ISEG Lisbon School of Economics & Management
Universidade de Lisboa

STATISTICS I

Bachelor's degrees in Economics and Finance

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Pratical Class 7

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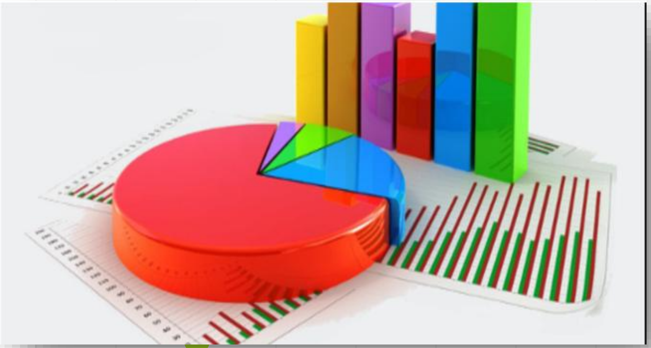

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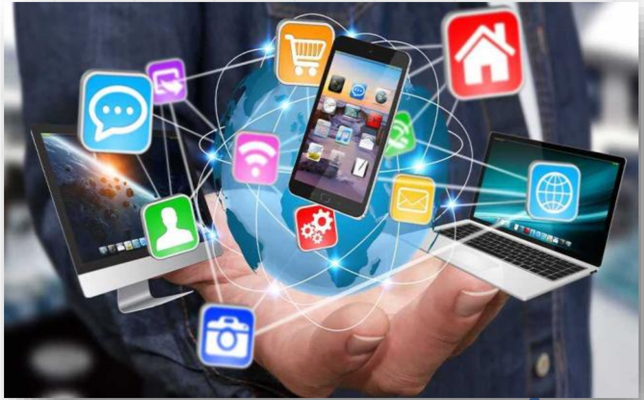

https://basiccode.com.br/produto/informatica-basica/

1. Basic Probability Theory.
2. Univariate random variables.
3. Expected Values.
4. Multivariate random variables (random vectors).
5. Expected Values of Functions of Random Vectors.
6. Special Random Variables and Repeated Sampling Distributions.

Bibliography:

- Miller & Miller, John E. , Freund’s Mathematical Statistics with applications , 8th Edition, Pearson Education, [MM], 2013
- P. Newbold, W. Carlson, B. Thorne, , Statistics for Business and Economics , 8th Edition, Pearson Education, [N], 2012

Exercises: 2, 3, 5, 7, 8, 9, 10, 12

Two-Dimensional Random Variables: Exercises

Joint Distribution, Marginal Distribution and Conditional Distribution Functions

Chapter 4

1

1. Let (X, Y) be a two-dimensional random variable such that its cumulative distribution function is given by

$$F_{X,Y}(x, y) = \begin{cases} 0, & x < 0 \text{ or } y < 0 \\ \frac{x+y}{2}, & 0 \leq x < 1 \text{ and } 0 \leq y < 1 \\ \frac{1+y}{2}, & x \geq 1 \text{ and } 0 \leq y < 1 \\ \frac{x+1}{2}, & 0 \leq x < 1 \text{ and } y \geq 1 \\ 1, & x \geq 1 \text{ and } y \geq 1 \end{cases}$$

Compute the marginal cumulative distribution functions of X and Y: F_X and F_Y.

- The Marginal cumulative distribution function of X :
  $F_X(x) = P(X \leq x) = P(X \leq x, Y \leq +\infty) = \lim_{y \to +\infty} F_{X,Y}(x, y)$.
- The Marginal cumulative distribution function of Y :
  $F_Y(y) = (Y \leq y) = P(X \leq +\infty, Y \leq y) = \lim_{x \to +\infty} F_{X,Y}(x, y)$.

Exercise 1

$$F_{X,Y}(x,y) = \begin{cases} 0, & x<0 \text{ or } y<0 \\ \frac{x+y}{2}, & 0\le x<1 \text{ and } 0\le y<1 \\ \frac{1+y}{2}, & x\ge 1 \text{ and } 0\le y<1 \\ \frac{x+1}{2}, & 0\le x<1 \text{ and } y\ge 1 \\ 1, & \underbrace{x\ge 1}_{x \to +\infty} \text{ and } y\ge 1 \end{cases}$$

$y \to +\infty$

$$F_X(x) = \lim_{y\to+\infty} F_{X,Y}(x,y) = \begin{cases} 0 & (x<0) \\ \frac{x+1}{2} & (0\le x<1) \\ 1 & (x\ge 1) \end{cases}$$

$$F_Y(y) = \lim_{x\to+\infty} F_{X,Y}(x,y) = \begin{cases} 0 & (y<0) \\ \frac{1+y}{2} & (0\le y<1) \\ 1 & (y\ge 1) \end{cases}$$

2. Let X and Y be two independent random variables such that

$$F_X(x) = \begin{cases} 0, & x < 0 \\ \frac{x}{2}, & 0 \le x < 2\,, \\ 1, & x \ge 2 \end{cases} \quad \text{and} \quad F_Y(y) = \begin{cases} 0, & y < 0 \\ \frac{y}{3}, & 0 < y < 3\,. \\ 1, & y \ge 3 \end{cases}$$

a) Find the joint density function of X and Y.

b) Compute the marginal density function of X and Y

Exercise 2 a)

a)

Because $X \perp Y$ we have:

$$F_{X,Y}(x,y) = F_X(x)\, F_Y(y) = \begin{cases} 0 & (x<0 \lor y<0) \\ \frac{xy}{6} & (0 \leq x < 2,\ 0 \leq y < 3) \\ \frac{x}{2} & (0 \leq x < 2,\ y \geq 3) \\ \frac{y}{3} & (x \geq 2,\ 0 \leq y < 3) \\ 1 & (x \geq 2,\ y \geq 3) \end{cases}$$

The joint density is:

↳ Note: The solutions only show $F_{X,Y}(x,y)$

$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y) = \begin{cases} \frac{1}{6} & (0<x<2,\ 0<y<3) \\ 0 & (\text{otherwise}) \end{cases}$$

Auxiliary calculations:

$$\frac{\partial^2}{\partial x \partial y}(0) = \frac{\partial^2}{\partial x \partial y}(1) = 0$$

$$\frac{\partial^2}{\partial x \partial y}\left(\frac{xy}{6}\right) = \frac{\partial}{\partial x}\left(\frac{x}{6}\right) = \frac{1}{6}$$

$$\frac{\partial^2}{\partial x \partial y}\left(\frac{x}{2}\right) = \frac{\partial^2}{\partial x \partial y}\left(\frac{y}{3}\right) = 0$$

