Mathematical Economics – 16/12/2025

Version A

Part I

a) $D_f = \{ y \geq 0 \wedge x \neq 0 \wedge x \neq 1 \wedge x \neq -1 \}$

b) $cl(A) = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4 \wedge y \geq x\}$

. A is not closed

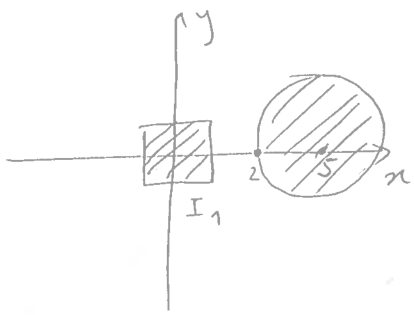
c) $\lim\limits_{n \to +\infty} f\left(-\frac{2}{n}\right) = 1$

$\lim\limits_{x \to -\infty} \frac{1}{f(x)} = -\infty$

d) $f(x,y) = \cos(x^2+y) + 7y$

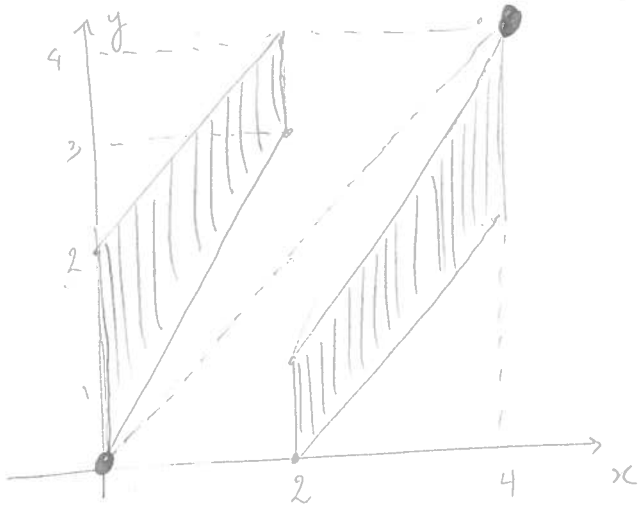
e) $\frac{\partial^2 f}{\partial x^{200}}(0,0) = 3^{200} e^{3.0 - 5.0} = 3^{200}$

f) non-empty, disjoint and convex.



$\boxed{x = 1}$ for example

(g.)



- closed graph property
- $\forall x \in [0,4]$: $H(\{x\})$ is Convex (~~for~~. $x = 2$, $H(\{x\})$ is not convex)
- $Fix(H) = \{0, 4\}$

h) . $f(x) = -2$ (for example)

- f is injective / monotonic

i) f is strictly concave ($\Delta_1 = -1$
$\Delta_2 = -1 + 4 = 3 > 0$)

$f(1,2)$ is an absolute maximum of f.

(j.)
$$\begin{cases} 6y-2-8\mu=0 \\ 6x-3\mu=0 \\ \mu\geq 0 \\ \mu(8x+3y-17)=0 \end{cases}$$

$\boxed{\mu=0}$

$\Downarrow$

$(x_0,y_0)\in \underline{int}\{(x,y)\in\mathbb{R}^2: 8x+3y\leq 17\}$

(K)

$$\underbrace{t+ye^{2ty}}_{\substack{f(t,y)\\ \parallel \\ \frac{\partial\phi}{\partial t}}}+\underbrace{ate^{2ty}}_{\substack{g(t,y)\\ \parallel \\ \frac{\partial\phi}{\partial y}}}\frac{dy}{dt}=0$$

$$\frac{\partial f}{\partial y}(t,y)=e^{2ty}+2tye^{2ty}$$

$$\frac{\partial g}{\partial t}(t,y)=ae^{2ty}+2yate^{2ty}$$

they are equal if $\boxed{a=1}$

(l) $y(x)=e^{5x}(\cos x+\sin x)$.

