Microeconomics

Chapter 3
Profit function

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Profit function

The previous chapter introduced the **profit function**:

$$\pi(\mathbf{p}, \mathbf{w}) = \max_{\mathbf{x}} \mathbf{py} - \mathbf{wx},$$

with $\mathbf{y} = \big(f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x})\big)$.

The **profit function** $\pi(\mathbf{p}, \mathbf{w})$ gives us the maximum profits as a function of the prices: For each $\mathbf{p}$ and $\mathbf{w}$ it uses the inputs $\mathbf{x}$ that maximizes profits.

Profit function has properties that follow from its definition. Properties are useful to analyze profit-maximizing behavior.

This chapter uses comparative statics to derive insights into the profit function (and some more).

What happens to the profit function if we weakly increase all output prices and weakly decrease all input prices?

Nondecreasing in output prices and nonincreasing in input prices

Nondecreasing in output prices and nonincreasing in input prices: If $\mathbf{p}' \geq \mathbf{p}$ and $\mathbf{w}' \leq \mathbf{w}$, then $\pi(\mathbf{p}', \mathbf{w}') \geq \pi(\mathbf{p}, \mathbf{w})$.

Proof:
Let (**y**,**x**) be the profit maximizing plan at (**p**,**w**) so $\pi(\mathbf{p}, \mathbf{w}) = \mathbf{py} - \mathbf{wx}$.

Let (**y**′,**x**′) be the profit maximizing plan at (**p**′,**w**′), so $\pi(\mathbf{p}', \mathbf{w}') = \mathbf{p}'\mathbf{y}' - \mathbf{w}'\mathbf{x}'$.

By WAPM, we have that $\mathbf{p}'\mathbf{y}' - \mathbf{w}'\mathbf{x}' \geq \mathbf{p}'\mathbf{y} - \mathbf{w}'\mathbf{x}$. That is, at (**p**′,**w**′) it must be that (**y**′,**x**′) is a weakly better choice then (**y**,**x**).

Moreover, since $\mathbf{p}' \geq \mathbf{p}$ and $\mathbf{w}' \leq \mathbf{w}$, we also have that $\mathbf{p}'\mathbf{y} - \mathbf{w}'\mathbf{x} \geq \mathbf{py} - \mathbf{wx}$.

Putting these two inequalities together, we have that:

$$\pi(\mathbf{p}', \mathbf{w}') = \mathbf{p}'\mathbf{y}' - \mathbf{w}'\mathbf{x}' \geq \mathbf{p}'\mathbf{y} - \mathbf{w}'\mathbf{x} \geq \mathbf{py} - \mathbf{wx} = \pi(\mathbf{p}, \mathbf{w}).$$

Homogeneous of degree one

What happens to the profit function if we multiply all prices by a factor $t > 0$? So we want to know how $\pi(t\mathbf{p}, t\mathbf{w})$ and $\pi(\mathbf{p}, \mathbf{w})$ are related.

Profit function is homogeneous of degree one: If we multiply all prices by $t > 0$, the profit is multiplied by t:

$$\pi(t\mathbf{p}, t\mathbf{w}) = t\pi(\mathbf{p}, \mathbf{w}).$$

This statement can be proved with the WAPM logic.

Exercise

Proof that the profit function is homogeneous of degree one.

Demand and supply functions from the profit function

Consider a firm with one input and one output, so the prices are (p, w).

If you were given the factor demand function $x(p, w)$, finding the profit function is easy. Simply substitute the factor demand function into π:

$$\begin{aligned}\pi(p, w) =& pf(x) - wx \\ =& pf(x(p, w)) - wx(p, w).\end{aligned}$$

Can we reverse the operation and recover the factor demand and supply function from the profit function? Yes, this is what **Hotelling's lemma** shows us.

Hotelling's lemma

Hotelling's lemma shows that we can find the factor demand and supply function from the profit function as follows:

$$\frac{\partial \pi(p, w)}{\partial w} = -x(p, w).$$
$$\frac{\partial \pi(p, w)}{\partial p} = f(p, w).$$

In words, **Hotelling's lemma** is that the:
(1) derivative of the profit function wrt the input price gives the (negative) factor demand function
(2) derivative of the profit function wrt the output price gives the factor supply function.

You already saw the analogous results for the consumer – Roy's identity and Shephard's lemma – back in Chapter 7.

Proof Hotelling's lemma for the supply function

We know the profit function $\pi(p, w)$ is given by

$$\pi(p, w) = pf(x(p, w)) - wx(p, w).$$

Note that we write the profit function, after you substituted for $x(p, w)$, as $\pi(p, w)$: when you look at the function, only (p, w) is in it, hence $\pi(p, w)$.