Exercise 2 b)

b)

$$f_X(x) = \frac{d}{dx} F_X(x) = \begin{cases} \frac{1}{2} & (0 < x < 2) \\ 0 & (\text{otherwise}) \end{cases}$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \begin{cases} \frac{1}{3} & (0 < y < 3) \\ 0 & (\text{otherwise}) \end{cases}$$

Alternative solution:

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\, dy = \begin{cases} \int_0^3 \frac{1}{6}\, dy = \frac{1}{6}\left[y\right]_{y=0}^{y=3} = \frac{3}{6} = \frac{1}{2} & (0 < x < 2) \\ 0 & (\text{otherwise}) \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\, dx = \begin{cases} \int_0^2 \frac{1}{6}\, dy = \frac{1}{6}\left[x\right]_{x=0}^{x=2} = \frac{2}{6} = \frac{1}{3} & (0 < y < 3) \\ 0 & (\text{elsewhere}) \end{cases}$$

3. If the values of the joint probability function of X and Y are as shown in the table

Y \ X	0	1	2
0	1/12	1/6	1/24
1	1/4	1/4	1/40
2	1/8	1/20	0
3	1/120	0	0

(a) find:

i. $P(X = 1, Y = 2)$;

ii. $P(X = 0, 1 \leq Y < 3)$;

iii. $P(X + Y \leq 1)$;

iv. $P(X > Y)$.

(b) find the following values of the joint cumulative distribution function of the two random variables:

i. $F(1.2, 0.9)$;

ii. $F(-3, 1.5)$;

iii. $F(2, 0)$;

iv. $F(4, 2.7)$.

Exercise 3 a)

$f_{X,Y}(x,y)$:

Y \ X	0	1	2
0	1/12	1/6	1/24
1	1/4	1/4	1/40
2	1/8	1/20	0
3	1/120	0	0

(iii) (ii) (i) (iv)

a)

(i) $P(X=1, Y=2) = f_{X,Y}(1,2) = \frac{1}{20}$

(ii) $P(X=0,\ 1 \leq Y < 3) = \sum_{(x,y):\, x=0 \,\wedge\, 1\leq y<3} f_{X,Y}(x,y) = \sum_{y=1}^{2} f_{X,Y}(0,y) =$

$= f_{X,Y}(0,1) + f_{X,Y}(0,2) = \frac{1}{4} + \frac{1}{8} = \frac{3}{8}$

(iii) $P(X+Y \leq 1) = \sum_{(x,y)\in D_{X,Y}:\, x+y\leq 1} f_{X,Y}(x,y) = f_{X,Y}(0,0) + f_{X,Y}(1,0) + f_{X,Y}(0,1) = \frac{1}{12} + \frac{1}{6} + \frac{1}{4} =$

$= \frac{2}{24} + \frac{4}{24} + \frac{1}{4} = \frac{6}{24} + \frac{1}{4} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

(iv) $P(X>Y) = \sum_{(x,y)\in D_{X,Y}:\, x>y} f_{X,Y}(x,y) = f_{X,Y}(1,0) + f_{X,Y}(2,0) + f_{X,Y}(2,1) = \frac{1}{6} + \frac{1}{24} + \frac{1}{40} = \frac{5}{24} + \frac{1}{40} =$

$= \frac{200}{960} + \frac{24}{960} = \frac{224}{960} = \cdots = \frac{7}{30}$

Exercise 3 b)

b)

(i) $F_{X,Y}(1.2, 0.9) = F_X(1,0) = P(X \leq 1, Y \leq 0) = \sum_{(x,y) \in D_{X,Y} :\, x \leq 1 \wedge y \leq 0} f_{X,Y}(x,y) = f_{X,Y}(1,0) + f_X(0,0) = \frac{1}{6} + \frac{1}{12} = \frac{3}{12} = \frac{1}{4}$

(or:) $= \sum_{(x,y) \in D_{X,Y} :\, x \leq 1.2 \wedge y \leq 0.9} f_{X,Y}(x,y) =$

(ii) $F_{X,Y}(-3, 1.5) = F_X(-3,1) = P(X \leq -3, Y \leq 1) = \sum_{(x,y) :\, D_{X,Y} :\, x \leq -3 \wedge y \leq 1} f_{X,Y}(x,y) = 0$

(or:) $= \sum_{(x,y) \in D_{X,Y} :\, x \leq -3 \wedge y \leq 1.5} f_{X,Y}(x,y) =$

(iii) $F_{X,Y}(2,0) = P(X \leq 2, Y \leq 0) = \sum_{(x,y) \in D_{X,Y} :\, x \leq 2 \wedge y \leq 0} f_{X,Y}(x,y) = f_{X,Y}(2,0) + f_{X,Y}(1,0) + f_{X,Y}(0,0) = \frac{1}{24} + \frac{1}{6} + \frac{1}{12} = \frac{1}{24} + \frac{4}{24} + \frac{2}{24} = \frac{7}{24}$

(iv) $F_{X,Y}(4, 2.7) = F_{X,Y}(2,2) = P(X \leq 2, Y \leq 2) = 1 - P(X > 2 \vee Y > 2) = 1 - f_{X,Y}(0,3) = 1 - \frac{1}{120} = \frac{119}{120}$

(or:) $= \sum_{(x,y) \in D_{X,Y} :\, x \leq 4 \wedge y \leq 2.7} f_{X,Y}(x,y) = \sum_{(x,y) \in D_{X,Y} :\, x \leq 2 \wedge y \leq 2} f_{X,Y}(x,y) =$

Y \ X	0	1	2
0	1/12	1/6	1/24
1	1/4	1/4	1/40
2	1/8	1/20	0
3	1/120	0	0

4. If the joint probability distribution of X and Y is given by

$$f_{X,Y}(x,y) = c(x^2 + y^2), \quad \text{for } x = 1, 3; \quad y = -1, 2.$$

a) Find the value of c;

b) Compute $P(X + Y > 2)$;

c) Compute the cumulative distribution function.

$$F_{X,Y}(x,y) = P(X \leq x, Y \leq y)$$

Properties of the joint cumulative distribution function:

- $0 \leq F_{X,Y}(x,y) \leq 1$
- $F_{X,Y}(x,y)$ is non decreasing with respect to x and y :
  - $\Delta_x > 0 \Rightarrow F_{X,Y}(x + \Delta x, y) \geq F_{X,Y}(x,y)$
  - $\Delta_y > 0 \Rightarrow F_{X,Y}(x, y + \Delta y) \geq F_{X,Y}(x,y)$
- $\lim_{x \to -\infty} F_{X,Y}(x,y) = 0$, $\lim_{y \to -\infty} F_{X,Y}(x,y) = 0$ and $\lim_{x \to +\infty, y \to +\infty} F_{X,Y}(x,y) = 1$
- $P(x_1 < X \leq x_2, y_1 < Y \leq y_2) =$
  $F_{X,Y}(x_2,y_2) - F_{X,Y}(x_1,y_2) - F_{X,Y}(x_2,y_1) + F_{X,Y}(x_1,y_1)$.
- $F_{X,Y}(x,y)$ is right continuous with respect to x and y:
  $\lim_{x \to a^+} F_{X,Y}(x,y) = F_{X,Y}(a,y)$ and $\lim_{y \to b^+} F_{X,Y}(x,y) = F_{X,Y}(x,b)$.