Solutions: $\lambda_1=5+i$ $\quad\lambda_2=5-i$

$\lambda_1+\lambda_2=10$

$\lambda_1.\lambda_2=25+1=26$

$$\boxed{y''-10y'+26y=0}$$

$\boxed{y(0)=1}$

$$y'(x) = 5e^{5x}(\cos x + \sin x) + e^{5x}(-\sin x + \cos x)$$

$\boxed{y'(0) = 5+1 = 6}$

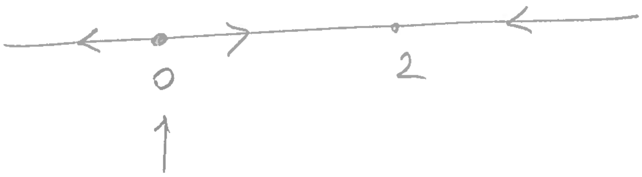
(m) $\begin{cases} \dot{x} = -x \\ \dot{y} = y \end{cases}$

(0,0) is unstable, Hartman-Grobmann

(n) $p' = 10p - 5p^2$

$p' = 5p(2-p)$

Equilibria: $\{0, 2\}$



$f(p) = 10p - 5p^2$

$f'(p) = 10 - 10p$

$f'(0) = 10 > 0$ $\quad f'(2) = -10 < 0$

- monotonic increasing
- $\lim_{t\to-\infty} p(t) = 0$ $\quad$ • inflexion point when $p = 1$.

o) $\det A > 0$ ✓

$\operatorname{Tr} A < 0$ ✓

(p) $F(t, x, \dot{x}) = 1 - x^2 - \dot{x}^2$

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}} = 0 \Leftrightarrow -2x - \frac{d}{dt}\left(-2\dot{x}\right) = 0$$

$$\Leftrightarrow -2x + 2\ddot{x} = 0 \Leftrightarrow$$

$$\Leftrightarrow \boxed{\ddot{x} - x = 0}$$ ✓

Solution: $x(t) = C_1 e^{t} + C_2 e^{-t}.$

$$\begin{cases} x(0) = 1 \\ x(1) = e \end{cases} \Leftrightarrow \begin{cases} C_1 + C_2 = 1 \\ C_1 e + C_2 e^{-1} = e \end{cases}$$

$$\Leftrightarrow \begin{cases} C_2 = 0 \\ C_1 = 1 \end{cases} \Leftrightarrow \boxed{x(t) = e^{t}}$$ ✓

F is <u>concave</u> in $(x, \dot{x})$

✓

Part II

a) $\cdot\ [\frac{3}{2}, +\infty[$ is a complete metric space

$\cdot\ |F'(x)| = \left|-\frac{1}{x^2}\right| \leq 1 \quad \forall x \in [\frac{3}{2}, +\infty[$

$\Downarrow$

F is a contraction.

b) $F(x) = x \Leftrightarrow 1 + \frac{1}{x} = x \Leftrightarrow x + 1 = x^2 \Leftrightarrow x^2 - x - 1 = 0$

$\Leftrightarrow x = \frac{1 \pm \sqrt{1+4.1}}{2} \Leftrightarrow \boxed{x = \frac{1+\sqrt{5}}{2}} \vee x = \frac{1-\sqrt{5}}{2} \notin D_F$ (X)

(golden number).

c) $\cdot\ (x_{n+1})_n$ is a subsequence of $(x_n)_n$ $\Big\}$ F is continuous $\Rightarrow (x_{n+1})_n \longrightarrow F(\ell) = \ell$

$\cdot\ (x_n)_n \longrightarrow \ell$

$\lim_{n \to +\infty} x_n = \frac{1+\sqrt{5}}{2}$ ✓

2) $f(x,y) = xy e^{x+y}$

a) $\nabla f(x,y) = \left(y.e^{x+y} + xye^{x+y} \, ; \, xe^{x+y} + xye^{x+y} \right)$

$$\nabla f(x,y) = \bar{0} \iff \begin{cases} y + xy = 0 \\ x + xy = 0 \end{cases} \iff \begin{cases} y(1+x) = 0 \\ x(1+y) = 0 \end{cases}$$

$$\iff \begin{cases} y = 0 \\ x = 0 \end{cases} \vee \begin{cases} x = -1 \\ y = -1 \end{cases}$$

Critical points: $(0,0)$ and $(-1,-1)$.