But if we are taking derivatives, it is useful to realize that p affects π directly and indirectly via x. Hence, it is useful to write the profit function as:

$$\pi(p, w) = \pi(p, w, x(p, w)).$$

Then, remember that partially differentiating the profit function, after you substituted for $x(p, w)$, is similar to taking the total derivative towards p,

$$\frac{\partial \pi(p, w, x(p, w))}{\partial p} = \frac{\partial \pi(p, w, x)}{\partial p} + \frac{\partial \pi(p, w, x)}{\partial x} \frac{\partial x(p, w)}{\partial p} = \frac{d\pi(p, w, x)}{dp}.$$

Proof Hotelling's lemma for the supply function

Let us label the two effects of the derivative as the "direct effect" and the "indirect effect":

$$\frac{\partial \pi(p,w)}{\partial p} = \underbrace{\frac{\partial \pi(p,w,x)}{\partial p}}_{\text{direct effect}} + \underbrace{\frac{\partial \pi(p,w,x)}{\partial x}\frac{\partial x(p,w)}{\partial p}}_{\text{indirect effect}}.$$

Using that the profit function is $\pi(p,w) = pf(x(p,w)) - wx(p,w)$, we can now write the derivative as:

$$\begin{aligned}\frac{\partial \pi(p,w)}{\partial p} &= f(x(p,w)) + p\frac{\partial f(x)}{\partial x}\frac{\partial x(p,w)}{\partial p} - w\frac{\partial x(p,w)}{\partial p} \\ &= \underbrace{f(x(p,w))}_{\text{direct effect}} + \underbrace{\left(p\frac{\partial f(x)}{\partial x} - w\right)\frac{\partial x(p,w)}{\partial p}}_{\text{indirect effect}}.\end{aligned}$$

Proof Hotelling's lemma for the supply function

The derivative of the profit function is:

$$\frac{\partial \pi(p,w)}{\partial p} = \underbrace{f(x(p,w))}_{\text{direct effect}} + \underbrace{\left(p\frac{\partial f(x)}{\partial x} - w\right)\frac{\partial x(p,w)}{\partial p}}_{\text{indirect effect}}.$$

From the FOC of profit maximization we know that the indirect effect is zero at $x = x^* = x(p,w)$,

$$p\frac{\partial f(x)}{\partial x} - w = 0.$$

So that **only the direct effect matters**:

$$\frac{\partial \pi(p,w)}{\partial p} = f(x(p,w)).$$

Exercise

Proof Hotelling's Lemma for the factor demand function. That is, show that

$$\frac{\partial \pi(p, w)}{\partial w} = -x(p, w)$$

The envelope theorem

Hotelling's lemma follows from a more general result known as the envelope theorem.

Envelope theorem: if you want to know how an optimized function (e.g., $\pi(p, w)$) changes when an exogenous variable changes (e.g., p), only the *direct* effect of this exogenous variable needs to be considered, even if the exogenous variable also enters the optimized function *indirectly* as part of the solution to endogenous choice variables (e.g., $x(p, w)$).

The envelope theorem

Consider an arbitrary maximization problem

$$\max_x f(a, x),$$

where x is the endogenous choice variable and a is the exogenous variable. The solution to this maximization problem is a function $x(a)$: the optimal x as function of the exogenous variable a.

We can substitute $x(a)$ in $f(a, x)$ which will give us the maximum value of f, $f(a, x(a)) = f(a)$, as function of the exogenous variable a.

Note the analogy: $f(a, x(a)) = f(a)$ is the profit function, $x(a)$ is the factor demand, and a is the exogenous price.

We are often interested in how the maximum value $f(a, x(a))$ changes when a changes. That is:

$$\frac{\partial f(a, x(a))}{\partial a}.$$

This is a similar question to Hotelling's lemma: how do (maximal) profits change when prices change?

The envelope theorem

Remember that partially differentiating the objective function, after you substituted for $x(a)$, is similar to taking the total derivative,

$$\frac{\partial f(a, x(a))}{\partial a} = \underbrace{\frac{\partial f(a, x)}{\partial a}}_{\text{direct effect}} + \underbrace{\frac{\partial f(a, x)}{\partial x} \frac{\partial x(a)}{\partial a}}_{\text{indirect effect}} = \frac{df(a, x)}{da}.$$

The FOC of maximizing $f(a, x)$ is

$$\frac{\partial f(a, x)}{\partial x} = 0.$$

Hence, we know that the indirect effect is zero at $x = x^* = x(a)$, and only the direct effect remains:

$$\frac{\partial f(a, x(a))}{\partial a} = \frac{\partial f(a, x)}{\partial a}.$$

Intuitively, the partial derivative of $f(a, x(a))$ with respect to a is given by the partial derivative of f with respect to a "holding x fixed at the optimal choice $x(a)$". Indeed, since $x(a)$ is chosen optimally, f will not change when $x(a)$ changes slightly because of a change in a.

Exercise

Consider the function $f(a, x) = ln(x) - ax$.

1. Find the function $x(a)$ that maximizes $f(a, x)$.
2. Plug $x(a)$ into $f(a, x)$ to find $f(a, x(a))$. Explicitly show that:

$$\frac{\partial f(a, x(a))}{\partial a} = \frac{\partial f(a, x)}{\partial a}.$$

Homework exercises

Exercises: 3.3, 3.4 (only give the factor demand functions), and exercises on the slides