Exercise 4 a)

$f_{X,Y}(x,y) = c(x^2+y^2) \quad (x = 1, 3;\ y = -1, 2)$

a)

$f_{X,Y}(x,y)$ is a f.m.f. so:

$$\sum_{(x,y)\in D_{X,Y}} f_{X,Y}(x,y) = 1 \Leftrightarrow c\ 1^2 + (-1)^2 + c(1^2+2^2) + c(3^2+(-1)^2) + (3^2+2^2) = 1 \Leftrightarrow$$

$$\Leftrightarrow c(1+1+1+4+9+1+9+4) = 1 \Leftrightarrow$$

$$\Leftrightarrow 30c = 1 \Leftrightarrow c = \frac{1}{30}$$

Exercise 4 b)

b)

$$P(X+Y>2) = \sum_{(x,y)\in D_{X,Y}:\ x+y>2} f_{X,Y}(x,y) = f_{X,Y}(1,2) + f_{X,Y}(3,2) =$$

$$= \frac{1}{30}(1^2+2^2) + \frac{1}{30}(3^2+2^2) = \frac{1}{30}(1+4+9+4) = \frac{18}{30} = \frac{9}{15} = \frac{3}{5}$$

Exercise 4 c)

e)

$$F(x, y) = \begin{cases} 0 & (x < -1 \lor y < -1) \\ f_{X,Y}(-1,-1) = \frac{2}{30} = \frac{1}{15} & (-1 \leq x < 3;\ -1 \leq y < 2) \\ f_{X,Y}(-1,-1) + f_{X,Y}(-1,2) = \frac{2}{30} + \frac{5}{30} = \frac{7}{30} & (-1 \leq x < 3;\ y \geq 2) \\ f_{X,Y}(-1,-1) + f_{X,Y}(3,-1) = \frac{2}{30} + \frac{10}{30} = \frac{12}{30} = \frac{6}{15} = \frac{2}{5} & (x \geq 3;\ -1 \leq y < 2) \\ 1 & (x \geq 3;\ y \geq 2) \end{cases}$$

5. Given the values of the joint probability distribution of X and Y shown in the table

	$X=-1$	$X=1$
$Y=-1$	$\frac{1}{8}$	$\frac{1}{2}$
$Y=0$	0	$\frac{1}{4}$
$Y=1$	$\frac{1}{8}$	0

a) Find the marginal probability function of X.

b) Find the marginal probability function of Y.

c) Find the conditional probability function of X given $Y=-1$.

d) Compute the conditional cumulative distribution function of X given $Y=-1$.

e) Verify if X and Y are independent.

Exercise 5 a)

$f_{X,Y}(x,y)$	$X=-1$	$X=1$
$Y=-1$	$\frac{1}{8}$	$\frac{1}{2}$
$Y=0$	0	$\frac{1}{4}$
$Y=1$	$\frac{1}{8}$	0

$D_X = \{-1, 1\}$

$D_Y = \{-1, 0, 1\}$

a)

$$f_X(x) = \sum_{y \in D_Y} f_{X,Y}(x,y) = \begin{cases} \frac{1}{8} + \frac{1}{8} = \frac{2}{8} = \frac{1}{4} & (x=-1) \\ \frac{1}{2} + \frac{1}{4} = \frac{3}{4} & (x=1) \\ 0 & (\text{elsewhere}) \end{cases}$$

Exercise 5 b)

b)

$$f_Y(y) = \sum_{x \in D_X} f_{X,Y}(x,y) = \begin{cases} \frac{1}{8} + \frac{1}{2} = \frac{1}{8} + \frac{4}{8} = \frac{5}{8} & (y = -1) \\ \frac{1}{4} & (y = 0) \\ \frac{1}{8} & (y = 1) \\ 0 & (\text{elsewhere}) \end{cases}$$

Exercise 5 c)

c)

$$f_{X|Y=-1}(x) = \frac{f_{X,Y}(x,-1)}{f_Y(-1)} = \begin{cases} \frac{1}{8} / \frac{5}{8} = \frac{1}{5} & (x = -1) \\ \frac{1}{2} / \frac{5}{8} = \frac{4}{5} & (x = 1) \\ 0 & (\text{elsewhere}) \end{cases}$$

Exercise 5 d)

d)

$$\boxed{F_{X|Y=-1}(x) = \sum_{x' \in D_x :\, x' \leq x} f_{X|Y=-1}(x')}$$

$x = -1$

$x = 1$

<u>Auxiliary calculations:</u>

$$F_{X|Y=-1}(-1) = f_{X|Y=-1}(-1) = \frac{1}{5}$$

$$F_{X|Y=-1}(1) = f_{X|Y=-1}(-1) + f_{X|Y=-1}(1) = \frac{1}{5} + \frac{4}{5} = 1$$

<u>Therefore:</u>

$$F_{X|Y=-1}(x) = \begin{cases} 0 & (x < -1) \\ \frac{1}{5} & (-1 \leq x < 1) \\ 1 & (x \geq 1) \end{cases}$$

Exercise 5 e)

e)

$$f_{X,Y}(-1,0) = 0 \neq f_X(-1)\, f_Y(0) = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$$

Therefore X and Y are not independent.

Also, we could have used $f_X(x) \neq f_{X|Y=-1}(x)$

6. If the values of the joint probability function of X and Y are as shown in the table

Y \ X	0	1	2
0	1/12	1/6	1/24
1	1/4	1/4	1/40
2	1/8	1/20	0
3	1/120	0	0

find

a) the marginal probability function of X;

b) the marginal probability function of Y;

c) the conditional probability function of X given $Y = 1$;

d) the conditional probability function of Y given $X = 0$.

