Lecture 5: Factor pricing models

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Overview

- The discount factor of the simple consumption-based model does not perform well empirically
- Linear factor pricing models are *more or less ad-hoc* ways to solve that problem
- A k factor model explains the expected excess return on portfolio i according to

$$E\left(R_i\right) - R^f = \beta_{i,1} E\left(f_1\right) + ... + \beta_{i,k} E\left(f_k\right)$$

where the $E\left(f\right)$ are expected premiums and the factor loadings, the β's, are the slopes in the time-series regression

$$R_{i,t} - R_t^f = \alpha_i + \beta_{i,1} f_{1,t} + ... + \beta_{i,k} f_{k,t} + \varepsilon_{i,t}$$

Overview

- For instance the Fama-Fench 3 factor model considers:
  - the market excess return: $R_m - R_f$
  - the SMB (**S**mall **M**inus **B**ig), the difference between the return on a portfolio of small stocks minus the return on a portfolio of large stocks
  - the HML (**H**igh **M**inus **L**ow), the difference between the return on a portfolio of high-book-to-market stocks minus the return on a portfolio of low-book-to-market stocks.
  - These factors are calculated with combinations of portfolios composed by ranked stocks
  - The Capitalization ranking, the Book-to-Market ranking and the available historical market data may be accessed on Kenneth French's web page

Discount factor and single factor model (or single beta model)

Proposition:

$$p = E(mx) \text{ implies } E(R^i) = \alpha + \beta_{i,m}\lambda_m.$$

Proof:

$$1 = E\left(mR^i\right) = E\left(m\right)E\left(R^i\right) + cov\left(m, R^i\right)$$

$$E\left(R^i\right) = \frac{1}{E\left(m\right)} - \frac{cov\left(m, R^i\right)}{E\left(m\right)}$$

multiply and divide by $var(m)$ and $\alpha \equiv \frac{1}{E(m)}$.

$$\begin{aligned} E\left(R^i\right) &= \alpha + \left(-\frac{cov\left(m, R^i\right)}{var(m)}\right)\left(\frac{var(m)}{E\left(m\right)}\right) \\ &= \alpha + \beta_{i,m}\lambda_m \end{aligned}$$

Discount factor and single factor model

- The $\beta_{i,m}$ is the symmetric of the regression coefficient of returns R^i on the discount factor.

Alternatively, can write the formula for excess returns:

$$
\begin{aligned}
E\left(R^i\right) &= \frac{1}{E\left(m\right)} - \frac{cov\left(m, R^i\right)}{var(m)}\left(\frac{var(m)}{E\left(m\right)}\right) \\
E\left(R^i\right) &= R^f - \frac{cov\left(m, R^i - R^f\right)}{var(m)}\left(\frac{var(m)}{E\left(m\right)}\right) \\
E\left(R^{ie}\right) &= \beta_{i,m}\lambda_m
\end{aligned}
$$

where $R^{ie} = R^i - R^f$, using the fact that

$$
R^f = \frac{1}{E\left(m\right)} \text{ and } cov\left(m, R^i - R^f\right) = cov\left(m, R^i\right)
$$

Discount factor and single beta model

- **Definition**: CAPM is a model such that the mean asset returns are linear in the regression betas of asset returns on $m_t = \frac{\beta u'(c^i_{t+1})}{u'(c^i_t)}$.

$$R^i_t = a_i + \beta_i m_t + \varepsilon_{it}$$

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$$R^i_t = a_i + \beta_i m_t + \varepsilon_{it}$$

- (time series regression, one for each i to get the β_i's)

$$E\left(R^i\right) = \alpha_i + \beta_{i,m}\lambda_m$$

Discount factor and single beta model

- **Definition**: CAPM is a model such that the mean asset returns are linear in the regression betas of asset returns on $m_t = \frac{\beta u'(c_{t+1}^i)}{u'(c_t^i)}$.

