Prob. theory and Stochastic Processes I ①

– First part – Exam 16/12/2025

a) $\# \mathcal{A}(I) = 2^4 = 16$

b) $\{5\} = \bigcap_{n=1}^{+\infty} \left]5 - \frac{1}{n}, 5\right]$

c) $0 = \mu(A) \neq \lim_{n \to +\infty} \mu(A_n) = +\infty$

$\lim \mu(A_n) = +\infty, \forall n \in \mathbb{N}.$

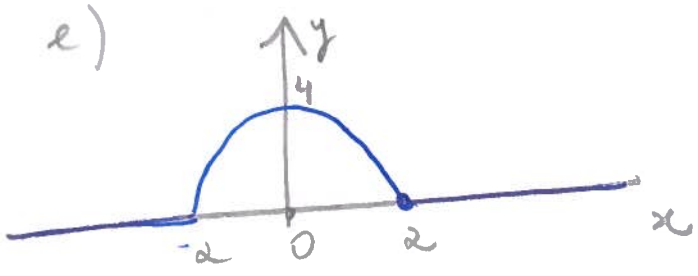
d) $m(B) = 1$

$\sum_{n=1}^{+\infty} \delta_{\frac{1}{n}}(A) = 4$

$\left\{\frac{\sqrt{2}}{2}\right\}$ for example

Vitali

e)



I ②

f) 1) 0 and $3/4$

2) $\Lambda_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$

3) 2^n

4) $m(\Lambda_n) = \frac{2^n}{3^n}$

5) Cantor, 0

6) uncountable

g) $f(x) = 5x$

h) $\int_A f\, d\mu = f(a_1) + f(a_2)$

i) $0 = \int_{[1,+\infty[} \lim_{n \to +\infty} \varphi_n \, dm \neq \lim_{n \to +\infty} \int_{[1,+\infty[} \varphi_n \, dm = 1$

$\varphi_n \nearrow 0$ does not hold.

j) 1) $f(r, \theta) = \frac{\cos(2\theta)}{r^2}$

2) $\int_0^{2\pi} |\cos 2\theta| d\theta \cdot \int_0^1 \frac{1}{r} dr.$

3) Fubini

k) 1) $\frac{2}{3} + \frac{1}{9} = \frac{7}{9}$

2) absolutely continuous

3) Radon-Nikodym

4) $f(0) = \frac{d\delta_0}{d\mu}(0) = 3/2$

l) 1) $\sum_{n=1}^{+\infty} P \circ X^{-1}(\{a_n\}) = 1 - \frac{1}{e^5}$

2) $E(e^{itX}) = e^{5(e^{it}-1)}$

3) $P(t) = 1 + 5it - \frac{30}{2}t^2$

m) absolutely continuous., 1

n) 1) $\sigma(X) = \{X^{-1}(B), B \in \mathcal{F}\}$

2) independent, $E(X|\mathcal{G}) = E(X)$

I 4

a) 1) recurrent positive

2) $T = \left(\frac{3}{10}, \frac{10}{13} \right)$

3) $\frac{10}{3}$ and $\frac{13}{10}$.

Probability theory and Stochastic Processes

Regular Assessment – 16/12/2025

II ①

Part II

[1]

$f \geqslant 0$ $\qquad A = \{x \in \mathbb{R} : f(x) \geqslant \lambda\}$, $\lambda > 0$.

$$\int f(x)\,d\mu(x) = \int_{\mathbb{R}} f(x)\,d\mu$$

$$\geqslant \int_A f(x)\,d\mu.$$

$$\geqslant \int_A \lambda\,d\mu. = \mu(A) \cdot \lambda$$

$$\therefore \boxed{\mu(A) \leqslant \frac{1}{\lambda} \int f(x)\,d\mu(x)}$$

II ②

2.a)

$$f_n(x) = n\, e^{-x^3} \sin\left(\frac{x^2}{n}\right) =$$

$$= x^2 e^{-x^3} \frac{\sin\left(\frac{x^2}{n}\right)}{\left(\frac{x^2}{n}\right)} \qquad x \neq 0$$

$$\xrightarrow{n \to +\infty} x^2 e^{-x^3}.$$

$$f_n(0) = 0.$$

Therefore: $f_n(x) \xrightarrow{n} x^2 e^{-x^3}, \; x \in \mathbb{R}_0^+$

b) $$|f_n(x)| = \left| x^2 e^{-x^3} \cdot \frac{\sin\left(\frac{x^2}{n}\right)}{\left(\frac{x^2}{n}\right)} \right| \leq x^2 e^{-x^3}$$

$|\sin x| \leq |x|, \forall x \in \mathbb{R}$

c) Let us use the dominated convergence theorem.