Exercise 6 a)

X	0	1	2
Y			
0	1/12	1/6	1/24
1	1/4	1/4	1/40
2	1/8	1/20	0
3	1/120	0	0

a)

$$f_X(0) = \frac{1}{12} + \frac{1}{4} + \frac{1}{8} + \frac{1}{120} = \cdots = \frac{7}{15}$$

$$f_X(1) = \frac{1}{6} + \frac{1}{4} + \frac{1}{20} = \cdots = \frac{7}{15}$$

$$f_X(2) = \frac{1}{24} + \frac{1}{40} = \cdots = \frac{1}{15}$$

<u>Conclusion</u>:

$$f_X(x) = \begin{cases} \frac{7}{15} & (x = 0, 1) \\ \frac{1}{15} & (x = 2) \\ 0 & (\text{elsewhere}) \end{cases}$$

Exercise 6 b)

b)

$$f_Y(0) = \frac{1}{12} + \frac{1}{6} + \frac{1}{24} = \frac{7}{24}$$

$$f_Y(1) = \frac{1}{4} + \frac{1}{4} + \frac{1}{40} = \frac{21}{40}$$

$$f_Y(2) = \frac{1}{8} + \frac{1}{20} = \frac{7}{40}$$

$$f_Y(3) = \frac{1}{120}$$

<u>Conclusion</u>:

$$f_Y(y) = \begin{cases} \frac{7}{24} & (y = 0) \\ \frac{21}{40} & (y = 1) \\ \frac{7}{40} & (y = 2) \\ \frac{1}{120} & (y = 3) \\ 0 & (\text{elsewhere}) \end{cases}$$

Exercise 6 c)

c)

$$f_{X|Y=1}(x) = \frac{f_{X,Y}(x,1)}{f_Y(1)} = \begin{cases} f_{X,Y}(0,1) / f_Y(1) = \frac{1}{4} / \frac{21}{40} = \frac{10}{21} & (x=0) \\ f_{X,Y}(1,1) / f_Y(1) = \frac{1}{4} / \frac{21}{40} = \frac{10}{21} & (x=1) \\ f_{X,Y}(2,1) / f_Y(1) = \frac{1}{40} / \frac{21}{40} = \frac{1}{21} & (x=2) \\ 0 & (\text{elsewhere}) \end{cases}$$

Therefore:

$$f_{X|Y=1}(x) = \begin{cases} \frac{10}{21} & (x = 1) \\ \frac{1}{21} & (x = 2) \\ 0 & (\text{elsewhere}) \end{cases}$$

Exercise 6 d)

d)

$$f_{Y|X=0}(y) = \frac{f_{X,Y}(0,y)}{f_X(0)} = \begin{cases} f_{X,Y}(0,0)\,/\,f_X(0) = \frac{1}{12}\Big/\frac{7}{15} = \frac{5}{28} & x = 0 \\ f_{X,Y}(0,1)\,/\,f_X(0) = \frac{1}{4}\Big/\frac{7}{15} = \frac{15}{28} & x = 1 \\ f_{X,Y}(0,2)\,/\,f_X(0) = \frac{1}{8}\Big/\frac{7}{15} = \frac{15}{56} & x = 2 \\ f_{X,Y}(0,3)\,/\,f_X(0) = \frac{1}{120}\Big/\frac{7}{15} = \frac{1}{56} & x = 3 \\ 0 & (\text{elsewhere}) \end{cases}$$

Therefore:

$$f_{Y|X=0}(y) = \begin{cases} \frac{5}{28} & (y=0) \\ \frac{15}{28} & (y=1) \\ \frac{15}{56} & (y=2) \\ \frac{1}{56} & (y=3) \\ 0 & (\text{elsewhere}) \end{cases}$$

7. If the joint cumulative distribution function of X and Y is given by

$$F_{X,Y}(x,y) = \begin{cases} (1-e^{-x^2})(1-e^{-y^2}) & \text{, for } x \geq 0,\ y \geq 0 \\ 0 & \text{, elsewhere} \end{cases}$$

(a) Find the marginal cumulative distribution functions of the two random variables X and Y.

(b) Find the joint density function of the two random variables X and Y.

(c) Find $P(1 < X \leq 2, 1 < Y \leq 2)$.

(d) Verify if X and Y are independent random variables.

Exercise 7 a)

$$F_{X,Y}(x,y) = \begin{cases} (1-e^{-x^2})(1-e^{-y^2}) & \text{, for } x \geq 0,\ y \geq 0 \\ 0 & \text{, elsewhere} \end{cases}$$

a)

$$F_X(x) = \lim_{y \to +\infty} F_{X,Y}(x,y) = \begin{cases} 1-e^{-x^2} & (x \geqslant 0) \\ 0 & (x < 0) \end{cases}$$

Auxiliary calculation:

$$\lim_{y \to +\infty} \left((1-e^{-x^2})(1-e^{-y^2}) \right) = 1 - e^{-x^2}$$

$$F_Y(y) = \lim_{x \to +\infty} F_{X,Y}(x,y) = \begin{cases} 1-e^{-y^2} & (y \geqslant 0) \\ 0 & (y < 0) \end{cases}$$

Auxiliary calculation:

$$\lim_{x \to +\infty} \left((1-e^{-x^2})(1-e^{-y^2}) \right) = 1 - e^{-y^2}$$

Exercise 7 b)

b)

$$f_{X,Y}(x,y) = \frac{\partial}{\partial x}\frac{\partial}{\partial y} F_{X,Y}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left((1-e^{-x^2})(1-e^{-y^2})\right)\right) =$$

$$= \frac{\partial}{\partial x}\left((1-e^{-x^2})\frac{\partial}{\partial y}(1-e^{-y^2})\right) =$$

$$= \frac{\partial}{\partial x}\left((1-e^{-x^2})\ 2ye^{-y^2}\right) = 2ye^{-y^2}\frac{\partial}{\partial x}(1-e^{-x^2}) =$$

$$= 2ye^{-y^2}\,2xe^{-x^2} = 4xye^{-y^2}e^{-x^2} \quad (x>0,\ y>0);$$

$f_{X,Y}(x,y) = 0$ otherwise

Exercise 7 c)

e)

$$
\begin{aligned}
P(1<X<2,\ 1<Y<2) &= \int_1^2\int_1^2 f_{X,Y}(x,y)\,dx\,dy = \\
&= \int_1^2\int_1^2 4xy\,e^{-y^2}e^{-x^2}\,dy\,dx = 4\int_1^2 x\,e^{-x^2}\int_1^2 y\,e^{-y^2}\,dy\,dx = \\
&= 4\int_1^2 x\,e^{-x^2}\,\tfrac{1}{2}\left[-e^{-y^2}\right]_1^2 dx = -2\int_1^2 x\,e^{-x^2}(e^{-4}-e^{-1})\,dx = \\
&= -2(e^{-4}-e^{-1})\int_1^2 x\,e^{-x^2}\,dx = -2(e^{-4}-e^{-1})\,\tfrac{1}{2}\left[-e^{-x^2}\right]_1^2 = \\
&= (e^{-4}-e^{-1})(e^{-4}-e^{-1}) = (e^{-4}-e^{-1})^2 \simeq 0.122
\end{aligned}
$$

Exercise 7 c)