$$R_t^i = a_i + \beta_i m_t + \varepsilon_{it}$$

- (time series regression, one for each i to get the β_i's)

$$E\left(R^i\right) = \alpha_i + \beta_{i,m}\lambda_m$$

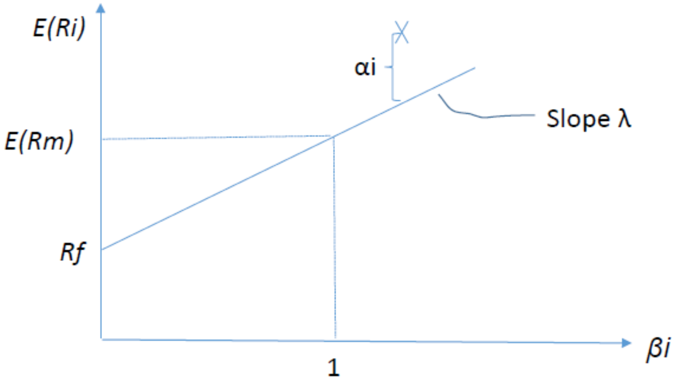
- (cross section regression of $E\left(R^i\right)$ on the β_i to get the λ_m)

Discount factor and single beta model

- λ_m is the slope of this cross-sectional relationship and model implies $\alpha_i = \alpha = R^f$
- The $cov\left(m, R^i\right)$ is in general negative. Positive expected returns are associated with positive correlation with consumption growth, and hence negative correlation with marginal utility growth (m).
- Thus, we expect

$$\beta_{i,m} = -\frac{cov\left(m, R^i - R^f\right)}{var(m)} > 0.$$

Discount factor and single beta model



Multiple Factor pricing and multiple beta models

- The single beta model does not work well in practice
- Average excess returns rise from growth (low book-to-market, "high price") to value (high book-to-market, "low price").
- Figure below includes the results of multiple regressions on the market excess return and Fama and French's hml factor,

$$R^e_{i,t} = a_i + b_i \times rmrf_t + h_i \times hml_t + \varepsilon_{i,t}$$

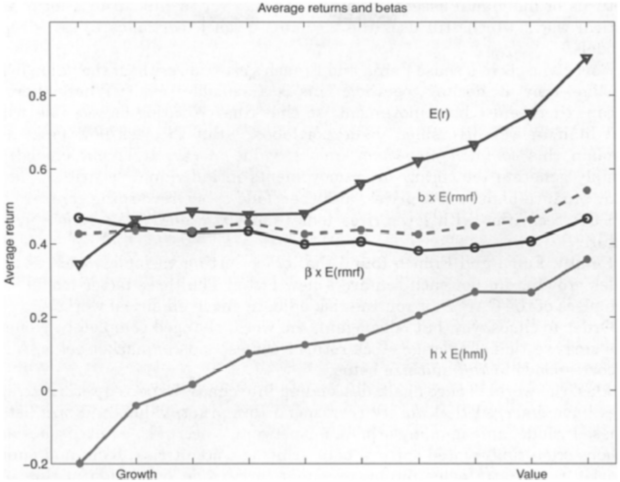


Figure 6. Average returns and betas. 10 Fama–French book-to-market portfolios. Monthly data, 1963–2010.

TABLE 1

Summary Statistics and Three-Factor Regressions for Simple Monthly Percent Excess Returns on 25 Portfolios Formed on Size and BE/ME: 7/63–12/93, 366 Months

	Book-to-market equity (BE/ME) quintiles									
Size	Low	2	3	4	High	Low	2	3	4	High
	Panel A: Summary statistics									
	Means					Standard deviations				
Small	0.31	0.70	0.82	0.95	1.08	7.67	6.74	6.14	5.85	6.14
2	0.48	0.71	0.91	0.93	1.09	7.13	6.25	5.71	5.23	5.94
3	0.44	0.68	0.75	0.86	1.05	6.52	5.53	5.11	4.79	5.48
4	0.51	0.39	0.64	0.80	1.04	5.86	5.28	4.97	4.81	5.67
Big	0.37	0.39	0.36	0.58	0.71	4.84	4.61	4.28	4.18	4.89

FIGURE 1 Fama and French (1996), Table 1.

