STATISTICAL LABORATORY

Lecture 10: Trend Analysis of a Time Series (Theory and Practice)

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Applied Mathematics for Economics and Management
1st Year / 1st Semester – 2026/2027

Program

1. Fundamental Concepts of Statistics
2. Exploratory Data Analysis
3. Organizing and Summarizing Data
4. Association and Relationships Between Variables
5. Index Numbers
6. **Time Series Analysis**

Trend analysis of a time series

In this section, we study the trend component of a time series using two methods:

1. **Analytical Method**
2. **Moving Average Method**

We assume that the series does not present seasonality, and that the cyclical component is not being considered.

Therefore, the model is:

$$x_t = T_t + e_t$$

where:

- x_t = observed value of the time series at time t
- T_t = trend
- e_t = irregular/random component
- $t = 1, 2, \ldots, n$

Analytical method – linear trend

We assume that the time series follows a linear trend: $T_t = a + b \times t$, such that

$$x_t = T_t + e_t = a + bt + e_t$$

where:

- x_t = observed value at time t
- T_t = trend component
- e_t = irregular/random component (random error)
- $t = 1, 2, \ldots, n$

Analytical method – linear trend

Estimation of Parameters: Least Squares Method

The parameters a and b are determined using the Least Squares Method. Thus, the parameters a and b are chosen to minimize the sum of squared residuals (also called sum of squared estimation errors):

$$\sum_{t=1}^{n}(x_t - a - bt)^2$$

Solving the first order condition gives the estimators:

$$\hat{b} = \frac{n\sum_{t=1}^{n} tx_t - \left(\sum_{t=1}^{n} t\right)\left(\sum_{t=1}^{n} x_t\right)}{n\sum_{t=1}^{n} t^2 - \left(\sum_{t=1}^{n} t\right)^2} \quad \text{and} \quad \hat{a} = \overline{x} - \hat{b}\overline{t}$$

where:

$$\overline{x} = \frac{1}{n}\sum_{t=1}^{n} x_t \quad \text{and} \quad \overline{t} = \frac{1}{n}\sum_{t=1}^{n} t.$$

Estimation of Parameters: Least Squares Method

Alternatively, the slope b can be calculated as the ratio of the covariance between x_t and t to the variance of t:

$$b = \frac{s_{xt}}{s_t^2}$$

where:

- $s_{xt} = \frac{1}{n}\sum_{t=1}^{n}(t - \overline{t})(x_t - \overline{x})$ (covariance between t and x_t)
- $s_t^2 = \frac{1}{n}\sum_{t=1}^{n}(t - \overline{t})^2 = \text{Var}(t)$

Analytical method – linear trend

- Note that this is the same formula when we introduce linear regression.
- Previously, in linear regression, we are choosing a and b to minimize

$$\sum_{i=1}^{n} (y_i - a - bx_i)^2$$

- Now, y_i becomes x_t, and x_i becomes t. So we are choosing a and b to minimize

$$\sum_{t=1}^{n} (x_t - a - bt)^2$$

Analytical method – linear trend

Estimation of Parameters: Least Squares Method

If $t = 1, 2, \ldots, n$ – mean and variance of t:

$$\bar{t} = \frac{1}{n}\sum_{t=1}^{n} t = \frac{n+1}{2}$$

$$s_t^2 = \text{Var}(t) = \frac{1}{n}\sum_{t=1}^{n}(t-\bar{t})^2 = \frac{n^2-1}{12}$$

Residuals:

$$\hat{e}_t = x_t - (\hat{a} + \hat{b}t)$$

where $\hat{e}_t$ measures the deviation of the observed value from the estimated trend.

Note:

Other functional forms may be adopted for the trend (for example, quadratic, exponential, or logarithmic) if the data suggest a different pattern.

This approach described above corresponds to the first analytical method (see Silvestre, 2007).

Analytical method – linear trend (Optional)

In the previous slide, we used the result $\sum_{t=1}^{n} t = \frac{n(n+1)}{2}$. Let $S = \sum_{t=1}^{n} t$, it can be written as

$$S = 1 + 2 + 3 + \cdots + (n-1) + n \tag{1}$$

or

$$S = n + (n-1) + (n-2) + \cdots + 2 + 1 \tag{2}$$

Summing (1) and (2), we get 2S,

$$\begin{aligned} 2S &= (n+1) + (n-1+2) + (n-2+3) + \cdots + (n+1) \\ &= \underbrace{(n+1) + (n+1) + \cdots + (n+1)}_{n \text{ terms}} \\ &= n(n+1) \end{aligned}$$

So $S = \frac{(n+1)n}{2}$.