Or:

$$P(1<X<2,\ 1<Y<2) = F_{X,Y}(2,2) - F_{X,Y}(2,1) - F_{X,Y}(1,2) + F_{X,Y}(1,1)$$

$$= (1-e^{-4})(1-e^{-4}) - (1-e^{-4})(1-e^{-1}) - (1-e^{-1})(1-e^{-4}) + (1-e^{-1})(1-e^{-1}) =$$

$$= (1-e^{-4})^2 - 2(1-e^{-1})(1-e^{-4}) + (1-e^{-1})^2 \simeq 0.122$$

Exercise 7 d)

d)

$$F_X(x) = \begin{cases} 1-e^{-x^2} & (x \geq 0) \\ 0 & (x < 0) \end{cases}$$

$$F_Y(y) = \begin{cases} 1-e^{-y^2} & (y \geq 0) \\ 0 & (y < 0) \end{cases}$$

$$F_X(x)\,F_Y(y) = \begin{cases} (1-e^{-x^2})(1-e^{-y^2}) & (x \geq 0,\ y \geq 0) \\ 0 & (\text{elsewhere}) \end{cases} = F_{X,Y}(x,y)$$

Conclusion: $X \perp Y$ because $\forall (x,y) \in \mathbb{R}^2 \rightarrow F_{X,Y}(x,y) = F_X(x)\,F_Y(y)$

8. If the joint probability density of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{4}(2x+y) & \text{, for } 0 < x < 1,\ 0 < y < 2 \\ 0 & \text{, elsewhere} \end{cases}$$

find

a) the marginal density of X;

b) the conditional density of Y given $X = 1/4$

c) the marginal density of Y;

d) the conditional density of X given $Y = 1$;

e) $P(X > \frac{1}{4} | X < \frac{3}{4})$.

f) the joint cumulative distribution function of X and Y.

Exercise 8 a)

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{4}(2x+y) & \text{, for } 0 < x < 1,\ 0 < y < 2 \\ 0 & \text{, elsewhere} \end{cases}$$

a)

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dy = \int_0^2 \frac{1}{4}(2x+y)\,dy =$$

$$= \frac{1}{4}\int_0^2 2x + y\,dy = \frac{1}{4}\left[2xy + \frac{y^2}{2}\right]_{y=0}^{y=2} =$$

$$= \frac{1}{4}(4x+2) = x + \frac{1}{2} \qquad (0 < x < 1)$$

Exercise 8 b)

b)

$$f_{Y|X=\frac{1}{4}}(y) = \frac{f_{X,Y}(\frac{1}{4}, y)}{f_X(\frac{1}{4})} = \frac{\frac{1}{4}(\frac{1}{2}+y)}{\frac{1}{4}+\frac{1}{2}} = \frac{\frac{1}{8}+\frac{1}{4}y}{\frac{3}{4}} =$$

$$= \frac{1}{6} + \frac{1}{3}y \quad (0 < y < 2)$$

Exercise 8 c)

c)

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx = \int_0^1 \frac{1}{4}(2x+y)\,dx =$$

$$= \frac{1}{4}\int_0^1 2x + y\,dx = \frac{1}{4}\left[x^2 + xy\right]_{x=0}^{x=1} =$$

$$= \frac{1}{4}(1+y) = \frac{1}{4} + \frac{1}{4}y \qquad (0 < y < 2)$$

Exercise 8 d)

d)

$$f_{X|Y=1}(x) = \frac{f_{X,Y}(x,1)}{f_Y(1)} = \frac{\frac{1}{4}(2x+1)}{\frac{1}{4}+\frac{1}{4}} = \frac{\frac{1}{4}+\frac{1}{2}x}{\frac{1}{2}} =$$

$$= \frac{1}{2} + x \qquad (0 < x < 1)$$

Exercise 8 e)

$$P\left(X > \tfrac{1}{4} \mid X < \tfrac{3}{4}\right) = \frac{P\left(\frac{1}{4} < X < \frac{3}{4}\right)}{P\left(X < \frac{3}{4}\right)} = \frac{\frac{1}{2}}{\frac{21}{32}} = \frac{32}{42} = \frac{16}{21}$$

$$P\left(\tfrac{1}{4} < X < \tfrac{3}{4}\right) = \int_{\frac{1}{4}}^{\frac{3}{4}} f_X(x)\,dx = \int_{\frac{1}{4}}^{\frac{3}{4}} x + \tfrac{1}{2}\,dx =$$

$$= \left[\frac{x^2}{2} + \frac{1}{2}x\right]_{1/4}^{3/4} = \frac{1}{2}\left(\left(\frac{3}{4}\right)^2 + \frac{3}{4} - \left(\left(\frac{1}{4}\right)^2 + \frac{1}{4}\right)\right) =$$

$$= \frac{1}{2}\left(\frac{9}{16} + \frac{3}{4} - \left(\frac{1}{16} + \frac{1}{4}\right)\right) = \frac{1}{2}\left(\frac{9}{16} + \frac{12}{16} - \frac{1}{16} - \frac{4}{16}\right) =$$

$$= \frac{1}{2}(1) = \frac{1}{2}$$

$$P\left(X < \tfrac{3}{4}\right) = \int_{-\infty}^{\frac{3}{4}} f_X(x)\,dx = \int_{0}^{\frac{3}{4}} x + \tfrac{1}{2}\,dx = \left[\frac{x^2}{2} + \frac{1}{2}x\right]_{0}^{\frac{3}{4}} =$$

$$= \frac{1}{2}\left(\left(\frac{3}{4}\right)^2 + \frac{3}{4}\right) = \frac{1}{2}\left(\frac{9}{16} + \frac{12}{16}\right) = \frac{1}{2}\left(\frac{21}{16}\right) = \frac{21}{32}$$

Exercise 8 f)

Auxiliary calculations:

$0 < x < 1, \; 0 < y < 2$

$$F_{X,Y}(x,y) = \int_0^x \int_0^y f_{X,Y}(u,v)\,dv\,du = \int_0^x \int_0^y \frac{1}{4}(2u+v)\,dv\,du =$$

$$= \frac{1}{4}\int_0^x \left[2uv + \frac{v^2}{2}\right]_{v=0}^{v=y} du = \frac{1}{4}\int_0^x 2uy + \frac{y^2}{2}\,du = \frac{1}{4}\left[2y\frac{u^2}{2} + \frac{y^2}{2}u\right]_{u=0}^{u=x} =$$

$$= \frac{1}{4}\left(yx^2 + \frac{y^2}{2}x\right) = \frac{1}{4}yx^2 + \frac{1}{8}xy^2$$

Exercise 8 f)