Table I–*Continued*

Size	Book-to-market equity (BE/ME) quintiles									
	Low	2	3	4	High	Low	2	3	4	High
Panel B: Regressions: $R_i - R_f = a_i + b_i\left(R_M - R_f\right) + s_i\text{smb} + h_i\text{hml} + e_i$										
			a					$t(a)$		
Small	−0.45	−0.16	−0.05	0.04	0.02	−4.19	−2.04	−0.82	0.69	0.29
2	−0.07	−0.04	0.09	0.07	0.03	−0.80	−0.59	1.33	1.13	0.51
3	−0.08	0.04	−0.00	0.06	0.07	−1.07	0.47	−0.06	0.88	0.89
4	0.14	−0.19	−0.06	0.02	0.06	1.74	−2.43	−0.73	0.27	0.59
Big	0.20	−0.04	−0.10	−0.08	−0.14	3.14	−0.52	−1.23	−1.07	−1.17
			b					$t(b)$		
Small	1.03	1.01	0.94	0.89	0.94	39.10	50.89	59.93	58.47	57.71
2	1.10	1.04	0.99	0.97	1.08	52.94	61.14	58.17	62.97	65.58
3	1.10	1.02	0.98	0.97	1.07	57.08	55.49	53.11	55.96	52.37
4	1.07	1.07	1.05	1.03	1.18	54.77	54.48	51.79	45.76	46.27
Big	0.96	1.02	0.98	0.99	1.07	60.25	57.77	47.03	53.25	37.18
			s					$t(s)$		
Small	1.47	1.27	1.18	1.17	1.23	39.01	44.48	52.26	53.82	52.65
2	1.01	0.97	0.88	0.73	0.90	34.10	39.94	36.19	32.92	38.17
3	0.75	0.63	0.59	0.47	0.64	27.09	24.13	22.37	18.97	22.01
4	0.36	0.30	0.29	0.22	0.41	12.87	10.64	10.17	6.82	11.26
Big	−0.16	−0.13	−0.25	−0.16	−0.03	−6.97	−5.12	−8.45	−6.21	−0.77
			h					$t(h)$		
Small	−0.27	0.10	0.25	0.37	0.63	−6.28	3.03	9.74	15.16	23.62
2	−0.49	0.00	0.26	0.46	0.69	−14.66	0.34	9.21	18.14	25.59
3	−0.39	0.03	0.32	0.49	0.68	−12.56	0.89	10.73	17.45	20.43
4	−0.44	0.03	0.31	0.54	0.72	−13.98	0.97	9.45	14.70	17.34
Big	−0.47	0.00	0.20	0.56	0.82	−18.23	0.18	6.04	18.71	17.57

Multiple Factor pricing and multiple beta models

- The table shows that small stocks tend to have higher returns than big stocks
- high book to market stocks tend to have higher returns than low book to market stocks
- the estimated intercepts say that the model leaves:
  - a large negative unexplained part for the portfolios of the smallest size and lowest book to market quintiles
  - and a large positive unexplained return for the portfolio of stocks in the largest size and lowest book to market quintiles
  - otherwise the intercepts are close to zero

Multiple Factor pricing and multiple beta models

- CAPM (single beta model) works when the stocks are grouped by size only
- Does not work when stocks are grouped by book to market ratio: does not price well value and growth stocks
- This observation motivates efforts to tie the discount factor m to other data
- Linear factor pricing models are the most popular models of this sort in finance

Multiple Factor pricing and multiple beta models

- **Factor pricing models** replace the consumption-based expression for marginal utility growth with a linear model of the form

$$m_{t+1} = a + \mathbf{b}'\mathbf{f}_{t+1}$$

a and $\mathbf{b}$ are free parameters and $\mathbf{f}_{t+1}$ are the factors.