$\cdot\ f_n(x) \xrightarrow{n} x^2 e^{-x^3}$

$\cdot\ |f_n(x)| \leq \underbrace{x^2 e^{-x^3}}$

this map is integrable. Indeed,

$$\int_0^{+\infty} x^2 e^{-x^3}\,dx = \lim_{M\to+\infty} \int_0^M x^2 e^{-x^3}\,dx$$

$$= \lim_{M\to+\infty} \left[\frac{e^{-x^3}}{-3}\right]_0^{+\infty} = \lim_{M\to+\infty}\left(\frac{e^{-M^3}}{-3} + \frac{e^0}{3}\right)$$

$$= 1/3 < +\infty.$$

$\therefore \displaystyle\lim_{n\to+\infty} \int_{]0,+\infty[} f_n(x)\,dm(x) =$

$$\underset{\uparrow}{=} \int \lim_{n\to+\infty} f_n(x)\,dm(x) = \int x^2 e^{-x^3}\,dm(x)$$

$$= 1/3$$

Dominated Convergence theorem

(3).

$$F(X) = \begin{cases} 0 & x < 0 \\ \frac{x^2}{4} & 0 \le x < 2 \\ 1 & x \geq 2 \end{cases}$$

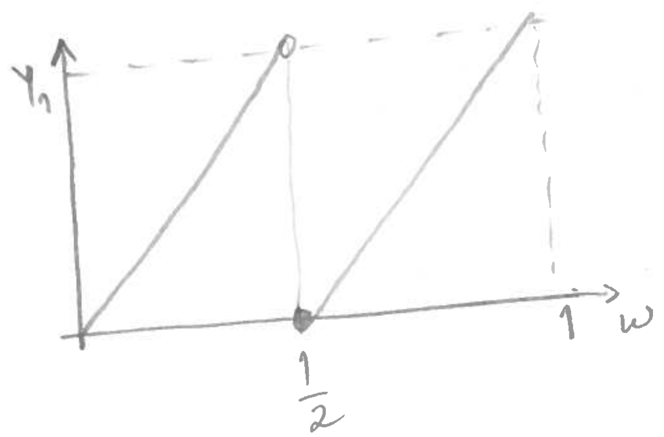
$$z = \sqrt{X}$$

$$P(Z \le z) = P(\sqrt{X} \le z) = P(X \le z^2) = F(z^2).$$

$$= \begin{cases} 0 & z^2 < 0 \\ \frac{z^4}{4} & 0 \le z^2 < 2 \\ 1 & z^2 \geq 2 \end{cases} = \begin{cases} 0 & z < 0 \\ \frac{z^4}{4} & 0 \le z < \sqrt{2} \\ 1 & z \geq \sqrt{2} \end{cases}$$

$\boxed{4}$

$$Y(\omega) = \begin{cases} 2\omega & 0 \le \omega < 1/2 \\ 2\omega - 1 & 1/2 \le \omega < 1 \end{cases}$$



II 5

$$\sigma(Y) = \{ Y^{-1}(B), B \in \beta([0,1[) \}$$

$$= A \dot{\cup} (A + 1/2), \; A \in \beta([0,\tfrac{1}{2}[).$$

$E(X|Y)$ is such that:

$$\int_B E(X|Y)\, dm = \int_B X \, dm, \quad \forall B \in \sigma(Y)$$

$$\Leftrightarrow \int_{A \dot{\cup} A+1/2} E(X|Y)\, dm = \int_A 2w^2\, dm + \int_{A+1/2} 2w^2\, dm$$

$$\Leftrightarrow 2\int_A E(X|Y)\, dm = \int_A 2w^2\, dm + \int_A 2(w+1/2)^2\, dm$$

$$(w+\tfrac{1}{2})^2 = w^2 + w + \tfrac{1}{4}$$

$$\Leftrightarrow 2\int_A E(X|Y)\, dm = \int_A 4w^2 + 2w + \tfrac{1}{2}\, dm$$

$$\Leftrightarrow \int_A E(X|Y)\, dm = \int_A 2w^2 + w + \tfrac{1}{4}\, dm$$

II (6)

$$E(X|Y) = \begin{cases} 2w^2 + w + \frac{1}{4} & w \in [0, \frac{1}{2}[\\ 2(w-\frac{1}{2})^2 + (w-\frac{1}{2}) + \frac{1}{4} & w \in [\frac{1}{2}, 1[\end{cases}$$

$$2(w-\tfrac{1}{2})^2 + (w-\tfrac{1}{2}) + \tfrac{1}{4} = 2w^2 - w + 1/4$$

$\boxed{5}$

$$\lim_{n\to+\infty} \sqrt[n]{x_1 \cdots x_n} = \lim_{n\to+\infty} x_1^{\frac{1}{n}} \cdots x_n^{1/n}$$

$$= \lim_{n\to+\infty} e^{Y_1/n} \cdots e^{Y_n/n} = \lim_{n\to+\infty} e^{\frac{Y_1+\cdots+Y_n}{n}} = (*)$$

$X_n \sim \mathcal{U}_{]0,1]}$

$$E(X_1) = \cdots = E(X_n) = \int_0^1 x \, dx$$

$$= \frac{x^2}{2}\Big|_0^1 = 1/2. \text{ (unnecessary)}$$

$$E(Y_1) = E(\ln X_1) = \int_0^1 \ln x \cdot 1 \, dx =$$

$$= \Big[-x + x \ln x\Big]_0^1 = -1$$

weak law of large numbers

$$(*) \overset{\downarrow}{=} e^{E(Y_1)} = e^{-1}$$