$\underline{0 < x < 1, \quad y \geq 2:}$

$$F_{X,Y}(x,y) = \int_0^x \underbrace{\int_{-\infty}^{+\infty} f_{X,Y}(u,y)\,dy}_{f_X(u)}\,du =$$

$$= \int_0^x f_X(u)\,du = \int_0^x u + \frac{1}{2}\,du =$$

$$= \left[\frac{u^2}{2} + \frac{1}{2}u\right]_0^x = \frac{1}{2}x^2 + \frac{1}{2}x$$

$\underline{x \geq 1, \quad 0 < y < 2}$

$$F_{X,Y}(x,y) = \int_0^y \underbrace{\int_{-\infty}^{+\infty} f_{X,Y}(x,v)\,dx}_{f_Y(v)}\,dv = \int_0^y \frac{1}{4} + \frac{1}{4}v\,dv =$$

$$= \left[\frac{1}{4}v + \frac{1}{8}v^2\right]_0^y = \frac{1}{4}y + \frac{1}{8}y^2$$

Exercise 8 f)

<u>Conclusion:</u>

$$F_{X,Y}(x,y) = \begin{cases} 0 & (x<0 \vee y<0) \\ \frac{1}{4}yx^2 + \frac{1}{8}xy^2 & (0\leq x<1,\ 0\leq y<2) \\ \frac{1}{2}x^2 + \frac{1}{2}x & (0\leq x<1 \wedge y\geq 2) \\ \frac{1}{4}y + \frac{1}{8}y^2 & (x\geq 1,\ 0\leq y<2) \\ 1 & (x\geq 1 \wedge y\geq 2) \end{cases}$$

9. Consider the joint probability density function of X and Y given by

$$f_{X,Y}(x,y) = \begin{cases} kxy & \text{, for } 0 < x < 2,\ 0 < y < 1 \\ 0 & \text{, elsewhere} \end{cases}$$

a) Determine k so that $f_{X,Y}$ can serve as a joint probability density function.

b) Determine the joint distribution function of X and Y.

c) Determine the marginal cumulative distribution function of X and Y.

d) Determine the marginal probability density function of X and Y.

e) Verify if X and Y are independent.

f) Compute Compute $P(X < 1.25, Y \leq 0.5)$, $P\left(Y > \frac{X}{4} + \frac{1}{2}\right)$ and $P\left(Y < \frac{X}{4} + \frac{1}{2}\right)$.

g) Compute $f_{X|Y=y}$ and $f_{Y|X=x}$.

Exercise 9 a)

$$f_{X,Y}(x,y) = \begin{cases} kxy & \text{, for } 0 < x < 2,\ 0 < y < 1 \\ 0 & \text{, elsewhere} \end{cases}$$

a) For $f_{X,Y}(x,y)$ to be a p.d.f. we must have:

$$\int_0^2 \int_0^1 f_{X,Y}(x,y)\,dy\,dx = 1 \iff$$

$$\iff \int_0^2 \int_0^1 Kxy\,dy\,dx = 1 \iff K\int_0^2 x \int_0^1 y\,dy\,dx = 1 \iff$$

$$\iff K\int_0^2 x\,\frac{1}{2}\left[y^2\right]_0^1 dx = 1 \iff K\int_0^2 x\,dx = 2 \iff$$

$$\iff K\,\frac{1}{2}\left[x^2\right]_0^2 = 2 \iff K(4-0) = 4 \iff K = 1$$

Exercise 9 b)

b) $f_{X,Y}(x,y) = \begin{cases} xy & (0 < x < 2,\ 0 < y < 1) \\ 0 & (\text{otherwise}) \end{cases}$

$$F_{X,Y}(x,y) = \int_0^x \int_0^y f_{X,Y}(u,v)\, dv\, du = \begin{cases} 0 & (x < 0 \vee y < 0) \\ \dfrac{x^2 y^2}{4} & (0 \leq x < 2,\ 0 \leq y < 1) \\ y^2 & (x \geq 2,\ 0 \leq y < 1) \\ \dfrac{x^2}{4} & (0 \leq x < 2,\ y \geq 1) \\ 1 & (x \geq 2,\ y \geq 1) \end{cases}$$

Exercise 9 b)

Auxiliary calculation:

<u>$0 < x < 2,\ 0 < y < 1$:</u>

$$F_{X,Y}(x,y) = \int_0^x \int_0^y u\, v \; dv\, du = \int_0^x \frac{u}{2} \left[v^2 \right]_{v=0}^{v=y} du =$$

$$= \frac{1}{2} \int_0^x u\, y^2 \, du = \frac{1}{4} \left[u^2 y^2 \right]_{u=0}^{u=x} = \frac{x^2 y^2}{4}$$

<u>$x \geq 2,\ 0 \leq y < 1$:</u> $\quad F_{X,Y}(x,y) = \frac{2^2 y^2}{4} = y^2$

<u>$0 \leq x < 2,\ y \geq 1$:</u> $\quad F_{X,Y}(x,y) = \frac{x^2 \cdot 1}{4} = \frac{x^2}{4}$

Exercise 9 c)

Auxiliary calculation: $F_X(x) = \lim_{y \to +\infty} F(x,y) = \frac{x^2}{4} \quad (0 < x < 2)$

$$F_X(x) = \begin{cases} 0 & (x < 0) \\ x^2/4 & (0 \leq x < 2) \\ 1 & (x \geq 2) \end{cases}$$

Auxiliary calculation: $F_Y(y) = \lim_{x \to +\infty} F_{X,Y}(x,y) = y^2 \quad (0 < y < 2)$

$$F_Y(y) = \begin{cases} 0 & (y < 0) \\ y^2 & (0 \leq y < 1) \\ 1 & (y \geq 1) \end{cases}$$

Exercise 9 d)

d)

$$f_X(x) = \frac{\partial}{\partial x} F_X(x) = \begin{cases} \frac{x}{2} & (0 < x < 2) \\ 0 & (\text{otherwise}) \end{cases}$$

$$f_Y(y) = \frac{\partial}{\partial y} F_Y(y) = \begin{cases} 2y & (0 < y < 1) \\ 0 & (\text{otherwise}) \end{cases}$$

Exercise 9 e)

$X \perp Y$ because:

$$f_X(x)\, f_Y(y) = \begin{cases} \frac{x}{2}\, 2y \\ 0 \end{cases} = \begin{cases} xy & (0 < x < 2,\ 0 < y < 1) \\ 0 & (\text{otherwise}) \end{cases} = f_{X,Y}(x, y)$$

Exercise 9 f)

f)

$$P(X < 1.25, Y \leq 0.5) = F_{X,Y}(1.25, 0.5) = \frac{1.25^2 \times 0.5^2}{4} = \frac{\left(\frac{5}{4}\right)^2 \times \left(\frac{1}{2}\right)^2}{4} = \frac{\frac{25}{16} \times \frac{1}{4}}{4}$$

$$= \frac{25}{64} \times \frac{1}{4} = \frac{25}{256}$$

$$P\left(Y > \frac{X}{4} + \frac{1}{2}\right) = 1 - P\left(Y \leq \frac{X}{4} + \frac{1}{2}\right) = 1 - \frac{17}{24} = \frac{7}{27}$$