- This specification is equivalent to a **multiple-beta model**

$$E(\mathbf{R}) = \alpha + \boldsymbol{\beta}'\boldsymbol{\lambda}_{\mathbf{f}}$$

- **Procedure:** Get the $\boldsymbol{\beta}^i$ by running the regression

$$R^i_{t+1} = a + \boldsymbol{\beta}^{i\prime}\mathbf{f}_{t+1} + \varepsilon_{it+1}.$$

After that, the $\boldsymbol{\lambda}_{\mathbf{f}}$ is obtained by running the regression of $E(R^i_{t+1})$ on the the $\boldsymbol{\beta}^{i\prime}$

Multiple Factor pricing and multiple beta models

Theorem:

$$m_{t+1} = a + \mathbf{b}'\mathbf{f}_{t+1} \Longleftrightarrow E(\mathbf{R}) = \alpha + \boldsymbol{\beta}'\boldsymbol{\lambda}_{\mathbf{f}}$$

- It is easier to prove for excess returns.
- In this case

$$E\left(mR^{e}\right) = 0$$

and we do not get the value for $E\left(m\right)$. Thus, we can normalize it to any constant for instance $E\left(m\right) = 1$ or

$$m = 1 + \mathbf{b}'[\mathbf{f} - E(\mathbf{f})]$$

Multiple Factor pricing and multiple beta models

Theorem: Given the model

$$m = 1 + \mathbf{b}'[\mathbf{f} - E(\mathbf{f})] \text{ with } E\left(mR^e\right) = 0 \tag{1}$$

one can find $\boldsymbol{\lambda}_{\mathbf{f}}$ such that

$$E(\mathbf{R}^e) = \boldsymbol{\beta}'\boldsymbol{\lambda}_{\mathbf{f}} \tag{2}$$

where $\boldsymbol{\beta}$ are the multiple regression coefficients of excess returns $\mathbf{R}^e$ on the factors.
Conversely, given $\boldsymbol{\lambda}_{\mathbf{f}}$ in (2), we can find $\mathbf{b}$ such that (1) holds.

Multiple Factor pricing and multiple beta models

Proof: From (1),

$$0 = E\left(mR^e\right) = E\left(R^e\right) + cov(R^e, \mathbf{f}')\mathbf{b}$$

Thus,

$$E\left(R^e\right) = -cov(R^e, \mathbf{f}')\mathbf{b}$$

Divide and multiply by $var(\mathbf{f})$

$$E(R^e) = -cov(R^e, \mathbf{f}')var(\mathbf{f})^{-1}var(\mathbf{f})\mathbf{b} = \boldsymbol{\beta}'\boldsymbol{\lambda}_{\mathbf{f}}$$

where $\boldsymbol{\lambda} = -var(\mathbf{f})\mathbf{b}$.

Multiple Factor pricing and multiple beta models

- **What should one use for factors f** ?
  - Factor pricing models look for variables that are good proxies for aggregate marginal utility growth, i.e., variables for which

$$\frac{\beta u'(c_{t+1})}{u'(c_t)} \approx a + \mathbf{b}'\mathbf{f}_{t+1}$$

-
  - Consumption is related to: (i) returns on broadbased portfolios, (ii) interest rates, (iii) GDP growth, (iv) investment, (v) other macroeconomic variables, and (vi) variables that forecast income in the future like: term premium, dividend/price ratio, stock returns, etc.

Multiple Factor pricing and multiple beta models

- **Conclusion:** Factors should be thought as proxies for marginal utility growth
- **Important:** All factor models are derived as specializations of the consumption-based model
- **The idea:**
    - (i) Start with a general equilibrium model which produces relations that express the determinants of consumption from exogenous variables and other endogenous variables; equations of the form

$$c_t = g(f_t).$$

-
    - (ii) use this kind of equation to substitute out for consumption in the basic first order conditions.

Capital Asset Pricing Model (CAPM)

- It was independently developed by Lintner (1965), Mossin (1964) and Sharpe (1964)
- The CAPM is

$$m = a + bR_w$$

R_w = wealth portfolio return

CAPM

- The CAPM is the first, most famous and was the most widely used model in asset pricing
- The values for the parameters a and b are found by requiring the discount factor m price any two assets
  - For instance with
    $$1 = E(mR_W)$$
    and
    $$1 = E(m)R_f$$
    get 2 equations and 2 unknowns
  - It is conventional to proxy R_W by the return on a broad-based stock portfolio such as a NYSE index or a S&P500 index.