$$H_f(x,y) = \begin{pmatrix} ye^{x+y} + ye^{x+y} + xye^{x+y} & ; \; e^{x+y} + ye^{x+y} + xe^{x+y} + xye^{x+y} \\ e^{x+y} + ye^{x+y} + xye^{x+y} + xe^{x+y} & xe^{x+y} + xe^{x+y} + xye^{x+y} \end{pmatrix}$$

$$H_f(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad H_f(-1,-1) = \begin{pmatrix} -e^{-2} & 0 \\ 0 & -e^{-2} \end{pmatrix}$$

$\Delta_1 = 0$

$\Delta_2 = -1 \neq 0$

$\Downarrow$

H_f defines an undefined QF

$\Downarrow$

$(0,0)$ is a saddle.

eigenvalues are negative

$\Downarrow$

H_f defines a NDQF

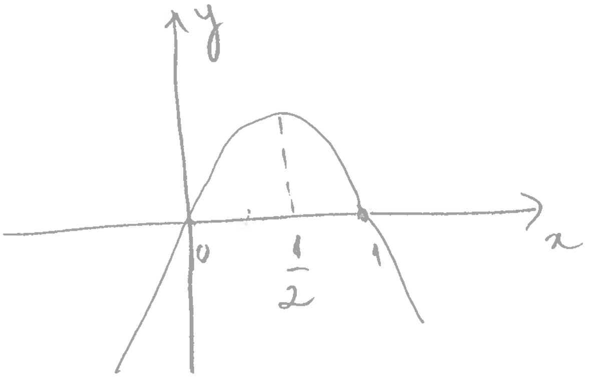
$\Downarrow$

$(-1,-1)$ is a local maximizer.

2.b)

$$f(x,y) = xye^{x+y}$$

$$f(x,1-x) = x(1-x)\,e^{\cancel{x}+1-\cancel{x}} = e\,x(1-x)$$



Extreme of f:

$$f\left(\tfrac{1}{2},\tfrac{1}{2}\right) = \left(\tfrac{1}{2}\right)^2 e^{1}$$

(maximum)

3)

$$\begin{cases} \boxed{x^2y' + xy = x^3} \\ y(1) = 4/3 \end{cases}$$

$$x^2y' + xy = x^3$$

$$\overset{x\neq 0}{\iff} y' + \underbrace{\frac{1}{x}}_{a(x)} y = \underbrace{x}_{b(x)}.$$

$$y(x) = \frac{\int e^{\int a(x)\,dx}\cdot b(x)\,dx + C}{e^{\int a(x)\,dx}}$$

$$y(x) = \frac{\int e^{\int \frac{1}{x}\,dx}\cdot x\,dx + C}{e^{\int \frac{1}{x}\,dx}} \iff y(x) = \frac{\int x^2\,dx + C}{x}$$

$$y(x) = \frac{\frac{x^3}{3} + C}{x} \iff \boxed{y(x) = \frac{x^2}{3} + \frac{C}{x}} \quad x \neq 0.$$

$$y(1) = \frac{4}{3} \iff \frac{4}{3} = \frac{1}{3} + C \iff C = 1$$

Solution: $y(x) = \frac{x^2}{3} + \frac{1}{x}$, $\underline{x \in \mathbb{R}^+}$

$\boxed{4}$

a) $\begin{cases} \dot{x} = x + y \\ \dot{y} = 3x + 3y \end{cases}$ $\quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$

eigenvalues of A: $\{0, 4\}$

$$P(\lambda) = \det \begin{pmatrix} 1-\lambda & 1 \\ 3 & 3-\lambda \end{pmatrix} = (1-\lambda)(3-\lambda) - 3$$

$$= \cancel{3} - \lambda - 3\lambda + \lambda^2 - \cancel{3}$$

$$= \lambda^2 - 4\lambda = \lambda(\lambda - 4)$$

E_0: $\begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{cases} x + y = 0 \\ \text{———} \end{cases} \iff \begin{cases} y = -x \\ \text{———} \end{cases}$