Exercise 9 f)

Auxiliary calculation:

$$P\left(Y \leq \frac{X}{4} + \frac{1}{2}\right) = \int_0^2 \int_0^{\frac{x}{4}+\frac{1}{2}} f_{X,Y}(x,y) \, dy \, dx =$$

$$= \int_0^2 \int_0^{\frac{x}{4}+\frac{1}{2}} x y \, dy \, dx = \frac{1}{2} \int_0^2 x \left[y^2 \right]_{y=0}^{y=\frac{x}{4}+\frac{1}{2}} dx =$$

$$= \frac{1}{2} \int_0^2 x \left(\frac{x}{4} + \frac{1}{2} \right)^2 dx = \frac{1}{2} \int_0^2 x \left(\frac{x^2}{16} + \frac{1}{4} + \frac{x}{4} \right) dx =$$

$$= \frac{1}{2} \left(\frac{1}{16} \int_0^2 x^3 dx + \frac{1}{4} \int_0^2 x \, dx + \frac{1}{4} \int_0^2 x^2 dx \right) =$$

$$= \frac{1}{2} \left(\frac{1}{16} \left[\frac{x^4}{4} \right]_0^2 + \frac{1}{4} \left[\frac{x^2}{2} \right]_0^2 + \frac{1}{4} \left[\frac{x^3}{3} \right]_0^2 \right) =$$

$$= \frac{1}{2} \left(\frac{16}{64} + \frac{4}{8} + \frac{8}{12} \right) = \frac{1}{2} \left(\frac{48}{64} + \frac{8}{12} \right) =$$

$$= \frac{1}{2} \left(\frac{576 + 512}{768} \right) = \frac{1088}{1536} \underset{\div 8}{=} \frac{136}{192} \underset{\div 2}{=} \frac{68}{96} \underset{\div 4}{=} \frac{17}{24}$$

$$P\left(Y < \frac{X}{4} + \frac{1}{2}\right) = P\left(Y \leq \frac{X}{4} + \frac{1}{2}\right) = \frac{17}{24}$$

Exercise 9 g)

g)

$$f_{X|Y=y}(x) = \begin{cases} \dfrac{f_{X,Y}(x,y)}{f_Y(y)} = \dfrac{xy}{2y} = \dfrac{x}{2} & (0<x<2,\ \text{fixed } y) \\ 0 & (\text{otherwise}) \end{cases}$$

$$f_{Y|X=x}(y) = \begin{cases} \dfrac{f_{X,Y}(x,y)}{f_X(x)} = \dfrac{xy}{\frac{x}{2}} = 2y & (0<y<1,\ \text{fixed } x) \\ 0 & (\text{otherwise}) \end{cases}$$

10. Let X and Y be two independent random variables such that

$$f_X(x) = \begin{cases} \frac{1}{2}, & 0 < x < 2 \\ 0, & \text{elsewhere} \end{cases}, \quad \text{and} \quad f_Y(y) = \begin{cases} \frac{1}{3}, & 0 < y < 3 \\ 0, & \text{elsewhere} \end{cases}.$$

Find the joint density function of X and Y.

Exercise 10

$X \perp Y$

$$f_X(x) = \begin{cases} \frac{1}{2}, & 0 < x < 2 \\ 0, & \text{elsewhere} \end{cases}, \quad \text{and} \quad f_Y(y) = \begin{cases} \frac{1}{3}, & 0 < y < 3 \\ 0, & \text{elsewhere} \end{cases}.$$

Since $X \perp Y$ we have:

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y) = \begin{cases} \frac{1}{6} & (0<x<2,\ 0<y<3) \\ 0 & (\text{otherwise}) \end{cases}$$

11. If the joint probability density of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} 24xy & \text{, for } 0 < x < 1,\ 0 < y < 1,\ x + y < 1 \\ 0 & \text{, elsewhere} \end{cases}$$

a) Compute $P(X + Y < 1/2)$.

b) Compute $F_{X|Y=\frac{1}{2}}(x)$

Exercise 11 a)

$$f_{X,Y}(x,y) = \begin{cases} 24xy & \text{, for } 0 < x < 1,\ 0 < y < 1,\ x + y < 1 \\ 0 & \text{, elsewhere} \end{cases}$$

a) $x + y < \frac{1}{2} \Leftrightarrow x < \frac{1}{2} - y$ $\qquad$ $x + y < \frac{1}{2} \rightarrow \begin{cases} x < \frac{1}{2} \\ y < \frac{1}{2} \end{cases}$

$$P(X+Y < \tfrac{1}{2}) = \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}-y} 24xy \, dx \, dy =$$

$$= 24 \int_0^{\frac{1}{2}} y \int_0^{\frac{1}{2}-y} x \, dx \, dy = 24 \int_0^{\frac{1}{2}} y \, \tfrac{1}{2} \left[x^2 \right]_{x=0}^{x=\frac{1}{2}-y} dy =$$

$$= 12 \int_0^{\frac{1}{2}} y \left(\tfrac{1}{2} - y \right)^2 dy = 12 \int_0^{\frac{1}{2}} y \left(\tfrac{1}{4} + y^2 - y \right) dy =$$

$$= 12 \int_0^{\frac{1}{2}} \tfrac{1}{4} y + y^3 - y^2 \, dy = 12 \left[\tfrac{1}{8} y^2 + \tfrac{1}{4} y^4 - \frac{y^3}{3} \right]_{y=0}^{y=\frac{1}{2}}$$

$$= 12 \left(\tfrac{1}{8} \times \tfrac{1}{4} + \tfrac{1}{4} \times \tfrac{1}{16} - \tfrac{1}{3} \times \tfrac{1}{8} \right) = 12 \left(\tfrac{1}{32} + \tfrac{1}{64} - \tfrac{1}{24} \right) =$$

$$= \frac{12}{32} + \frac{12}{64} - \frac{12}{24} = \frac{24}{64} + \frac{12}{64} - \frac{12}{24} = \frac{36}{64} - \frac{12}{24} =$$

$$= \frac{864 - 768}{1536} = \frac{96}{1536} = \frac{1}{16}$$

Exercise 11 b)

b) $x + y < 1 \Leftrightarrow x < 1 - y$

$= P(X \leq x \mid Y = \frac{1}{2})$

$$F_{X|Y=\frac{1}{2}}(x) = \int_0^x f_{X|Y=\frac{1}{2}}(u)\, du = \int_0^x 8u \, du$$

$$= \frac{8}{2}\left[u^2\right]_{u=0}^{u=x} = 4x^2 \quad \left(0 < x < \frac{1}{2}\right)$$

because given $Y = \frac{1}{2}$ we have $x + y < 1 \Leftrightarrow$ $\Leftrightarrow x + \frac{1}{2} < 1 \Leftrightarrow x < \frac{1}{2}$