CAPM

- The CAPM is more often expressed in its beta representation

$$E(R_i) = \alpha + \beta_{i,R_w} \left[E\left(R_w\right) - \alpha \right]$$

- There are many derivations of the CAPM

1) Two period quadratic utility;

$$U(c_t, c_{t+1}) = -(c_t - c^*)^2 - \beta E[(c_{t+1} - c^*)^2]$$

- the quadratic utility assumption means marginal utility is *linear* in consumption

- the constraints are

$$c_{t+1} = W_{t+1}$$

$$W_{t+1} = R_{t+1}^{W}(W_t - c_t)$$

$$R^W \equiv \sum_{i=1}^{N} \theta_i R_i$$

$$\sum_{i=1}^{N} \theta_i = 1.$$

CAPM

$$\Rightarrow m_{t+1} = \beta \frac{c_{t+1} - c^*}{c_t - c^*} = \beta \frac{R^W_{t+1}\left(W_t - c_t\right) - c^*}{c_t - c^*}$$

$$= -\frac{\beta c^*}{c_t - c^*} + \frac{\beta\left(W_t - c_t\right)}{c_t - c^*} R^W_{t+1}$$

$$\Longleftrightarrow m_{t+1} = a_t + b_t R^W_{t+1}$$

CAPM

2) One period, exponential utility $u(c) = -e^{-\theta c}$, and normal returns;

$$E[u(c)] = E\left(-e^{-\theta c}\right)$$

θ is the coefficient of absolute risk aversion.

- If consumption is normally distributed, we have

$$Eu(c) = -e^{-\theta E(c) + \frac{1}{2}\theta^2 \sigma^2(c)}$$

the budget constraint is

$$c = y^f R_f + \mathbf{y}^T \mathbf{R}$$

$$W = y^f + \mathbf{y}^T \mathbf{1}$$

where $\left(y^f_, \mathbf{y}^T\right)$ is the vector of investments in the riskless and risky assets

CAPM

$$\Rightarrow Eu(c) = -e^{-\theta E(y^f R_f + \mathbf{y}^T \mathbf{R}) + \frac{1}{2}\theta^2 \sigma^2 (y^f R_f + \mathbf{y}^T \mathbf{R})}$$

$$\Rightarrow Eu(c) = -e^{-\theta(y^f R_f + \mathbf{y}^T E\mathbf{R}) + \frac{1}{2}\theta^2 \mathbf{y}^T \Sigma \mathbf{y}}$$

Lagrangian:

$$L = -e^{-\theta(y^f R_f + \mathbf{y}^T E\mathbf{R}) + \frac{1}{2}\theta^2 \mathbf{y}^T \Sigma \mathbf{y}} + \lambda \left[W - y^f - \mathbf{y}^T \mathbf{1} \right]$$

Maximization of this expression w.r.t. $\left(y^f_, \; \mathbf{y}^T \right)$

$$-\theta R_f e^{-\theta(y^f R_f + \mathbf{y}^T E\mathbf{R}) + \frac{1}{2}\theta^2 \mathbf{y}^T \Sigma \mathbf{y}} - \lambda = 0$$

$$\left(-\theta E R_i + \theta^2 \sum_{j=1}^{N} y_j cov\left(R_j, R_i\right) \right) e^{-\theta(y^f R_f + \mathbf{y}^T E\mathbf{R}) + \frac{1}{2}\theta^2 \mathbf{y}^T \Sigma \mathbf{y}} - \lambda = 0$$

CAPM

- Rewriting the second equation

$$\left(-\theta E\mathbf{R}+\theta^2\mathbf{y}^T\Sigma\right)e^{-\theta(y^f R_f+\mathbf{y}^T E\mathbf{R})+\frac{1}{2}\theta^2\mathbf{y}^T\Sigma\mathbf{y}}-\lambda\mathbf{1}=0$$

- Taking the ratio of the 2 equations

$$-\theta R_f\mathbf{1}=\left(-\theta E\mathbf{R}+\theta^2\mathbf{y}^T\Sigma\right)$$

$$\Rightarrow \frac{E\mathbf{R}-R_f\mathbf{1}}{\theta}\Sigma^{-1}=\mathbf{y}^T$$

Conclusion: investors invest more in risky assets if their expected return is higher, less if the risk aversion coefficient is higher, and less if assets are riskier