Analytical method – linear trend (Optional)

To show $\frac{1}{n}\sum_{t=1}^{n}(t-\bar{t})^2 = \frac{n^2-1}{12}$, we also need the result:

$$\sum_{t=1}^{n} t^2 = \frac{n(n+1)(2n+1)}{6}.$$

To prove this, consider the expansion of $(t+1)^3$, since $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, we have

$$(t+1)^3 - t^3 = 3t^2 + 3t + 1$$

Summing both sides from $t = 1$ to n, we have

$$\sum_{t=1}^{n}\left((t+1)^3 - t^3\right) = 3\sum_{t=1}^{n} t^2 + 3\sum_{t=1}^{n} t + n \tag{3}$$

Analytical method – linear trend (Optional)

Writing the LHS of (3) explicitly yields

$$\begin{aligned} & \sum_{t=1}^{n}\left((t+1)^3-t^3\right) \\ = \; & \left[(1+1)^3-1\right]+\left[(2+1)^3-2^3\right]+\cdots+\left[(n+1)^3-n^3\right] \\ = \; & \left[2^3-1\right]+\left[3^3-2^3\right]\cdots+\left[(n+1)^3-n^3\right]. \end{aligned}$$

Look at the first two brackets, we can see the2^3 in the first bracket is cancelled by the 2^3 in the second bracket. In fact, we can write the following **telescoping sum**:

$$\begin{aligned} & \left[2^3-1\right]+\left[3^3-2^3\right]\cdots+\left[(n+1)^3-n^3\right] \\ = \; & -1+2^3-2^3+3^3-3^3+\cdots+n^3-n^3+(n+1)^3 \\ = \; & (n+1)^3-1. \end{aligned} \tag{4}$$

Analytical method – linear trend (Optional)

This allows us to rewrite (3),

$$\begin{aligned}(n+1)^3 - 1 &= 3\sum_{t=1}^{n} t^2 + 3\sum_{t=1}^{n} t + n \\ &= 3\sum_{t=1}^{n} t^2 + 3\frac{n(n+1)}{2} + n \end{aligned} \tag{5}$$

Solving for $\sum_{t=1}^{n} t^2$ in the above equation gives us

$$\sum_{t=1}^{n} t^2 = \frac{n(n+1)(2n+1)}{6}$$

Analytical method – linear trend: example

Observed Time Series

Data (x_t):

t	x_t
1	5
2	7
3	9
4	11
5	13

Step 1 – Compute Means

$$\overline{t} = \frac{1+2+3+4+5}{5} = 3$$

$$\overline{x} = \frac{5+7+9+11+13}{5} = 9$$

Analytical method – linear trend: example

Observed Time Series

	t	x_t
Data (x_t)	1	5
	2	7
	3	9
	4	11
	5	13

Step 2 – Compute Covariance and Variance

$$s_{xt} = \frac{1}{n}\sum_{t=1}^{5}(t-\bar{t})(x_t-\bar{x})$$

$$= \frac{1}{5}[(-2)(-4)+(-1)(-2)+(0)(0)+(1)(2)+(2)(4)] = 4$$

$$s_t^2 = \frac{1}{5}\sum_{t=1}^{5}(t-\bar{t})^2 = \frac{1}{5}[(-2)^2+(-1)^2+0^2+1^2+2^2] = \frac{10}{5} = 2$$

Analytical method – linear trend: example

Observed Time Series

Data (x_t):

t	x_t
1	5
2	7
3	9
4	11
5	13

Step 3 – Compute Slope and Intercept

$$b = \frac{s_{xt}}{s_t^2} = \frac{4}{2} = 2$$

$$a = \overline{x} - b\overline{t} = 9 - 2 \cdot 3 = 3$$

Analytical method – linear trend: example

Observed Time Series

Data (x_t):