Conclusion: $F_{X|Y=\frac{1}{2}}(x) = \begin{cases} 0 & (x < 0) \\ 4x^2 & (0 \leq x < \frac{1}{2}) \\ 1 & (x \geq \frac{1}{2}) \end{cases}$

Auxiliary calculations:

$$f_{X|Y=\frac{1}{2}}(x) = \frac{f_{X,Y}(x, \frac{1}{2})}{f_Y(\frac{1}{2})} = \frac{12x}{\frac{3}{2}} = \frac{24}{3}x = 8x \quad \left(0 < x < \frac{1}{2}\right)$$

because $x + y < 1 \Leftrightarrow x < 1 - y$

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\, dx = \int_0^{1-y} 24xy \, dx = 24\left[\frac{x^2}{2} y\right]_{x=0}^{x=1-y} =$$

$$= 12y(1-y)^2 \quad (0 < y < 1)$$

$$f_Y\left(\tfrac{1}{2}\right) = 6\left(1 - \tfrac{1}{2}\right)^2 = \frac{6}{4} = \frac{3}{2}$$

12. If the joint probability density of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} 24y\,(1-x-y) & , \text{for } x>0, y>0, x+y<1 \\ 0 & , \text{elsewhere} \end{cases}$$

a) Find the marginal density of X;

b) Find the marginal density of Y;

d) Determine whether the two random variables are independent.

Exercise 12 a)

a)

$$x + y < 1 \iff \begin{cases} x < 1-y \\ y < 1-x \end{cases}$$

$$f_X(x) = \int_0^{1-x} f_{X,Y}(x,y)\,dy = \int_0^{1-x} 24y(1-x-y)\,dy =$$

$$= 24 \int_0^{1-x} y - xy - y^2\,dy = 24 \left[\frac{y^2}{2} - x\frac{y^2}{2} - \frac{y^3}{3} \right]_{y=0}^{y=1-x}$$

$$= 24 \left(\frac{(1-x)^2}{2} - x\frac{(1-x)^2}{2} - \frac{(1-x)^3}{3} \right) =$$

$$= 12(1-x)^2 - 12x(1-x)^2 - 8(1-x)^3 =$$

$$= (12 - 12x - 8(1-x))(1-x)^2 =$$

$$= (12 - 12x - 8 + 8x)(1-x)^2 =$$

$$= (4 - 4x)(1-x)^2 = 4(1-x)^3 \quad (0 < x < 1)$$

Note: $4(1-x)^3 = 4(1-x)^2(1-x) = -4(x-1)^2(-1)(1-x) =$
$= -4(x-1)^2(x-1) = -4(x-1)^3$ → Result in the answer sheet

b)

Exercise 12 b)

$$f_Y(y) = \int_0^{1-y} f_{X,Y}(x,y)\,dx = \int_0^{1-y} 24y(1-x-y)\,dx =$$

$$= 24y \int_0^{1-y} 1-x-y\,dx = 24y\left[x - \frac{x^2}{2} - xy\right]_{x=0}^{x=1-y}$$

$$= 24y\left(1-y-\frac{(1-y)^2}{2} - (1-y)y\right) =$$

$$= 24y\left(1-y-\frac{1+y^2-2y}{2} - y + y^2\right) =$$

$$= 24y\left(1-y-\frac{1}{2}-\frac{y^2}{2}+\cancel{y}-\cancel{y}+y^2\right) =$$

$$= 24y\left(\frac{1}{2}+\frac{y^2}{2}-y\right) = 12y + 12y^3 - 24y^2 =$$

$$= 12(y - 2y^2 + y^3) = 12y(1-2y+y^2) =$$

$$= 12y(1-y)^2 \qquad (0<y<1)$$

Exercise 12 c)

$$f_X(x)\, f_Y(y) = 4(1-x)^3\; 4(1-x)^3 = 8(1-x)^3 \neq f_{X,Y}(x,y) = 24y(1-x-y)$$

Conclusion: X and Y are not independent.

13. If X is the amount of money (in dollars) that a salesperson spends on gasoline during a day and Y is the corresponding amount of money (in dollars) for which he or she is reimbursed, the joint density of these two random variables is given by

$$f(x,y) = \begin{cases} \frac{1}{25}\left(\frac{20-x}{x}\right) & , \text{for } 10 < x < 20, \frac{x}{2} < y < x \\ 0 & , \text{elsewhere} \end{cases}$$

find

(a) the marginal density of X;

(b) the conditional density of Y given $X = 12$;

(c) the probability that the salesperson will be reimbursed at least \$8 when spending \$12.

Exercise 13 a)

a)

$$f_X(x) = \int_{\frac{x}{2}}^{x} f_{X,Y}(x, y)\,dy = \int_{\frac{x}{2}}^{x} \frac{1}{25}\left(\frac{20-x}{x}\right) dy =$$

$$= \frac{1}{25}\,\frac{20-x}{x}\,[\,y\,]_{y=\frac{x}{2}}^{y=x} = \frac{1}{25}\,\frac{20-x}{x}\left(x - \frac{x}{2}\right) =$$

$$= \frac{1}{25}\,\frac{20-x}{x}\,\frac{x}{2} = \frac{20x - x^2}{50x} = \frac{20-x}{50} \quad (10 < x < 20)$$

Exercise 13 b)

b)

$$f_{Y|X=12}(y) = \frac{f_{X,Y}(12,y)}{f_X(12)} = \frac{\frac{1}{25}\left(\frac{20-12}{12}\right)}{\frac{20-12}{50}} = \frac{\frac{1}{25}\left(\frac{8}{12}\right)}{\frac{8}{50}} =$$

$$= \frac{50}{8}\,\frac{1}{25}\,\frac{8}{12} = \frac{2}{12} = \frac{1}{6} \quad (6 < y < 12)$$

$$\left(x = 12 \Rightarrow \frac{x}{2} < Y < x \Leftrightarrow 6 < Y < 12\right)$$

c)

Exercise 13 c)

$$F_{Y|X=12}(8) = \int_{-\infty}^{8} f_{Y|X=12}(y)\, dy = \int_{6}^{8} \frac{1}{6}\, dy = \frac{1}{6}\left[\, y \,\right]_6^8 =$$

$$= \frac{8-6}{6} = \frac{2}{6} = \frac{1}{3}$$

$$P(Y \geqslant 8 \mid X = 12) = 1 - F_{Y|X=12}(8) = 1 - \frac{1}{3} = \frac{2}{3}$$

Thanks!

Questions?