List 3.5 - Radom-Nikodym Theorem

1. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $f : \Omega \to \Omega$ an integrable map with respect to μ. Define the map ν as:
$$\nu(A) = \int_A f d\mu, \quad \forall A \in \mathcal{F}.$$
   (a) Prove that ν is a finite signed measure.
   (b) Prove that $\nu \ll \mu$ and $\mu \ll \nu$. Are these two measures equivalent?

2. Show that if λ is a signed measure and there is $A \in \mathcal{F}$ such that $\lambda(A)$ is finite, then $\lambda(\emptyset) = 0$.

3. Consider the mensurable space $\Omega = [0, 1]$ endowed with the σ–algebra os borelians $\mathcal{B}([0, 1])$. Define the measure
$$\mu = \frac{1}{2}\delta_0 + \frac{1}{2}m,$$
where δ_0 is the Dirac measure centered at 0 and m is the usual Lebesgue measure in $[0, 1]$.
   (a) Show that $\delta_0 \ll \mu$ and $m \ll \mu$.
   (b) Show that if f is an integrable map with respect to μ then:
$$\int_A f d\mu = \frac{1}{2}\int_A f d\delta_0 + \frac{1}{2}\int_A f dm$$
   (c) Use the Radon-Nikodym Theorem to show that if $A \in \mathcal{B}([0, 1])$, then:
$$\delta_0(A) = \frac{1}{2}\int_A \frac{d\delta_0}{d\mu} d\delta_0 + \frac{1}{2}\int_A \frac{d\delta_0}{d\mu} dm$$
   and
$$m(A) = \frac{1}{2}\int_A \frac{dm}{d\mu} d\delta_0 + \frac{1}{2}\int_A \frac{dm}{d\mu} dm$$
   (d) Using the set $A = \{0\}$, compute $\frac{d\delta_0}{d\mu}(0)$ and $\frac{dm}{d\mu}(0)$.
   (e) Let $A \in \mathcal{B}([0, 1])$ for which $0 \notin A$. Show that
$$\int_A \frac{d\delta_0}{d\mu} dm = 0 \quad \text{and} \quad \int_A \frac{dm}{d\mu} dm = 2m(A) = \int_A 2dm.$$
   (f) Define the maps $\frac{d\delta_0}{d\mu}$ and $\frac{dm}{d\mu}$ in Ω.

4. Let ν, λ and μ be finite signed measures. Show that if $\mu \ll \lambda$ and $\lambda \ll \nu$, then $\mu \ll \nu$ and
$$\frac{d\nu}{d\mu} = \frac{d\nu}{d\lambda} \cdot \frac{d\lambda}{d\mu}.$$

5. Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ a measure space, $(\Omega_2, \mathcal{F}_2)$ a mensurable space, $f : \Omega_1 \to \Omega_2$ a mensurable map and $\mu_2 = \mu \circ f^{-1}$. Show that for any integrable map g with respect to μ_2, then:

$$\int_{\Omega_2} g d\mu_2 = \int_{\Omega_1} (g \circ f) d\mu_1.$$