CAPM

the expression above is

$$\Leftrightarrow E\mathbf{R} - R_f = \theta \mathbf{y}^T \Sigma$$

$$\Rightarrow ER_i - R_f = \theta cov(R_i, \mathbf{y}^T \mathbf{R})$$

$$\Rightarrow ER_i - R_f = \theta cov(R_i, y^f R_f + \mathbf{y}^T \mathbf{R})$$

Define the market rate of return

$$R_m = y^f R_f + \mathbf{y}^T \mathbf{R}$$

CAPM

$$\Rightarrow ER_i = R_f + \theta cov(R_i, R_m)$$

$$\Rightarrow \frac{(ER_m - R_f)}{var(R_m)} = \theta$$

$$\Rightarrow ER_i = R_f + \frac{cov(R_i, R_m)}{var(R_m)} (ER_m - R_f)$$

$$\Leftrightarrow ER_i = \alpha + \beta_{R_i, R_m} (ER_m - \alpha)$$

Conclusion: the coeficient of absolute risk aversion is proportional to the price of risk

3) Infinite horizon, log utility and normally distributed returns.
Suppose the investor has log utility

$$u(c_t) = \log c_t$$

$$\Leftrightarrow u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma} \text{ with } \gamma = 1$$

CAPM

the wealth portfolio is the claim to all consumption dividends

$$p_t\left(W\right) = E_t \sum_{j=1}^{\infty}\left(m_{t+j}c_{t+j}\right) = E_t \sum_{j=1}^{\infty} \frac{\beta^j u'\left(c_{t+j}\right)}{u'\left(c_t\right)} c_{t+j}$$

$$= E_t \sum_{j=1}^{\infty} \beta^j \left(\frac{c_t}{c_{t+j}}\right) c_{t+j} = \frac{\beta}{1-\beta} c_t$$

$$R_{w,t+1} = \frac{p_{t+1}\left(W\right) + c_{t+1}}{p_t\left(W\right)} = \frac{\left(\frac{\beta}{1-\beta} + 1\right) c_{t+1}}{\left(\frac{\beta}{1-\beta}\right) c_t} = \frac{1}{\beta}\frac{c_{t+1}}{c_t} = \left[\frac{\beta u'\left(c_{t+1}\right)}{u'\left(c_t\right)}\right]^{-1}$$

conclusion: the return on the wealth portfolio is proportional to consumption growth and discount factor equals the inverse of the wealth portfolio return

$$m_{t+1} = \left(R_{w,t+1}\right)^{-1}$$

Arbitrage Pricing Theory (APT)

- The APT was introduced by Ross (1976) as an alternative to the CAPM
- APT is

$$m_{t+1} = a + \mathbf{b}^T \mathbf{f}_{t+1}$$

"there is a discount factor linear in the vector **f** that prices returns"

- The APT is "more general" than the CAPM as it allows for multiple risk factors

Arbitrage Pricing Theory (APT)

- **Approximate APT**
- Consider a **statistical** characterization for the payoff of asset i

$$x_i = E(x_i) + \sum_{j=1}^{M} \beta_{i,j}\tilde{f}_j + \varepsilon_i, \; i = 1, 2, ...N$$

where

$$\tilde{f} \equiv f - E(f)$$

- the factor decomposition can be regarded as a regression equation with

$$E(\varepsilon_i) = cov(\varepsilon_i, \tilde{f}_j) = 0, \text{ all } i \text{ and } j$$

Arbitrage Pricing Theory (APT)