t	x_t
1	5
2	7
3	9
4	11
5	13

Step 4 – Trend Equation

$$\hat{T}_t = \hat{a} + \hat{b}t = 3 + 2t$$

Step 5 – Residuals

$$\hat{e}_t = x_t - \hat{T}_t$$

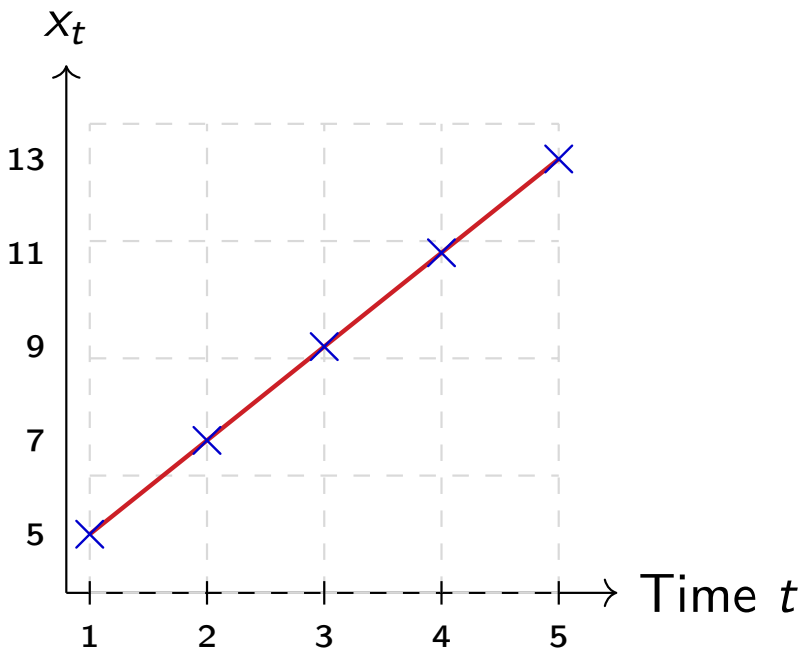
Analytical method – linear trend: example

Observed Time Series Data (x_t):

t	x_t	$\hat{T}_t$ (Trend)	$\hat{e}_t$ (Residuals)
1	5	5	0
2	7	7	0
3	9	9	0
4	11	11	0
5	13	13	0

All residuals are zero $\rightarrow$ the linear trend perfectly fits this simple data.

Analytical method – linear trend: example



t	x_t	$\hat{T}_t$	$\hat{e}_t$
1	5	5	0
2	7	7	0
3	9	9	0
4	11	11	0
5	13	13	0

Note:

Residuals are rarely all zero in practice. In our case, this occurs because it is an academic example.

Moving average method

Concept

- The moving average (MA) method smooths a time series by calculating averages over consecutive periods.
- The period (n) is the number of consecutive observations used to calculate each average.

Moving Average Formulas

a) Simple Moving Average (Period 3)

$$\frac{x_1 + x_2 + x_3}{3}, \quad \frac{x_2 + x_3 + x_4}{3}, \quad \frac{x_3 + x_4 + x_5}{3}, \ldots$$

b) Simple Moving Average (Period 4)

$$\frac{x_1 + x_2 + x_3 + x_4}{4}, \quad \frac{x_2 + x_3 + x_4 + x_5}{4}, \ldots$$

c) Centered Moving Average (Period 4)

$$\frac{1}{2}\left(\frac{x_1 + x_2 + x_3 + x_4}{4} + \frac{x_2 + x_3 + x_4 + x_5}{4}\right) = \frac{1}{8}(x_1 + 2x_2 + 2x_3 + 2x_4 + x_5), \ldots$$

Purpose of moving average method

1. **Reduce Random Fluctuations**
   - Original data can vary a lot from one period to another.
   - The moving average smooths these variations, making the overall pattern easier to see.
   - Distant past data is less useful, so we rely on the most recent observations.
2. **Highlight Trends**
   - Helps to identify whether values are increasing, decreasing, or stable over time.
3. **Facilitate Simple Forecasts**
   - The moving average can serve as a basis to predict future values, assuming the trend continues.
   - One step ahead forecast:

$$\hat{y}_{t+1|t} = \frac{1}{n}\sum_{i=0}^{n-1} y_{t-i} = \frac{y_t + y_{t-1} + \cdots + y_{t-n+1}}{n}$$

Note:

MA_3 denotes the moving averages with a period of 3.

