the rest voted for other water sports. In 2011, a similar survey was conducted; out of a sample of 200 respondents, 102 declared they prefer surfing, 82 chose scuba diving, and the remaining 16 selected other water sports. Is it possible to conclude at the 5% level that in 2011 these preferences remained the same?

Newbold et al (2013)

THANKS!

Questions?