- **The APT assumes** that

$$E(\varepsilon_i \varepsilon_k) = 0, \text{ for } i \neq k$$

This imposes a **restriction on the covariance matrix** of the payoffs $\mathbf{x}$

$$\begin{aligned} cov(x_i, x_k) &= E\left[(x_i - Ex_i)(x_k - Ex_k)\right] \\ &= E\left(\sum_{j=1}^{M} \beta_{i,j}\tilde{f}_j + \varepsilon_i\right)\left(\sum_{j=1}^{M} \beta_{k,j}\tilde{f}_j + \varepsilon_k\right) \\ &= \left(\sum_{j=1}^{M} \beta_{i,j}\beta_{k,j} E\left(\tilde{f}_j\right)^2 + \sum_{j \neq l}^{M}\sum_{j=1}^{M} \beta_{i,j}\beta_{k,l} E\left(\tilde{f}_j \tilde{f}_l\right) + E\left(\varepsilon_i \varepsilon_k\right)\right) \end{aligned}$$

where $E(\varepsilon_i \varepsilon_k) = 0$ for $i \neq k$ and $= \sigma^2(\varepsilon_i)$ for $i = k$

- That is the idiosyncratic terms ε_i must be uncorrelated

Arbitrage Pricing Theory (APT)

- The intuition behind the APT is that the completely idiosyncratic movements in asset returns should not carry any risk prices, since investors can diversify idiosyncratic returns away by holding diversified portfolios
- Therefore, risk prices or expected returns on a security should be related to the security's covariance with the common components or "factors" only
- It is important to explore under what conditions the idiosyncratic components have zero (or small) risk prices, so that only the common components matter to asset pricing
- If there were no residual, then we could price securities from the factors by arbitrage (by the law of one price)

Arbitrage Pricing Theory (APT)

- Can estimate a factor structure by running regressions if the factors are known.
  - For instance, the market (value-weighted portfolio), industry portfolios, size and book/market portfolios, small minus big portfolios, momentum portfolios, etc
- However, most of the time do not know the identities of the factor portfolios ahead of time.
  - In this case we have to use one of several statistical techniques under the broad heading of factor analysis (that is where the word "factor" came from) to estimate the factor model

Arbitrage Pricing Theory (APT)

- With multiple (orthogonalized) factors, we obtain

$$cov(x_i, x_k) = \left(\sum_{j=1}^{M} \beta_{i,j}\beta_{k,j} E\left(\tilde{f}_j\right)^2 + E\left(\varepsilon_i \varepsilon_k\right) \right)$$

Exact APT

- Suppose there was no idiosyncratic term i.e. we have an *exact factor model*

$$x_i = E\left(x_i\right) 1 + \boldsymbol{\beta}_i^T \tilde{\mathbf{f}}$$

then the price can only depend on the price of factors

$$p\left(x_i\right) = E\left(x_i\right) p\left(1\right) + \boldsymbol{\beta}_i^T p(\tilde{\mathbf{f}})$$

Arbitrage Pricing Theory (APT)

- with exact factor pricing and $x_i = R_i$

$$1 = E\left(R_i\right)\frac{1}{R_f} + \boldsymbol{\beta}_i^T p(\tilde{\mathbf{f}})$$

since $p\left(1\right) = E\left(m \cdot 1\right)$

$$\Rightarrow E\left(R_i\right) = R_f + \boldsymbol{\beta}_i^T\left[-p(\tilde{\mathbf{f}})R_f\right] = R_f + \boldsymbol{\beta}_i^T\boldsymbol{\lambda}$$

where $\boldsymbol{\lambda} = \left[-p(\tilde{\mathbf{f}})R_f\right]$

- expected returns are linear in the betas, and the constants $(\boldsymbol{\lambda})$ are related to the prices of the factors

In Practice

- Actual returns do not display an exact factor structure
- There is always some idiosyncratic or residual risk
- *But*, factor model regressions often have very high R^2, i.e. the idiosyncratic risks are small
- Thus, there is reason to hope that the APT holds approximately, especially for reasonably large portfolios

Arbitrage Pricing Theory (APT)

Formally:

- Assume:

$$x_i = E(x_i) + \boldsymbol{\beta}_i^T \widetilde{\mathbf{f}} + \varepsilon_i$$

$$p(x_i) = E(x_i)\, p(1) + \boldsymbol{\beta}_i^T p(\widetilde{\mathbf{f}}) + p(\varepsilon_i)$$

- what is the value of $p(\varepsilon_i)$? Is it small?