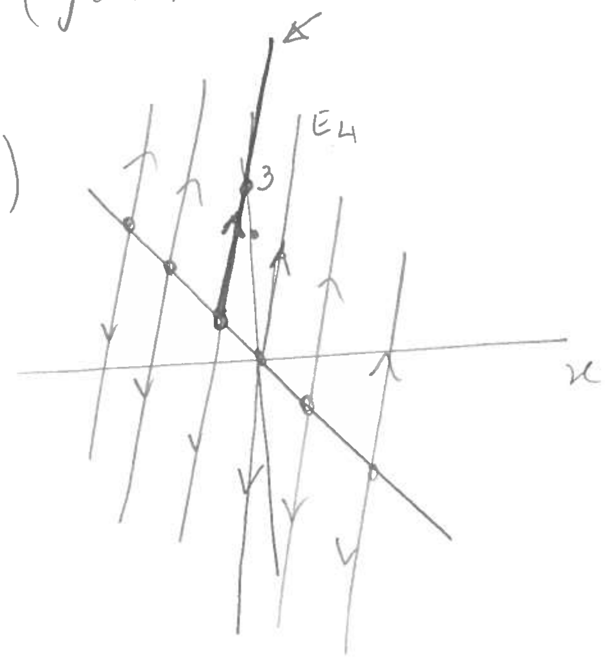
$\boxed{E_0 = \langle (1, -1) \rangle}$ $\quad \boxed{E_4 = \langle (1, 3) \rangle}$

E_4: $\begin{pmatrix} -3 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{cases} -3x + y = 0 \\ \text{———} \end{cases} \iff \begin{cases} y = 3x \\ \text{———} \end{cases}$

general form:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{4t} \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad c_1, c_2 \in \mathbb{R}.$$

b)



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Equivalent problem:

$$(*) \quad \max_{u \in A} \int_0^1 (-tx - u^2 + 1)\, dt \qquad \dot{x} = u(t)$$

$$H(t, x, u, p) = -tx - u^2 + 1 + \underline{p \cdot u}$$

$$\frac{\partial H}{\partial u}(t, x, u, p) = -2u + p \qquad \boxed{-2u + p = 0} \text{ 1st condition}$$

$$2u = p$$

2nd condition

$$\begin{cases} \dot{x} = \dfrac{\partial H}{\partial p} \\ \dot{p} = -\dfrac{\partial H}{\partial x} \end{cases} \Leftrightarrow \begin{cases} \dot{x} = u \\ \dot{p} = +t \end{cases} \Leftrightarrow \begin{cases} \dot{x} = p/2 \\ \dot{p} = t \end{cases}$$

$$\begin{cases} \dot{x} = p/2 \\ \boxed{\dot{p} = t} \end{cases} \iff \begin{cases} \boxed{\dot{x} = \frac{t^2}{4} + \frac{c_1}{2}} \\ p(t) = \frac{t^2}{2} + c_1 \end{cases}$$

$$\Downarrow$$

$$p(t) = \frac{t^2}{2} + c_1$$

$$x(t) = \frac{t^3}{12} + \frac{c_1}{2} t + c_2$$

$c_1, c_2 \in \mathbb{R}$.

We know:

$$x(0) = 1 \implies c_2 = 1$$

$$x(1) \in \mathbb{R} \implies p(1) = 0 \iff \frac{1}{2} + c_1 = 0 \implies c_1 = -1/2.$$

Control:

$$u(t) = \frac{p(t)}{2} = \frac{t^2}{4} + \frac{c_1}{2} = \frac{t^2}{4} - \frac{1}{4}.$$

State:

$$x^*(t) = \frac{t^3}{12} + \frac{c_1}{2} t + c_2 = \frac{t^3}{12} - \frac{1}{4} t + 1$$

Co-state

$$p(t) = \frac{t^2}{2} - \frac{1}{2}.$$

$$\mathcal{H}(t, x, u, p) = -tx - u^2 + 1 + pu$$

$$\nabla H(x,u) = (-t \,;\, -2u + p)$$

$$H_H(x,u) = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}$$

eigenvalues: $0, -2$

$\Downarrow$

H is concave (not strictly)

$\Downarrow$

x^* is the solution of $(*)$, $t \in [0,1]$.

the End!