Choosing the moving average period

Shorter periods (e.g., 3):

- Respond quickly to recent changes.
- Less smoothing, may still show fluctuations.

Longer periods (e.g., 4, 5, or more):

- Produce smoother curves.
- Better for identifying long-term trends, but slower to respond to recent changes.

Rule of thumb:

- The period should reflect the seasonality or frequency of the data.
- For monthly data with seasonal cycles of 3 months, use period 3.
- For quarterly or yearly trends, use period 4, 5, or more.

What 3- and 4-period MAs show:

- They smooth the time series.
- They highlight the trend (upward, downward, or stable).
- $MA_3 \rightarrow$ reacts more quickly to short-term changes.
- $MA_4 \rightarrow$ produces a smoother trend line.

Moving average method: example

Observed Data (x_t):

t	x_t
1	10
2	12
3	15
4	18
5	20
6	23
7	25
8	28

a) Period 3 Moving Average

$$\frac{x_1 + x_2 + x_3}{3} = 12.33, \quad \frac{x_2 + x_3 + x_4}{3} = 15.00, \ldots$$

b) Period 4 Moving Average

$$\frac{x_1 + x_2 + x_3 + x_4}{4} = 13.75, \quad \frac{x_2 + x_3 + x_4 + x_5}{4} = 16.25, \ldots$$

c) Centered Period 4 Moving Average

$$\frac{1}{2}\left(\frac{x_1 + x_2 + x_3 + x_4}{4} + \frac{x_2 + x_3 + x_4 + x_5}{4}\right) = 15.00, \ldots$$

Difference between 4-period MA and Centered MA:

MA_4 lies between the 2nd and 3rd observations. To properly align with time periods, a centered moving average averages two consecutive MA_4 values.

Moving average (MA) method: example

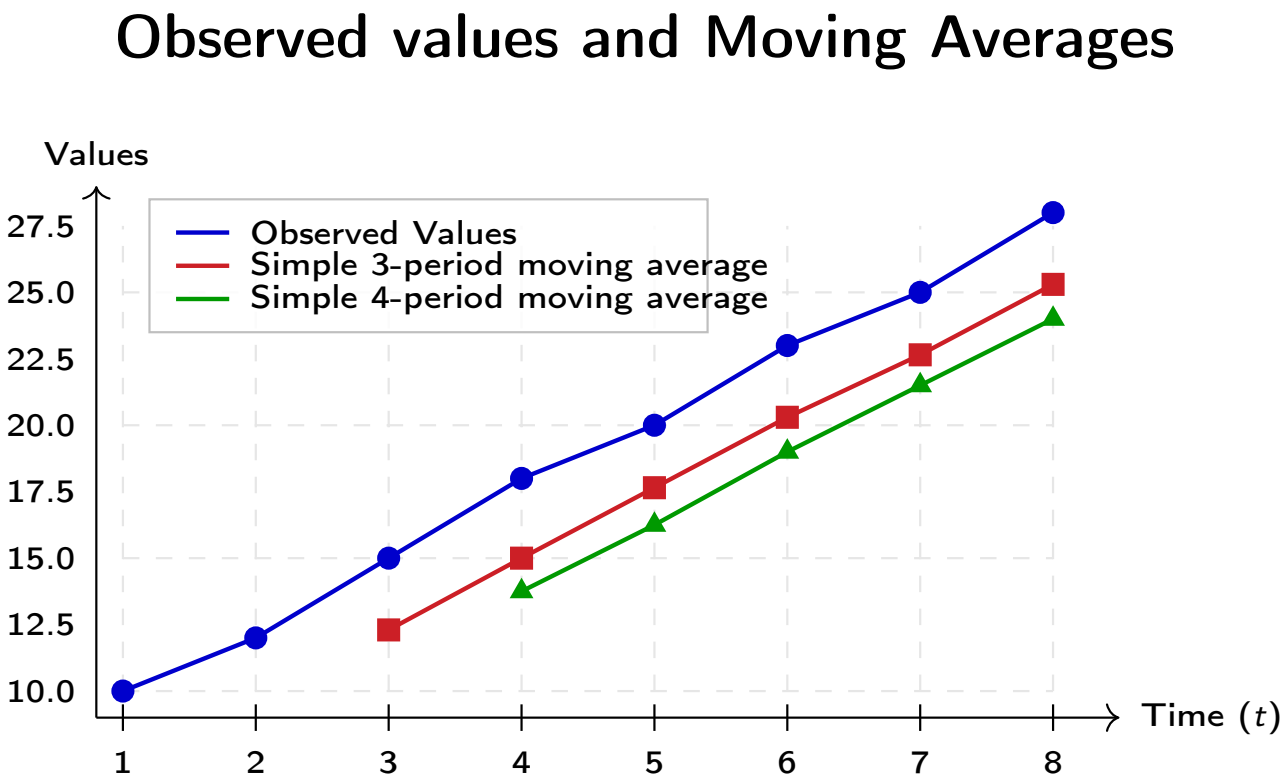
Observed Time Series Data (x_t):