Arbitrage Pricing Theory (APT)

- Next we state 2 theorems that can be interpreted to say that the APT holds approximately for portfolios that either have high R^2, or well-diversified portfolios

Arbitrage Pricing Theory (APT)

Theorem: Fix a discount factor m that prices the factors, implying that $p(x_i) = E(mx_i)$, and $\sigma^2(m) \leq A$. Then, as $var(\varepsilon_i) \to 0$, $p(x_i - E(x_i)) \to p(\boldsymbol{\beta}_i^T \mathbf{f})$.
By doing some algebra:

$$\begin{aligned} var(x_i) &= var\left(E(x_i) + \boldsymbol{\beta}_i^T \tilde{\mathbf{f}} + \varepsilon_i\right) \\ &= var(\boldsymbol{\beta}_i^T \mathbf{f}) + var(\varepsilon_i) \end{aligned}$$

which is related to the regression R^2. By definition:

$$\frac{var(\varepsilon_i)}{var(x_i)} = 1 - R^2$$

The theorem says that as $R^2 \to 1$, $var(\varepsilon_i) \to 0$. But $var(\varepsilon_i) \to 0$ means that $\varepsilon_i \to 0$, i.e. ε_i is a random variable that takes values almost always very close to 0.
If m is bounded then $p(\varepsilon_i) = E(m\varepsilon_i) \to 0$.

Arbitrage Pricing Theory (APT)

Theorem: As the number of primitive assets increases, the R^2 of well-diversified portfolios increases to 1.
Proof:
Consider an equally weighted portfolio (in fact it does not need to be equally weighted...)

$$x_p = \frac{1}{N}\sum_{i=1}^{N} x_i$$

$$\Rightarrow x_p = \frac{1}{N}\sum_{i=1}^{N}\left(a_i + \boldsymbol{\beta}_i^T \mathbf{f} + \varepsilon_i\right)$$

Arbitrage Pricing Theory (APT)

$$\Rightarrow x_p = \frac{1}{N}\sum_{i=1}^{N} a_i + \frac{1}{N}\sum_{i=1}^{N} \boldsymbol{\beta}_i^T \mathbf{f} + \frac{1}{N}\sum_{i=1}^{N} \varepsilon_i$$

$$\Rightarrow x_p = a_p + \boldsymbol{\beta}_p^T \mathbf{f} + \varepsilon_p$$

where

$$var(\varepsilon_p) = var\left(\frac{1}{N}\sum_{i=1}^{N} \varepsilon_i\right)$$

since $E(\varepsilon_i \varepsilon_k) = 0$, for $i \neq k$ and if $var(\varepsilon_i)$ is bounded then

$$\lim_{N\to\infty} var(\varepsilon_p) = 0.$$

Arbitrage Pricing Theory (APT)

- These two theorems can be interpreted to say that the APT holds approximately (in the usual limiting sense) for either portfolios that have high R^2, or well-diversified portfolios
- These 2 theorems say that if you fix m and take limits over N or ε we get a good approximation to an exact APT
- However in practice, you fix N or ε and look for an m that can price the portfolios
- **Important**: It may be possible that the approximate APT does not work because we can always choose an m sufficiently "far out" to generate an arbitrarily large price for an arbitrarily small ε_i

Solution: impose ad-hoc restrictions on the m
Example: impose a bound on $\sigma^2(m)$
consider only the $m \in [\underline{m}, \overline{m}]$, where

$$\underline{m} = \arg\min_m \left\{ p\left(x_i\right) = E\left(m x_i\right), \text{ s.t. } E(mf) = p(f), m > 0, \sigma^2(m) \leq A \right\}$$

$$\overline{m} = \arg\max_m \left\{ p\left(x_i\right) = E\left(m x_i\right), \text{ s.t. } E(mf) = p(f), \; m > 0, \sigma^2(m) \leq A \right\}$$

- If we impose a restriction on the volatility of m then we have an APT limit theorem
- **Theorem:** As $\varepsilon_i \longrightarrow 0$ and $R^2 \longrightarrow 1$, the price $p(x_i - Ex_i)$ assigned by any discount factor m that satisfies $E(mf) = p(f)$, $m > 0, \sigma^2(m) \leq A$ approaches $p(\boldsymbol{\beta}_i^T \mathbf{f})$