t	x_t	MA – Period 3	MA – Period 4	Centered MA – Period 4
1	10	–	–	–
2	12	–	–	–
3	15	12.33	–	–
4	18	15.00	13.75	–
5	20	17.67	16.25	15.00
6	23	20.33	19.00	17.63
7	25	22.67	21.50	20.25
8	28	25.33	24.00	22.75

Note & Interpretation:

MA_3 smooths short-term fluctuations. MA_4 is smoother than MA_3. Centered MA_4 aligns values with the center of the period. Shorter periods react faster; longer/centered averages reveal long-term trends more clearly.

Moving average (MA) method: example



Exercise 6.1

6.1 Consider the data from Exercise 5.5: The data below refer to the number of online orders received by a company over the past 10 years:

Year	1	2	3	4	5	6	7	8	9	10
Orders	25	35	50	70	99	145	230	340	500	800

a) Analyze the evolution of the variable under study using an appropriate graphical representation.

b) Determine the trend values using 3-period moving averages.

Seasonality

Seasonal movements repeat at regular intervals. There are two types of seasonal patterns:

Fixed Pattern

- The seasonal effect is constant over time.
- Example: Sales increase every December.
- *Note: In this study, we will only consider fixed patterns.*

Variable Pattern

- The seasonal effect changes over time, not perfectly repeated.
- Example: Tourist visits vary each year depending on weather events.

Seasonality Models:

- **Additive:** Seasonal variation is constant, independent of the level of the series.

$$X_t = T_t + S_t + E_t$$

- **Multiplicative:** Seasonal variation changes proportionally with the level of the series.

$$X_t = T_t \times S_t \times E_t$$

Key Difference:

In the additive model, the seasonal component does not depend on the trend. In the multiplicative model, the higher the trend, the larger the seasonal effect.

Studying seasonal movements

Two methods to study seasonal movements:

1. Monthly averages (or medians) method
2. Moving averages method

Note:

The seasonal component is not directly estimated under these methods. Unlike the trend, which can be modeled (for example, by fitting a straight line or curve), the seasonal effect is measured using **seasonal indices**, which reflect the recurring pattern over time.

These indices indicate how each period (month, quarter, etc.) deviates proportionally from the overall average.

Monthly averages method

Steps (Silvestre's Method):

1. Arrange the observations in a table, reserving rows for months and columns for years (or vice versa).
2. Calculate the sums for each month (row) and for each year (column), and record them in the respective row and column.
3. Compute the monthly averages and the overall average (average of the monthly averages).
4. Express each month's average as a proportion of the overall average (for multiplicative seasonality).
5. For additive seasonality, consider the difference from the overall average.

Note:

This method (Monthly Averages Method) is not used to estimate the seasonal component itself, but rather to identify and measure the seasonal pattern through seasonal indices. The indices can then be used to adjust the series for seasonality in further analysis.

Monthly averages method – multiplicative seasonality: example

Month	2021	2022	2023	Row Sum	Row Mean	Seasonal Index
Jan	120	130	125	375	125	0.82
Feb	140	135	145	420	140	0.91
Mar	150	155	160	465	155	1.01
Apr	160	165	170	495	165	1.08
May	180	175	185	540	180	1.18
Column Sum	750	760	785	2295	–	–
Column Mean	150	152	157	–	153	–

Seasonal Index Interpretation:

- Values $> 1 \rightarrow$ month above average activity
- Values $< 1 \rightarrow$ month below average activity

Explanations:

1. Row Sum / Mean: across years for each month.
2. Seasonal Index:
   Row Mean $\div$ Overall Mean(153).
3. Column Sum / Mean: across months for each year.

Monthly averages method – multiplicative seasonality: example

How to Calculate the Seasonal Index (Multiplicative Model)

1. Calculate the Row Mean (monthly average)

- For each month, sum the values across the years and divide by the number of years.
- Example: January

$$\text{Row Mean(Jan)} = \frac{120 + 130 + 125}{3} = 125$$

2. Calculate the Overall Mean (global average)

- Average of all row means.
- For 5 months:

$$\text{Overall Mean} = \frac{125 + 140 + 155 + 165 + 180}{5} = 153$$

Monthly averages method – multiplicative seasonality: example

3. Calculate the Seasonal Index

- For each month:

$$\text{Seasonal Index} = \frac{\text{Row Mean}}{\text{Overall Mean}}$$

- Example January:

$$SI_{Jan} = \frac{125}{153} \approx 0.82$$

- Example February:

$$SI_{Feb} = \frac{140}{153} \approx 0.91$$

- Example March:

$$SI_{Mar} = \frac{155}{153} \approx 1.01$$

Note:

Moving averages method for seasonal analysis

Definition:

- The Moving Averages Method is used to smooth the time series and isolate the seasonal component.
- Particularly useful for series with a multiplicative or additive seasonal pattern.

Objective:

- To smooth the time series, eliminate short-term fluctuations, and identify the seasonal component by isolating the trend using centered moving averages.

Assumption:

- At this stage, we assume that the time series is composed only of trend and seasonal components (T and S), that is, $X_t = T_t + S_t$ (additive model) or $X_t = T_t \times S_t$ (multiplicative model). The irregular component is not considered.

Moving averages method for seasonal analysis

Steps of the Method

1. Arrange the observations

- Place the data in a table by year (columns) and quarters (rows), e.g. $X_{1\mathrm{I}}, X_{1\mathrm{II}}, X_{1\mathrm{III}}, X_{1\mathrm{IV}}, X_{2\mathrm{I}}, X_{2\mathrm{II}}, \ldots$

2. Compute the 4-quarter Moving Averages (MA(4))

- Each moving average is obtained as:

$$\mathrm{MA}(4)_t = \frac{X_t + X_{t-1} + X_{t-2} + X_{t-4}}{4}$$

Note:

In this case, we are considering quarterly data (four quarters per year), so a 4-term moving average is used to smooth out seasonal effects and estimate the trend component.

3. Calculate the Centered Moving Averages (CMA)

Since a 4-term moving average lies between two quarters, we must center it to align with the actual time period:

$$CMA_t = \frac{MA(4)_t + MA(4)_{t+1}}{2}$$

Or using Silvestre's method:

$$CMA = \frac{\left(\frac{X_{1I}+X_{1II}+X_{1III}+X_{1IV}}{4}\right) + \left(\frac{X_{2I}+X_{2II}+X_{2III}+X_{2IV}}{4}\right)}{2}$$

4. Compute the Seasonal Component

- **Multiplicative model:** $S_t = \frac{X_t}{CMA_t}$
- **Additive model:** $S_t = X_t - CMA_t$
- We assume that CMA_t is a good approximation of T_t.

These values represent the seasonal effect for each quarter.

5. Average the Seasonal Effects by Quarter

Group all the seasonal effects of the same quarter (I, II, III, IV) and compute their mean to obtain the **Seasonal Indices**:

$$SI_{\text{I}} = \text{average of all } S_{\text{I}} \text{ values}$$

$$SI_{\text{II}} = \text{average of all } S_{\text{II}} \text{ values}$$

$$\ldots \text{and so on.}$$

Estimation of seasonal indices using MA method: example

Assumption

- At this stage, we assume that the time series contains only trend (T) and seasonal (S) components, i.e., $X_t = T_t \times S_t$.
- This is a multiplicative model, and the irregular component (I) is ignored.
- We also assume that the data are quarterly, so each year is divided into four quarters: Q1, Q2, Q3, Q4.

Year	Quarter	Observed Value (X_t)
2023	Q1	120
2023	Q2	150
2023	Q3	180
2023	Q4	160
2024	Q1	130
2024	Q2	160
2024	Q3	200
2024	Q4	170

Estimation of seasonal indices using MA method: example

Step 1 – Arrange the observations

Arrange the observed values X_t in chronological order, reserving one line for each period (e.g., quarter) and one column for each year.

Year	Q1	Q2	Q3	Q4
2023	120	150	180	160
2024	130	160	200	170

Estimation of seasonal indices using MA method: example

Step 2 – Compute the Moving Averages $MA(4)_t$

To smooth out the seasonal effect, compute 4-term moving averages:

$$MA(4)_t = \frac{X_t + X_{t-1} + X_{t-2} + X_{t-3}}{4}$$

Period	$MA(4)_t$
Between Q1–Q4 2023	$(120 + 150 + 180 + 160)/4 = 152.5$
Between Q2–Q1 2024	$(150 + 180 + 160 + 130)/4 = 155.0$
Between Q3–Q2 2024	$(180 + 160 + 130 + 160)/4 = 157.5$
Between Q4–Q3 2024	$(160 + 130 + 160 + 200)/4 = 162.5$
Between Q1–Q4 2024	$(130 + 160 + 200 + 170)/4 = 165.0$

Estimation of seasonal indices using MA method: example

Step 3 – Center the Moving Averages

Because 4 is an even number of terms, the moving averages must be centered:

$$CMA_t = \frac{MA(4)_t + MA(4)_{t+1}}{2}$$

Centered at	CMA_t
Q3 2023	$(152.5 + 155.0)/2 = 153.75$
Q4 2023	$(155.0 + 157.5)/2 = 156.25$
Q1 2024	$(157.5 + 162.5)/2 = 160.00$
Q2 2024	$(162.5 + 165.0)/2 = 163.75$

Note:

1. Window of 4 periods $\rightarrow$ no exact center.
2. Take the average of two consecutive MAs $\rightarrow$ centralizes on the middle period.
3. In your case, the CMA of Q1–Q4 2023 and Q2–Q1 2024 is centered on Q3-2023.

Estimation of seasonal indices using MA method: example

Step 4 – Compute the Ratios X_t/CMA_t

The ratio of each observation to its centered moving average provides the seasonal factor (and possibly a small irregular component):

Quarter	X_t	CMA_t	X_t/CMA_t
Q3 2023	180	153.75	1.17
Q4 2023	160	156.25	1.02
Q1 2024	130	160.00	0.81
Q2 2024	160	163.75	0.98

Estimation of seasonal indices using MA method: example

Step 5 – Average Ratios by Period Group the ratios by quarter and calculate their mean $\rightarrow$ this gives the seasonal index for each quarter:

Quarter	Average Ratio (Seasonal Index)
Q1	0.81
Q2	0.98
Q3	1.17
Q4	1.02

Note:

In this example, there is only one ratio available per quarter (due to the limited number of years and centered moving averages). Therefore, the seasonal index for each quarter is equal to that single ratio.

These indices indicate how each quarter typically deviates from the trend: above (> 1) or below (< 1).

Estimation of seasonal indices using MA method: example

Step 6 – Normalize the Indices

Normalize so that their average equals 1 (for the multiplicative model):

$$S_i^* = \frac{S_i}{\text{Mean of all indices}}$$

$$\text{Mean of all indices} = (0.81 + 0.98 + 1.17 + 1.02)/4 = 0.995$$

Quarter	Normalized S_i^*
Q1	0.81
Q2	0.99
Q3	1.18
Q4	1.03

Estimation of seasonal indices using MA method: example

Step 7 – Interpretation

- $Q3 = 1.18 \rightarrow$ On average, values in the 3rd quarter are 18% higher than the yearly average.
- $Q1 = 0.81 \rightarrow$ On average, values in the 1st quarter are 19% lower than the average.

Notes:

- In this example, quarterly data for two consecutive years (2023–2024) were used.
- It is assumed that the series contains only trend (T) and seasonal (S) components:

$$X_t = T_t \times S_t$$

- The irregular component is not considered at this stage.

Exercise 6.5

The number of mobile phones sold in a retail chain is shown below[cite: 6]:

Year / Quarter	1	2	3	4
1	500	440	580	600
2	450	400	500	550

a) Graphically analyse the evolution of mobile phone sales.

b) Verify the existence of seasonality using an appropriate method and remove it (assuming additive seasonality).

c) Identify which components of the series are associated with the following events, briefly justifying: c1) the budget deficit stabilization agreement with the EU; c2) the Christmas season.

THANKS!

Questions?